Model Reduction for forward solutions

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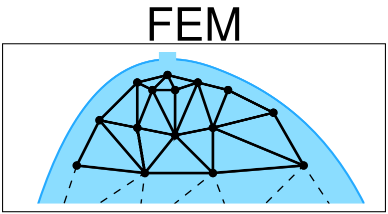
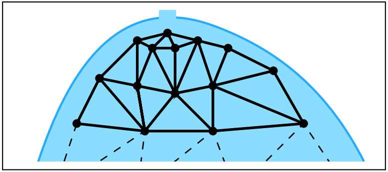
Finite Element Models (FEM)

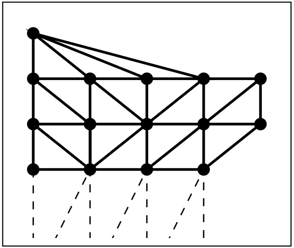


Finite Element Models (FEM)

FEM



Resistor Mesh



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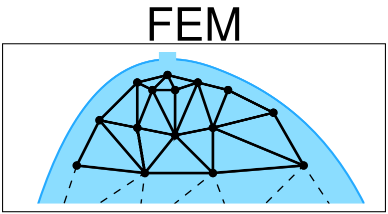
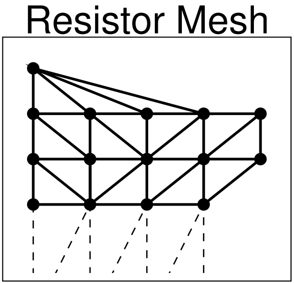
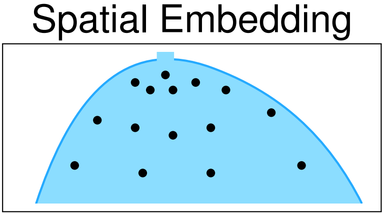
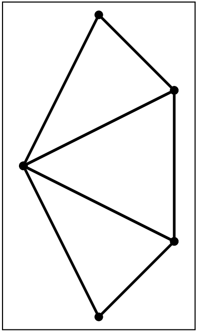


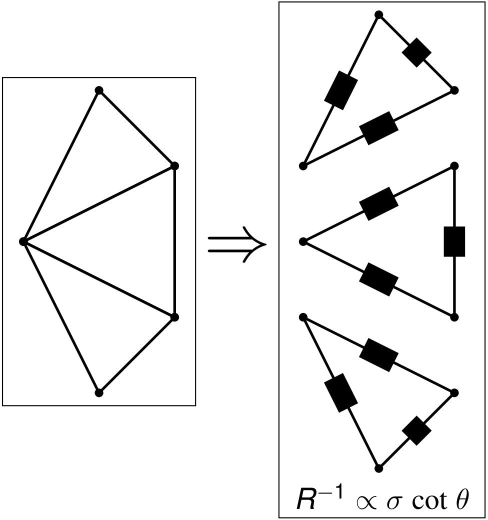




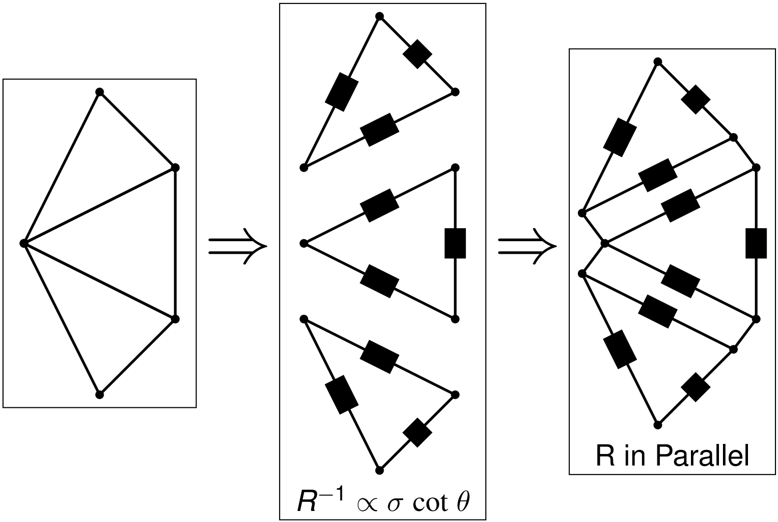
FEM and Resistor Meshes



FEM and Resistor Meshes

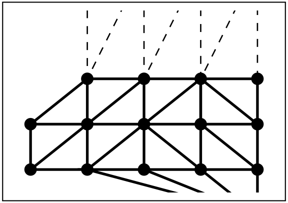


FEM and Resistor Meshes



Equivalent N-terminal resistances

An N-terminal equivalent can be created with $\frac{1}{2}N(N-1)$ resistances.



