Carleton University, Systems and Computer Engineering
SYSC 4405: Midterm Exam, October 23, 2007
Exam Number: 1

Background: Showers can be a waste of energy, because water that is heated in the hot water tank might never be used. One way to solve this is to heat the water as it passes the through the shower. An electrical heater is placed in the water pipe, and (to control the system) a temperature sensor is placed in the shower head, just as the water sprays out.

You are the electronics engineer working for a company that plans to build this new type of shower. It is decided to build a DSP based control system. The first task (covered by this exam) is to characterize the DSP properties of the system.

Input: The input $x(t)$ is the control to the shower heater. It can be either *off* ($x(t) = 0$) or *on* ($x(t) = 1$).

Output: The output $y(t)$ is the water temperature, measured in °C.

You conduct two tests: 1) Turning the system *on*, $x(t) = u(t)$, gives

$$y(t) = \begin{cases} 10\ °\text{C} & \text{if } t \leq 1.75\ s \\ 50 - 40exp(-\frac{t-1.75\ s}{1.25\ s})\ °\text{C} & \text{if } t > 1.75\ s \end{cases} \tag{1}$$

and, 2) Turning the system *off*, $x(t) = 1 - u(t)$, gives

$$y(t) = \begin{cases} 50\ °\text{C} & \text{if } t \leq 1.75\ s \\ 10 + 40exp(-\frac{t-1.75\ s}{1.25\ s})\ °\text{C} & \text{if } t > 1.75\ s \end{cases} \tag{2}$$

1. (1 point) Your exam is exam number **1**. Write down this number.

   ANSWER:
   This should be easy

2. (5 points) From these results, characterize your system for the following properties. (for each item, write "no", only if the data so far show the system cannot have that property)

   (a) linear
   (b) memoryless
   (c) shift-invariant
   (d) LSI
   (e) stable
   (f) causal

   ANSWER:

   (a) linear *NO – input can only be 0 or 1, and has offset of* 10 °C
   (b) memoryless *NO – time shift*
   (c) shift–invariant *YES – only depends on delay after experiment*
   (d) LSI *NO – not linear*
   (e) stable *YES – small change in input only results in small changes in output*
   (f) causal *YES – only depends on delay after experiment*

3. (5 points) Using a sample rate of $T_s = 0.25\ s$, calculate and sketch the sample sequence of the output, $y[n]$, for a unit step input, $x[n] = u[n]$. Show the value of $y[n]$ at three points on the exponential curve.

   ANSWER:
   Use eqn #1 and $n = \frac{t}{T_s}$. Thus $n_0 = \frac{1.75\ s}{0.25\ s} = 7.0$ and $\alpha = \frac{0.25\ s}{1.25\ s} = 0.2$. It does not matter if n_0 and τ are not integers.

$$y[n] = \begin{cases} 10\ °\text{C} & \text{if } n \leq n_0 \\ 50 - 40e^{-\alpha(n-n_0)}\ °\text{C} & \text{if } n > n_0 \end{cases}$$

4. (5 points) If necessary, create a signal $y'(t)$ which is LSI to the input $x(t)$ by adding a constant value to $y(t)$. Calculate the Z Transform of the input, $x[n]$, and output, $y'[n]$. Show the Region of Convergence with poles and zeros for $X(z)$ and $Y'(z)$.

   ANSWER:
   The Z transform of $x[n] = u[n]$ is $X(z) = \frac{1}{1-z^{-1}}$. $X(z)$ has a pole at $z = 1$ and the ROC is $|z| > 1$ (since $x[n]$ is a right sided sequence.

   We need to create $y'(t)$ and $y'[n]$. The Z transform of $y'[n]$ is

$$y'[n] = 40u[n-n_0](1-e^{-\alpha(n-n_0)})\ ^\circ\mathrm{C}$$
$$y'[n] = 40u[n-n_0] - 40u[n-n_0](e^{-\alpha})^{(n-n_0)}\ ^\circ\mathrm{C}$$
$$Y'(z) = 40z^{-n_0}\left(\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-\alpha}z^{-1}}\ ^\circ\mathrm{C}\right)$$

   $Y'(z)$ has poles at $z = 1$ and $z = e^{-\alpha}$ and the ROC is $|z| > 1$ (since $e^{-\alpha} < 1$ and $y'[n]$ is a right sided sequence).

