Resistor networks and transfer resistance matrices

W. R. B. Lionheart, K. Paridis, * A Adler †

Resistor networks are important for Electrical Impedance Tomography for two reasons. They are used to provide convenient stable test loads or phantoms for EIT systems [6, 7, 8] and they provide a lumped approximation to a conductive body that includes Finite Element, Finite Difference and Finite Volume Method as special cases. In this paper we give a consistency condition on transfer resistance matrices of networks derived from n-port theory and review necessary and sufficient conditions for a matrix to be the transfer resistance of a planar network. We give an example to show that there are three dimensional conductivity distributions for which the transfer resistance matrix for electrodes on a plane cannot be represented by a planar resistor network.

1 Transfer resistance matrices

Given a system of L electrodes attached to a conductive body to which a vector of currents $\mathbf{I} \in \mathbb{R}^L$, $\sum_{\ell=1}^{L} I_\ell = 0$ is applied the resulting vector of voltages $\mathbf{V} \in \mathbb{R}^L$ satisfies

$$\mathbf{V} = \mathbf{RI}, \tag{1}$$

where $\mathbf{R}$ is the (real symmetric) transfer resistance matrix. Without loss of generality this is chosen so that $\sum_{\ell=1}^{L} V_\ell = 0$. Restricted to this subspace $\mathbf{R}$ has an inverse – the transfer conductance matrix.

In EIT of course $\mathbf{R}$ represents the complete data that can be obtained with this system of electrodes at zero frequency, and it is typically calculated for a known conductivity using the complete electrode model and the finite element method. Such a finite element model gives rise to a resistor network of which the electrodes are considered as terminals. In the general case of a body of arbitrary topology in three dimensional space we can deduce some properties of the matrix $\mathbf{R}$ from general results in circuit theory. In particular the theory of n-port networks.

2 n-port networks

An n-port network is a connected resistor network with $m > 2n$ terminals in which n pairs of terminals have been chosen, and within each pair one is labeled + and one −. The *open circuit resistance matrix* of this n-port network is the matrix $\mathbf{S}$ such that

$$\mathbf{V} = \mathbf{SI} \tag{2}$$

where here $\mathbf{I} \in \mathbb{R}^n$ is a current applied across each pair of terminals and $\mathbf{V} \in \mathbb{R}^n$ the resulting voltages across those terminals. Here $\mathbf{S}$ is a real symmetric $n \times n$ matrix and indeed

$$\mathbf{S} = \mathbf{C}^T\mathbf{RC}, \tag{3}$$

where $\mathbf{R}$ is the transfer resistance of the network with the $L = 2n > 4$ distinguished terminals and where the i-th column of the matrix $\mathbf{C}$ has a 1 in the row corresponding to the + terminal of the i-th port and −1 in the row corresponding to the − terminal and is otherwise zero.

Cederbaum [1] noticed that the open circuit resistance matrix of an n-port has a property known as paramountcy (see Fig 1).

Definition: *Let* $\mathbf{A}$ *be real symmetric* $n \times n$ *matrix with elements* a_{ij}*. Let* $I = (i_1, i_2, ..., i_k)$ *be an ordered set* $k < n$ *of indices between* 1 *and* n *and* A_{II} *the determinant of the submatrix of rows and columns indexed*

*School of Mathematics, University of Manchester, U.K.
†Systems and Computer Engineering, Carleton University, Canada