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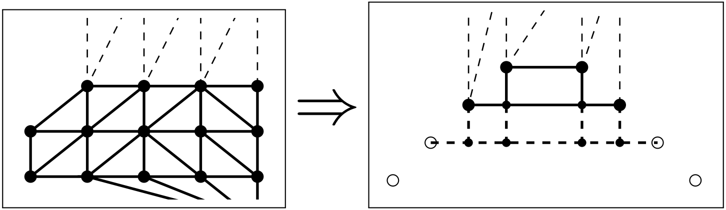


For $N = 4$, equivalent has $\frac{1}{2}(4 \times 3) = 6$ resistances.

Block Matrix Inverse

We have a FEM with two regions A and B.

$$\begin{bmatrix} V_A \\ V_B \end{bmatrix} = \begin{bmatrix} A & C^t \\ C & B \end{bmatrix}^{-1} \begin{bmatrix} I_A \\ I_B \end{bmatrix}$$

Current into nodes are (I_A, I_B) and nodal voltages are (V_A, V_B).

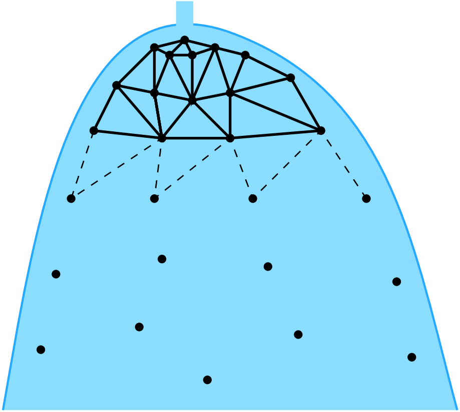
If we know that $I_B = 0$, then

$$\begin{bmatrix} V_A \end{bmatrix} = \begin{bmatrix} \tilde{A} \end{bmatrix}^{-1} \begin{bmatrix} I_A \end{bmatrix}$$

where

$$\tilde{A} = A - C^t B^{-1} C$$

Extended region in FEM



R	Region of Interest	
S	Shared Nodes	
E	Extended Region (Not of interest)	

Connections: R⇔S C_R
R⇔E C_E

Block Matrix Inverse

FEM region A is $R \cup S$, B is E.

$$\begin{bmatrix} V_R \\ V_S \\ V_E \end{bmatrix} = \begin{bmatrix} R & C_R^t & 0 \\ C_R & S & C_E^t \\ 0 & C_E & E \end{bmatrix}^{-1} \begin{bmatrix} I_R \\ I_S \\ I_E \end{bmatrix}$$

Block Matrix Inverse

FEM region A is $R \cup S$, B is E.

$$\begin{bmatrix} \boxed{\begin{matrix} V_R \\ V_S \end{matrix}} \\ V_E \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} R & C_R^t \\ C_R & S \end{matrix}} & \begin{matrix} 0 \\ C_E^t \end{matrix} \\ \begin{matrix} 0 & C_E \end{matrix} & E \end{bmatrix}^{-1} \begin{bmatrix} \boxed{\begin{matrix} I_R \\ I_S \end{matrix}} \\ I_E \end{bmatrix}$$

Thus $\tilde{A}$ for $R \cup S$ is

$$A - C^t B^{-1} C = \begin{bmatrix} R & C_R^t \\ C_R & S \end{bmatrix} - \begin{bmatrix} 0 & C_E \end{bmatrix} E^{-1} \begin{bmatrix} 0 \\ C_E^t \end{bmatrix}$$

Block Matrix Inverse

FEM region A is $R \cup S$, B is E.

$$\begin{bmatrix} \boxed{\begin{matrix} V_R \\ V_S \end{matrix}} \\ V_E \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} R & C_R^t \\ C_R & S \end{matrix}} & \begin{matrix} 0 \\ C_E^t \end{matrix} \\ \begin{matrix} 0 & C_E \end{matrix} & E \end{bmatrix}^{-1} \begin{bmatrix} \boxed{\begin{matrix} I_R \\ I_S \end{matrix}} \\ I_E \end{bmatrix}$$

Thus $\tilde{A}$ for $R \cup S$ is

$$\begin{aligned} A - C^t B^{-1} C &= \begin{bmatrix} R & C_R^t \\ C_R & S \end{bmatrix} - \begin{bmatrix} 0 & C_E \end{bmatrix} E^{-1} \begin{bmatrix} 0 \\ C_E^t \end{bmatrix} \\ &= \begin{bmatrix} R & C_R^t \\ C_R & S \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \boxed{C_E E^{-1} C_E^t} \end{bmatrix} \end{aligned}$$

$C_E E^{-1} C_E^t$ is size $N \times N$ (N shared nodes)

