EIT Reconstruction Using $L1$ Norms for Data and Image Terms

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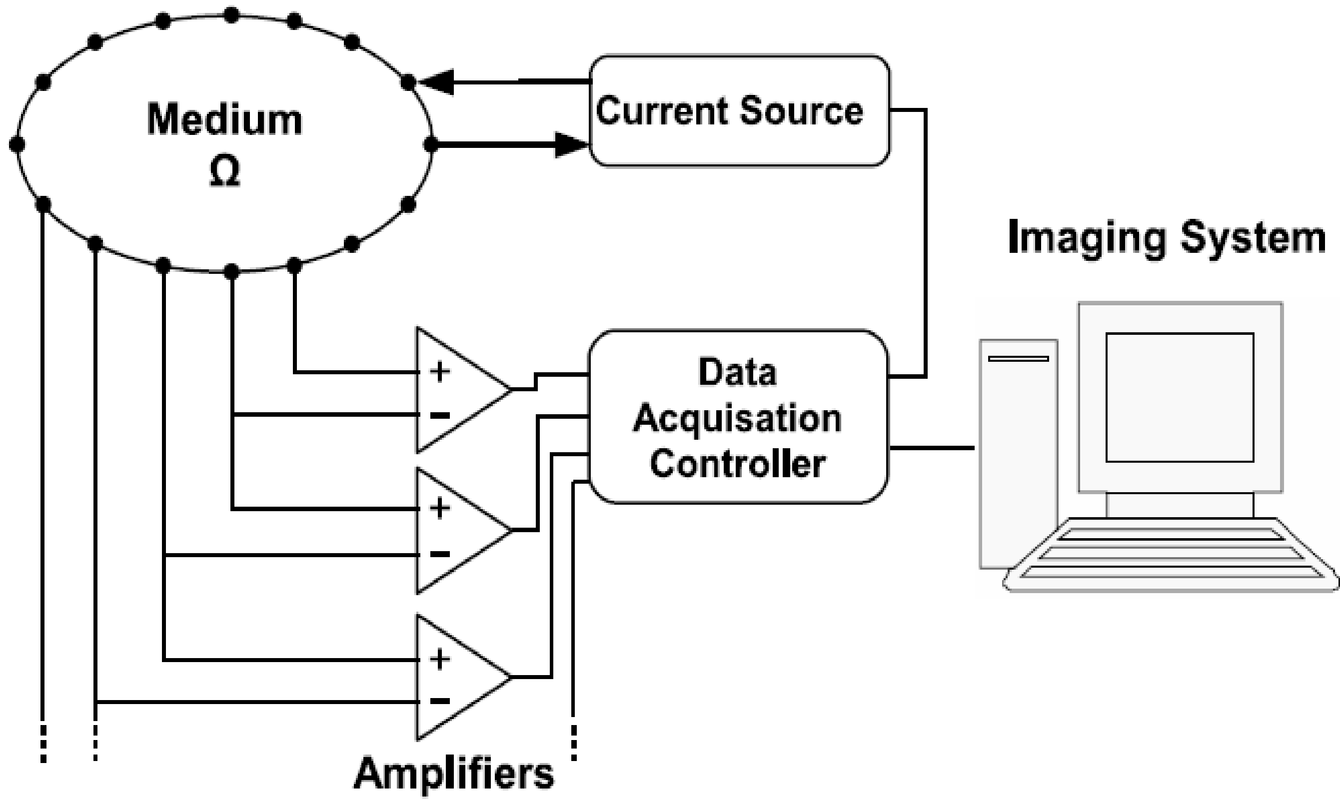
Electrical Impedance Tomograph