5. (5 points) Calculate the impulse response $h'[n]$, where $H'(z) = \frac{Y'(z)}{X(z)}$. Show the Region of Convergence with poles and zeros for $H'(z)$.

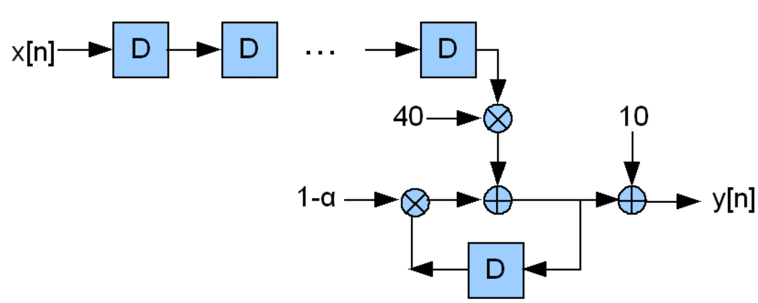
   ANSWER:
   First, write $Y'(z)$ as a single fraction

$$\begin{aligned}
Y'(z) &= 40z^{-n_0}\frac{1-e^{-\alpha}z^{-1}-1+z^{-1}}{(1-z^{-1})(1-e^{-\alpha}z^{-1})}\ ^\circ\mathrm{C} \\
&= 40z^{-n_0-1}\frac{1-e^{-\alpha}}{(1-z^{-1})(1-e^{-\alpha}z^{-1})}\ ^\circ\mathrm{C} \\
H'(z) = \frac{Y'(z)}{X(z)} &= \frac{40z^{-n_0-1}\frac{1-e^{-\alpha}}{(1-z^{-1})(1-e^{-\alpha}z^{-1})}\ ^\circ\mathrm{C}}{\frac{1}{1-z^{-1}}} \\
\frac{Y'(z)}{X(z)} &= 40z^{-n_0-1}\frac{1-e^{-\alpha}}{1-e^{-\alpha}z^{-1}}\ ^\circ\mathrm{C}
\end{aligned}$$

   $H'(z)$ has a pole at $z = e^{-\alpha}$ and the ROC is $|z| > e^{-\alpha}$ (since $h'[n]$ is a right sided sequence).

6. (5 points) Show a block diagram of a LCCDE system which models the shower. (the output should be $y[n]$, not $y'[n]$). Discuss briefly (≤ 50 words) how this model relates to the physical system.

   ANSWER:
   The LCCDE model shows a delay for the pipe between the heater and the shower head, and a feedback for the mixing in the pipe and shower head. The last addition shows the temperature offset of the cold water in the system.



7. (5 points) Is this sampling rate, $T_s = 0.25\ s$ sufficient for $y(t)$ if we don't use low pass filter before the ADC? Sketch a graph to show how much aliasing we expect in the signal. (You may use this relationship, $FT\{e^{-at}u(t)\} = \frac{1}{a+j2\pi f}$, and the FT of a sampling pulse sequence, with sample rate T_s, is a pulse sequence with separation $\Delta f = \frac{1}{T_s}$)

   ANSWER:

   The curve of equation 1 has a term $e^{-t/\tau}$. We can ignore the 40 (since this doesn't change the relative amount of aliasing). We can ignore the delay, since this is just a phase change. Thus $a = 1/\tau = 1/1.25\ s = 0.8s$.

   In $a + j2\pi f$, the $-3dB$ point is at $|a| = |2\pi f|$. Thus $f = \frac{a}{2\pi} = 0.13$

   The amount of aliasing is controlled by the relative level of this curve at the crossover ($2/T_s = 2Hz$). The signal level is down by $\frac{2}{0.13} = 15.4$ at this frequency.