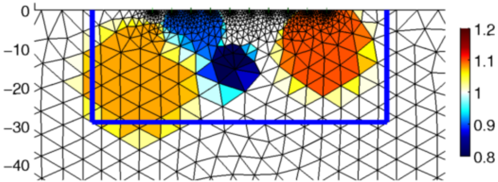
Interpretation

- Modified FEM
  FEM of region A, but connected nodes S are modified:
  $S \longrightarrow S - C_E E^{-1} C_E^t$
- Original FEM, connected to a modified extended region

- Either way, slow calc of E^{-1} is only done once
  (in practice using LU or Cholesky)
- Computational benefits for repeated calculations (i.e. iterative EIT solvers)

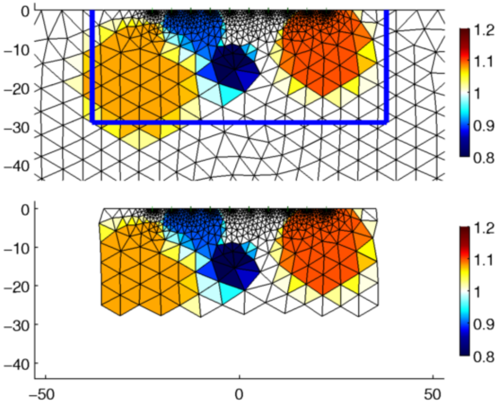
Example

FEM with and without the extended region.



Example

FEM with and without the extended region.



Region E removed & $S \longrightarrow S - C_E E^{-1} C_E^t$;
same electrode voltages.

Discussion

- A way to find an equivalent resistor network with fewer DoF.
- To benefit from this
  - Don't care about internal voltages within E
  - Multiple solutions (so that cost of E^{-1} is amortized)
- Works in 3D (although examples are 2D)
- Can be formulated in terms of an equivalent Dirichlet-to-Neumann map
- Possible applications:
  - fit the resistor network to E rather than a fixed outer region
  - set upper/lower bounds on effect of region E

Model reduction for forward solutions

Andy Adler, William R.B. Lionheart

Introduction: EIT estimates the internal conductivity distribution from body surface electrodes via the solution of an inverse problem. Most approaches require solution of a forward problem based on a finite element model (FEM) of the medium of interest. For iterative solvers, the solution time is typically dominated by the time to solve the FEM (both for estimation of the measurements and the Jacobian) at each iteration. There is thus a strong incentive to develop techniques to reduce the solution time.

To obtain accurate forward solutions, it is typically necessary to model a large region far away from the electrodes. This is most severe in geophysical or endoscopic EIT applications where the medium is effectively infinite. However, it is also true of traditional EIT applications where, for example, the chest extends above and below the electrode plane(s). In these extended regions, we do not actually need so solve for the internal voltages; we simply need to model the effect of the extended region on the ROI. This effect can be thought of in two ways: 1) the extended region's effect can be represented by its Dirichlet to Neumann map, or 2) in a resistor model representation of the FEM, the extended region can be replaced by an equivalent N-port resistor mesh (with $0.5 \times N \times (N+1)$ resistors)

Objective: To implement the model reduction scheme and verify its accuracy and improved calculation time.

Methods: Consider an FEM with regions (R = ROI, E = Extended). With appropriate reordering of the FEM node numbering, unconnected FEM system matrix is block diagonal blocks [R, E]. The connectivity matrix has three regions: 1) nodes connected only to R (represented as [A; 0]), 2) nodes on the boundary of R and E (and connected to both, represented as [B; C]), and 3) nodes connected only to E ([0 ; D]). The connected FEM system matrix is

$$\begin{bmatrix} A^t & 0 \\ B^t & C^t \\ 0 & D^t \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} A & B & 0 \\ 0 & C & D \end{bmatrix} = \begin{bmatrix} A^tRA & A^tRB & 0 \\ B^tRA & B^tR^tRB + C^tEC & C^tED \\ 0 & DEC^t & D^tED \end{bmatrix}$$

The 2×2 block matrix inverse, $[A|B;\, C|D]^{-1}$ has a block $(A - BD^{-1}C)^{-1}$ in the A position. The equivalent connected system matrix for its effect on nodes in R is

$$\begin{bmatrix} A^tRA & A^tRB \\ B^tRA & B^tRB + (C^tEC - DEC^t(D^tED)^{-1}C^tED) \end{bmatrix}$$

The term, $C^tEC - DEC^t(D^tED)^{-1}C^tED$, models the effect of the extended region. It depends only on connectivity matrices and the system matrix E and can thus be precalculated.

Results: Software was implemented to verify these calculations. The calculation of the exended region term is approximately the time for the full model calculations. At each subsequent step, the model calculation is approximately the same as for the ROI model alone. Results show a 2D FEM with and without the extended region.

Discussion: Model reduction techniques allow more rapid forward calculations. The code will be made part of the next EIDORS release.