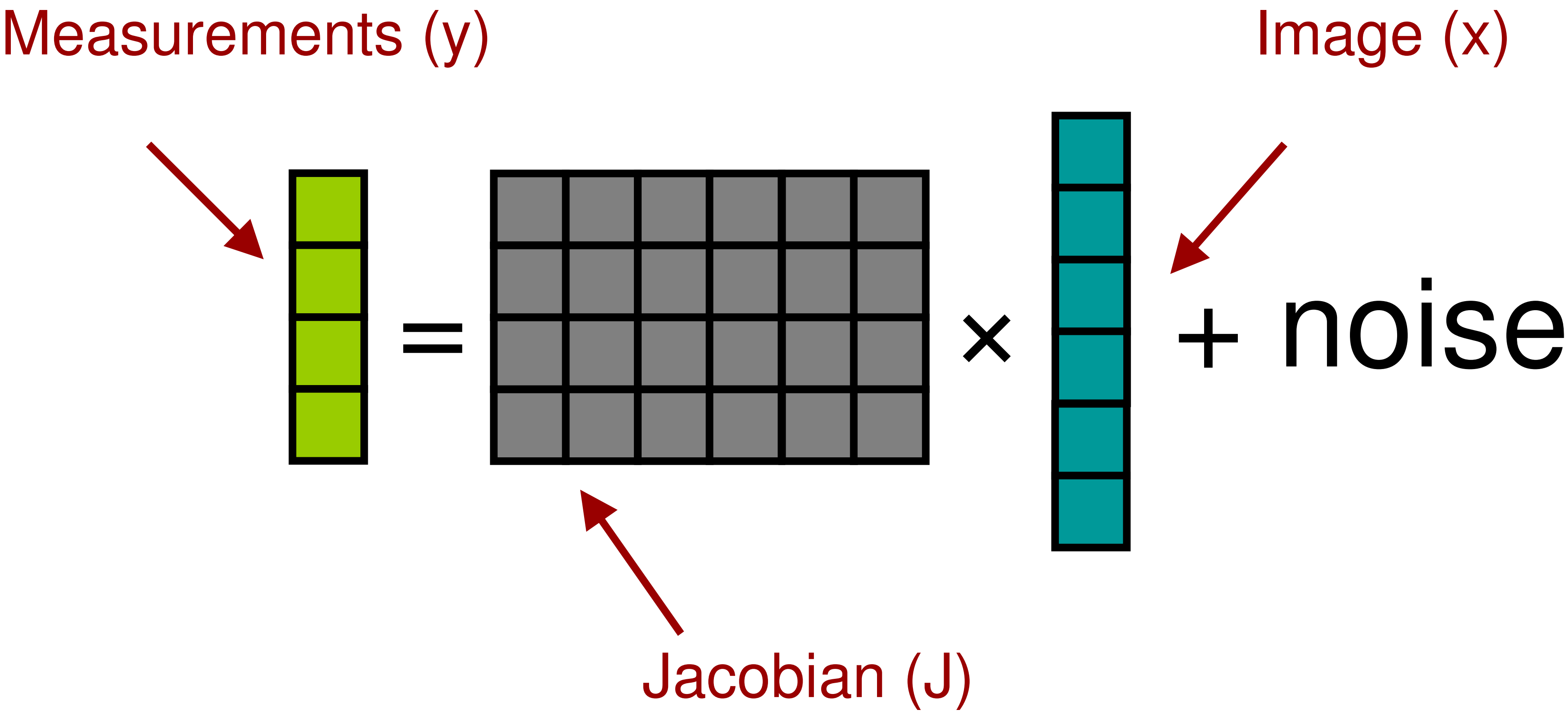
- Goal: reconstruct conductivity in Ω.
- Problem: ill-conditioned problem, since measurements most sensitive to boundary
- Solution: use of regularization techniques to penalize un-realistic solutions

Motivation

- The popular Gauss-Newton type reconstruction of EIT seeks Least Squares (L2 norm) solution
  - Smoothed edges
  - Sensitive to measurement outliers
- L1 norm solution can increase edge sharpness
  - Sharpened edges
  - Robust against measurement outliers

