Efficient computations of the Jacobian matrix using different approaches are equivalent

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Abstract: Two main approaches have been used to calculate the Jacobian, $\mathbf{J}$, or sensitivity matrix: the "adjoint field" method and differentiation of the system matrix. While some investigations have sought to test which is more efficient, we show the approaches are equivalent, and an efficient implementation of either produces the same underlying algorithm.

1 Introduction

The sensitivity matrix, $\mathbf{J}$, describes how small internal conductivity changes relate to measurements; it is key to image reconstruction and also to understanding the characteristics of EIT configurations. Efficient calculation of the $\mathbf{J}$ matrix is essential. Fig. 1 shows a body where data element $d_{i,j}$ is recorded using measurement $\mathbf{m}_i$ and current $\mathbf{c}_j$ patterns.

When a region k undergoes a change $\sigma_k \to \sigma_k + \delta\sigma$, the sensitivity is defined

$$\mathbf{J}_{i,j,k} = \frac{\partial}{\partial\sigma_k} d_{i,j} \tag{1}$$

Normally, $\mathbf{J}$ is represented as a matrix, by selecting rows corresponding to (i,j) pairs of (measurement, stimulation) in the order applied by the EIT hardware.

Numerical methods are used to solve the forward problem on arbitrary geometries, and the finite element method (FEM) is widely used because it facilitates refinement in regions of high electric field, such as near the electrodes. We assume piecewise-constant conductivity on each element.

Two approaches to the calculation of $\mathbf{J}$ have been used in EIT: adjoint-field methods[1, 3], and differentiation of the FEM system (admittance) matrix[5], which we call the "admittance matrix differentiation method". It is also possible to approximate $\mathbf{J}$ using a "perturbation Jacobian."
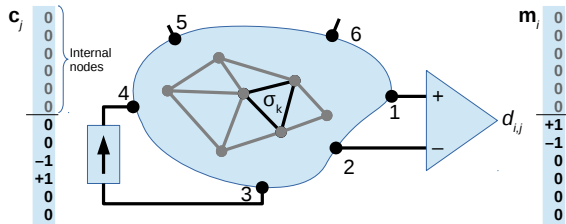
Figure 1: FEM with six electrodes and internal nodes, using current pattern $\mathbf{c}_j$ and measurement pattern $\mathbf{m}_i$. Electrodes nodes are enumerated after internal FEM nodes in this example.

2 Jacobian calculations

Measurements are $d_{i,j} = \mathbf{m}_i^\mathsf{T}\mathbf{v}_j$, using the notation: continuous values $\mathcal{V}$, values on FEM nodes $\mathbf{v}$, and values on nodes $\dot{\mathbf{v}}_k$ in region k. After choosing a ground node, the node voltages due to current pattern $\mathbf{c}_j$ are $\mathbf{v}_j = \mathbf{Y}^{-1}\mathbf{c}_j$ with admittance

matrix $\mathbf{Y}(\boldsymbol{\sigma}) = \mathbf{C}^\mathsf{T}\mathbf{S}(\boldsymbol{\sigma})\mathbf{C}$, where $\boldsymbol{\sigma}$ is the vector of conductivities in each finite element, $\mathbf{C}$ is a connectivity matrix, and $\mathbf{S}(\boldsymbol{\sigma})$ is a block diagonal matrix with blocks $\sigma_k\mathbf{B}_k$ for each region k

$$[\mathbf{B}_k]_{\ell,m} = \int_k \phi_\ell \phi_m dA \tag{2}$$

integrated over the volume of region k, where ϕ_ℓ, ϕ_m are shape functions.

2.1 Adjoint-field method

The adjoint-field method calculates[1]

$$\mathbf{J}_{i,j,k} = \int_k \nabla\mathcal{V}_i \cdot \nabla\mathcal{V}_j dA, \tag{3}$$

where $\mathcal{V}_j, \mathcal{V}_i$ are the body voltages from applying a current $\mathbf{c}_j$ and $\mathbf{m}_i$ (interpreted as a current). Defining $\mathbf{v}_j = \mathbf{Y}^{-1}\mathbf{c}_j$ and $\mathbf{v}_i = \mathbf{Y}^{-1}\mathbf{m}_i$, eqn (3) may be represented as $\mathbf{J}_{i,j,k} = \dot{\mathbf{v}}_{k,i}^\mathsf{T}\mathbf{B}_k\dot{\mathbf{v}}_{k,j}$, where node voltages in region k are $\dot{\mathbf{v}}_{k,i} = \dot{\mathbf{C}}_k\mathbf{v}_i$, for a selection matrix $\dot{\mathbf{C}}_k$ (and similarly for j). The Jacobian is then efficiently calculated for each region k

$$\mathbf{J}_{i,j,k} = (\dot{\mathbf{C}}_k\mathbf{Y}^{-1}\mathbf{m}_i)^\mathsf{T} \mathbf{B}_k (\dot{\mathbf{C}}_k\mathbf{Y}^{-1}\mathbf{c}_j) \tag{4}$$

2.2 Admittance-matrix differentiation method

From (1) and $d_{i,j} = \mathbf{m}_i^\mathsf{T}\mathbf{v}_j = \mathbf{m}_i^\mathsf{T}\mathbf{Y}^{-1}\mathbf{c}_j$ [5],

$$
\begin{aligned}
\mathbf{J}_{i,j,k} &= \frac{\partial}{\partial\sigma_k}\mathbf{m}_i^\mathsf{T}\mathbf{Y}^{-1}\mathbf{c}_j = \mathbf{m}_i^\mathsf{T}\mathbf{Y}^{-1}\left(\frac{\partial}{\partial\sigma_k}\mathbf{Y}\right)\mathbf{Y}^{-1}\mathbf{c}_j \\
&= \mathbf{m}_i^\mathsf{T}\mathbf{Y}^{-1}\mathbf{C}^\mathsf{T}\left(\frac{\partial}{\partial\sigma_k}\mathbf{S}(\boldsymbol{\sigma})\right)\mathbf{C}\mathbf{Y}^{-1}\mathbf{c}_j \\
&= (\dot{\mathbf{C}}_k\mathbf{Y}^{-1}\mathbf{m}_i)^\mathsf{T}\mathbf{B}_k(\dot{\mathbf{C}}_k\mathbf{Y}^{-1}\mathbf{c}_j) \tag{5}
\end{aligned}
$$

where $\mathbf{B}_k$ is the only non-zero block in $\frac{\partial}{\partial\sigma_k}\mathbf{S}(\boldsymbol{\sigma})$, and $\dot{\mathbf{C}}_k$ is the corresponding reduced connectivity matrix.

3 Discussion

The equivalence of (4) and (5) illustrates that an implementation of both methods yields the same underlying calculation (see also [2], ch. 3). Efficient implementation strategies are also the same for both: pre-calculation of the matrices in parenthesis, and calculation of larger blocks $\mathbf{B}_k$ for model regions with the same parameter value in the inverse model.

References

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