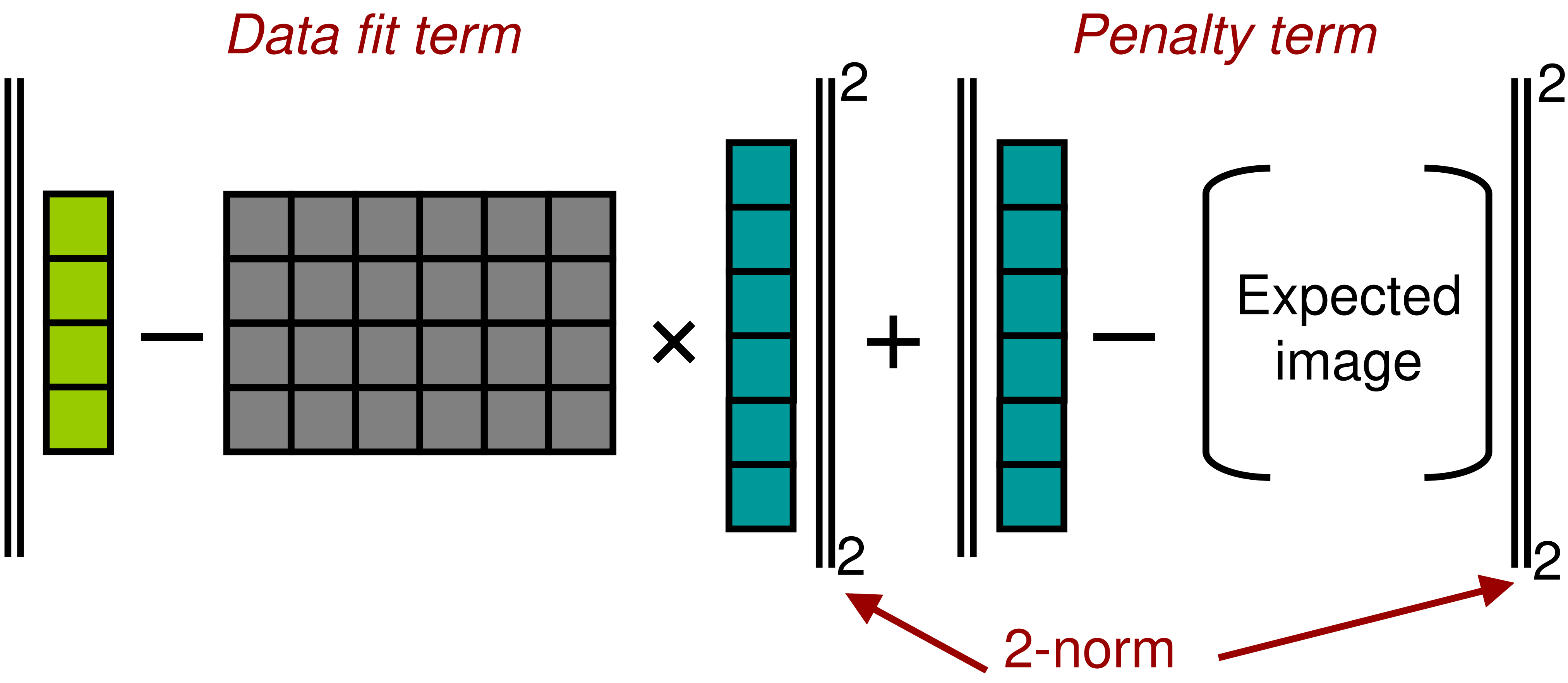
Forward Model (linearized)

$$\mathbf{y} = \mathbf{J}\mathbf{x}$$



Least Squares (L2 norm) solution

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{J}\mathbf{x}\|_2^2 + \|\mathbf{x} - \mathbf{x}_0\|_2^2$$



Total Variation (TV)

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{J}\mathbf{x}\|_2^2 + \|\mathbf{x} - \mathbf{x}_0\|_1$$

1-norm on
penalty term

- 1-norm allows non-smooth images
- L1 norm is not differentiable,
- Several iterative techniques

L1 norm solution

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{J}\mathbf{x}\|_1 + \|\mathbf{x} - \mathbf{x}_0\|_1$$

1-norm

Our proposed idea:

- Penalty 1-norm allows non-smooth images
- Data 1-norm gives reduced penalty for outliers

A Generalized and Weighted Iterative Solution for L1/L2 norms minimization

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{J}\mathbf{x}\|^{p_n}_{\mathbf{\Sigma}_{\mathbf{n}}^{-1}} + \|\mathbf{x} - \mathbf{x}_0\|^{p_x}_{\mathbf{\Sigma}_{\mathbf{x}}^{-1}}$$

- p_n and p_x are norm type:
  - 1 for L-1 norm; 2 for L-2 norm
- $\mathbf{\Sigma}_n$ and $\mathbf{\Sigma}_x$ are covariance matrices:
  - $\mathbf{\Sigma}_n$ for measurement noise
  - $\mathbf{\Sigma}_x$ for image elements

Develop iterative expression

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{J}\mathbf{x}\|^{p_n}_{\mathbf{\Sigma}_\mathbf{n}^{-1}} + \|\mathbf{x} - \mathbf{x}_0\|^{p_x}_{\mathbf{\Sigma}_\mathbf{x}^{-1}}$$

Reformulate as:

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}}$$

$$\left(\mathbf{y} - \mathbf{J}\mathbf{x}\right)^t \mathbf{D}_\mathbf{n}^t \mathbf{\Sigma}_\mathbf{n}^{-1} \mathbf{D}_\mathbf{n} \left(\mathbf{y} - \mathbf{J}\mathbf{x}\right)^t +$$
$$\left(\mathbf{x} - \mathbf{x}_0\right)^t \mathbf{D}_\mathbf{x}^t \mathbf{\Sigma}_\mathbf{x}^{-1} \mathbf{D}_\mathbf{x} \left(\mathbf{x} - \mathbf{x}_0\right)^t$$

Where:

$$[\mathbf{D}_\mathbf{n}]_{i,i} = \left([|\mathbf{y} - \mathbf{J}\mathbf{x}|]_i + \beta\right)^{\frac{1}{2}p_n - 1}$$

$$[\mathbf{D}_\mathbf{x}]_{i,i} = \left([|\mathbf{x} - \mathbf{x}_0|]_i + \beta\right)^{\frac{1}{2}p_x - 1}$$

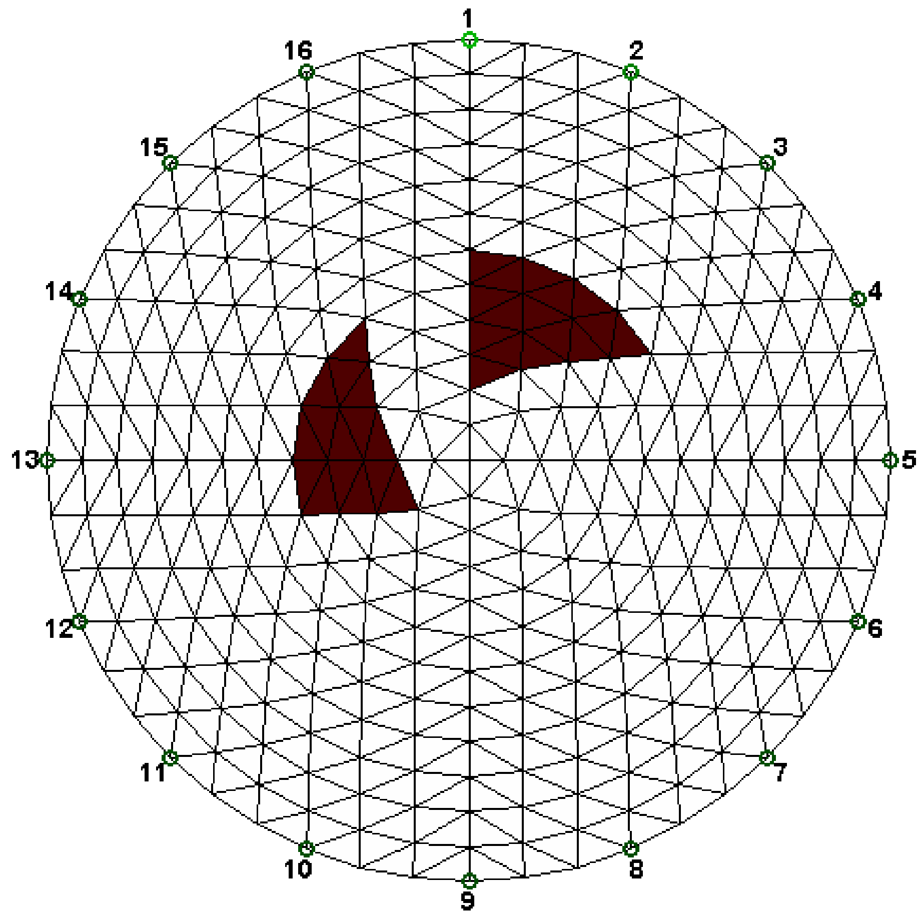
The General Iterative Solution

$$\hat{\mathbf{x}}^{k+1} = \hat{\mathbf{x}}^{k} + \underbrace{\left(\mathbf{J}^t\mathbf{W}(\hat{\mathbf{x}}^k)\mathbf{J} + \lambda^2\mathbf{R}(\hat{\mathbf{x}}^k)\right)^{-1}\mathbf{J}^t\mathbf{W}(\hat{\mathbf{x}}^k)\overbrace{\left(\mathbf{y} - \mathbf{J}\hat{\mathbf{x}}^k\right)}^{\text{Error signal}}}_{\text{Image update}}$$

Where:

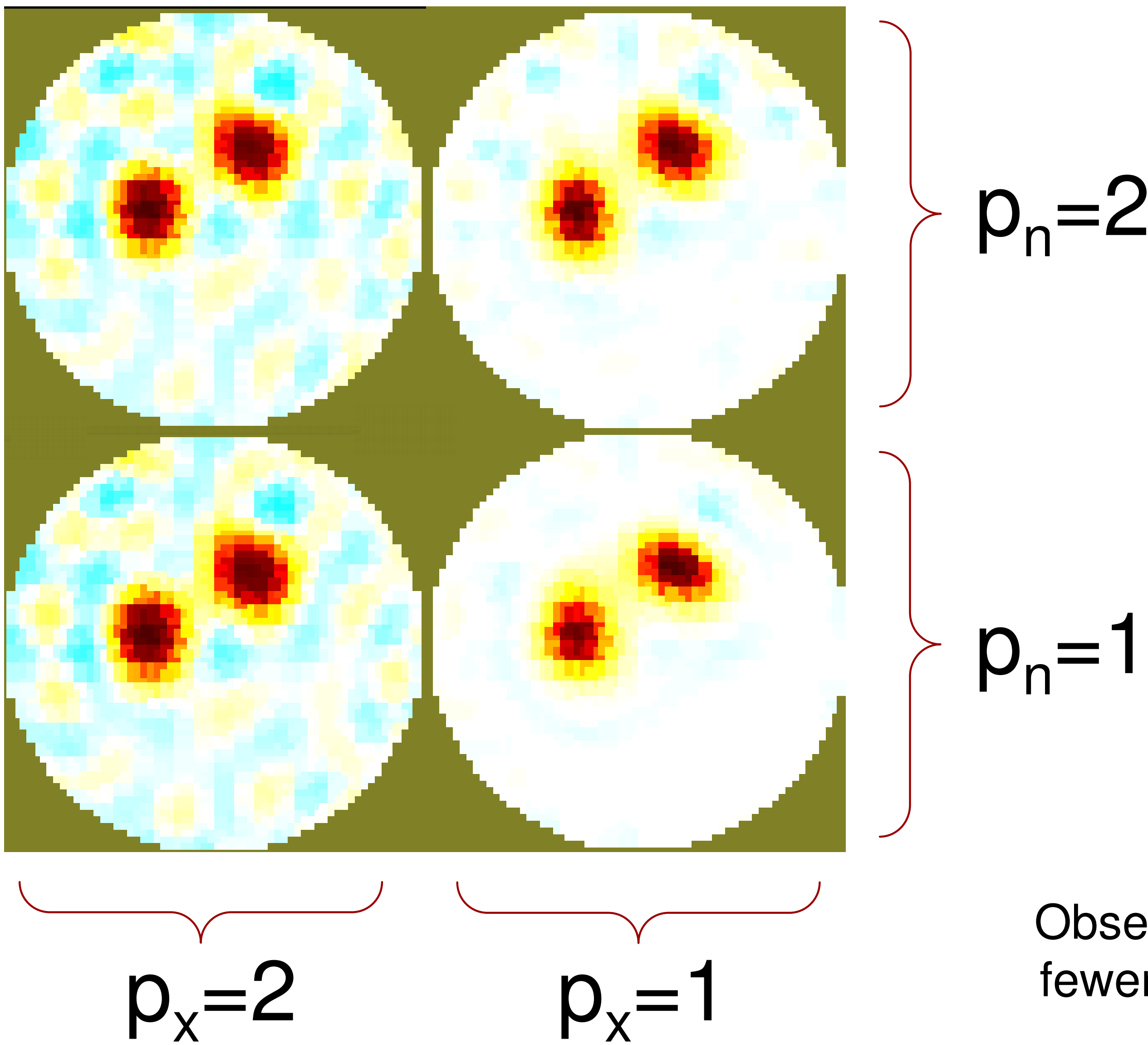
$$\mathbf{W}(\mathbf{x}) = \mathbf{D_n}(\mathbf{x})^t\boldsymbol{\Sigma}_{\mathbf{n}}^{-1}\mathbf{D_n}(\mathbf{x})$$
$$\mathbf{R}(\mathbf{x}) = \mathbf{D_x}(\mathbf{x})^t\boldsymbol{\Sigma}_{\mathbf{x}}^{-1}\mathbf{D_x}(\mathbf{x})$$

Simulation: Forward Model



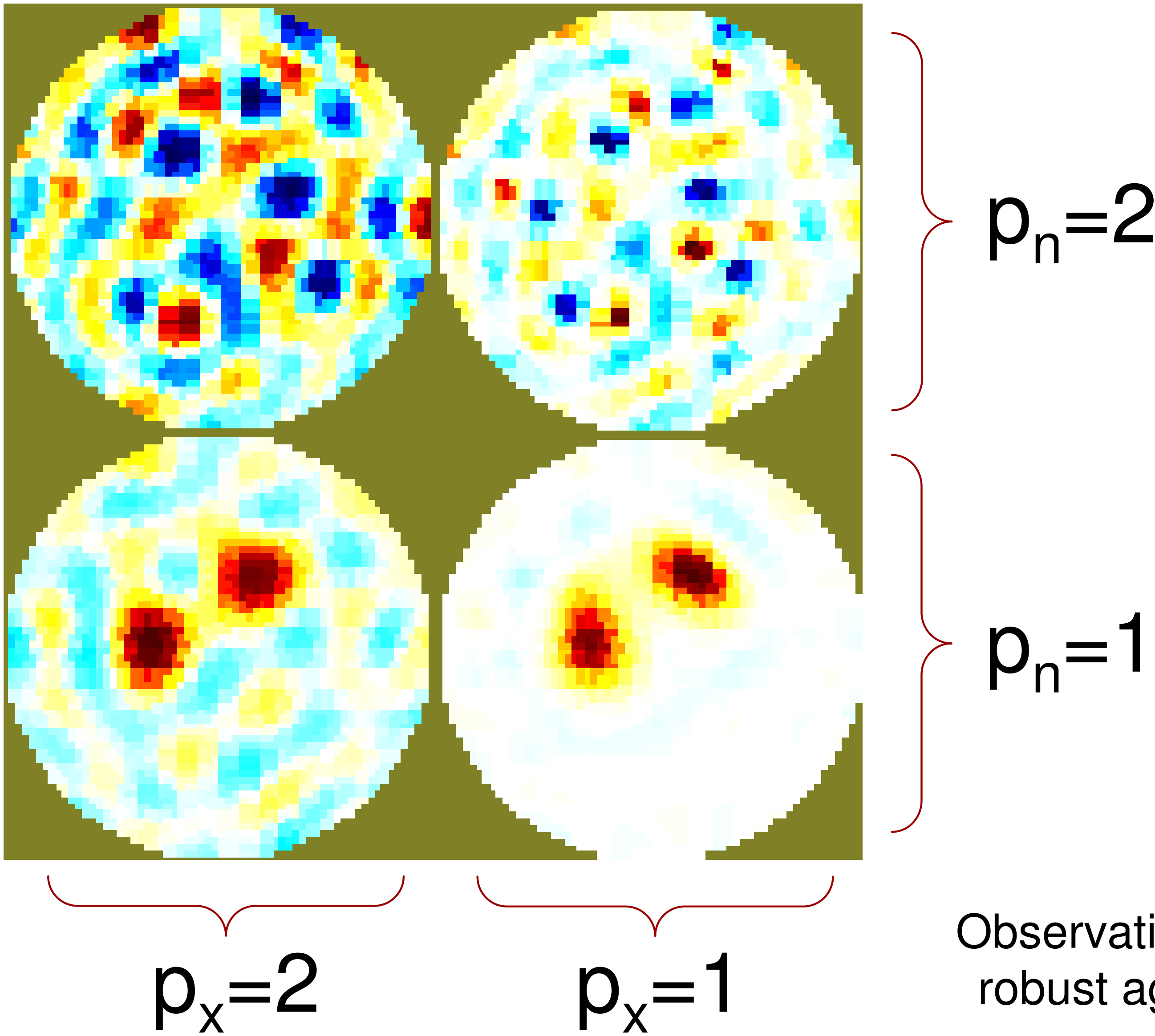
the forward model: finite element model with 576 elements. Electrodes are indicated by green dots. The background and inhomogeneities have conductivities 1:0 and 2:0, respectively.

Simulation: Comparison -No data outliers



Observation : L1 *prior* norm gives fewer artefacts and clearer edges

Simulation: Comparison --Data outliers added



Observation: L1 *data* norm gives more robust against meassurement errors.

Conclusion

- L1 norm on data term gives robustness against data error
- L1 norm on the regularization term helps achieve low noise/artefacts
- The proposed method can flexibly choose L2/L1 norms on the data fidelity term and the regularization term