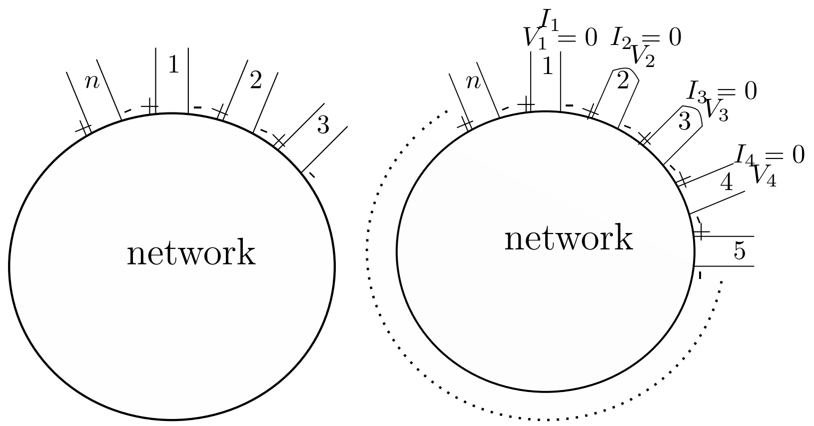

Figure 1: *Left*: An n-port network. Each port consists of two terminals of the network labeled + and $-$ but those terminals do not need to be in any sense adjacent and can be chosen arbitrarily. *Right*: The paramountcy condition is derived for applying a current through one port while the other ports are short circuited or open circuit

by I. Suppose J is another ordered subset of k indices and denote by A_{IJ} the determinant with rows indexed by I and columns by J. We say the matrix $\mathbf{A}$ is **paramount** *if $A_{II} \geq |A_{IJ}|$ for all such I and J.*

Most EIT systems use each electrode as both positive and negative current and voltage terminal, so to apply this definition we choose a subset of the measurements used to define an n-port open circuit resistance matrix where $n \leq L/2$. We thus have a consistency condition on a set of EIT data. For any such designation of a subset of electrodes as ports the resulting $\mathbf{S}$ must be paramount.

This condition may be useful for validating EIT data and fault finding in EIT systems. In particular if paramountcy fails for a given subset of electrodes chosen as ports but not for others suspicion falls on the current drive and voltage measurement circuits for the electrodes in those subsets.

3 Planar networks

The case of planar networks is much better understood. Consider a connected planar graph embedded in the unit disk in the plane such that L of the vertices fall on the unit circle. The resistor network that results from assigning non zero resistances has a transfer resistance matrix $\mathbf{R}$ with generalized inverse $\mathbf{A}$ (the transfer conductance or 'Dirichlet-to-Neumann' matrix). Given that the graph has sufficient connections between the electrodes (see [4, 2]) $\mathbf{A}$ is satisfies

$$\det(-1)^k \mathbf{A}_{P,Q} > 0, \tag{4}$$

where $\mathbf{A}_{P,Q}$ is the matrix restricted to subsets $P, Q \subset \{1, ..., L\}$, $P \cap Q = \emptyset$, $|P| = |Q| = k > 1$ and on the circle the electrodes in P and Q are ordered as $p_1, .., p_k, q_k, ..., q_1$. The sets P and Q should be thought of as two ordered and not interleaved sets of electrodes. The condition of 'sufficient connections' required is for all such P and Q there are disjoint paths through the resistor network joining each p_i to q_i. See Fig 2 for an example.

Indeed any matrix with this property is the transfer conductance matrix for some such planar resistor network and [5] give a canonical topology for this network. Of course given a network and transfer conductance other networks with the same transfer conductance can be derived using $Y - \Delta$ and resistors in series and parallel transformations, but up to such transformations the resistor mesh is determined by the transfer conductance.

For the continuum case of a simply connected conductive domain in the plane and assuming point electrodes [3] show that the transfer conductance matrix has the same property (4).

The well known consistency condition for two dimensional EIT data with adjacent pair drives, that the voltages decrease from the current source to the sink, is a consequence of (4), but the full set of inequalities provides a wider range of consistency conditions that might be applied to the data. Clearly the paramountcy is a weaker condition and might be expected to be less useful.

It was claimed by [8] that a planar resistor network could be used as a realistic test for an EIT system applied to a three dimensional body. The system in question was intended to be used with the electrodes in a single plane on the chest. It is not known if there are conditions on a three dimensional conductivity that

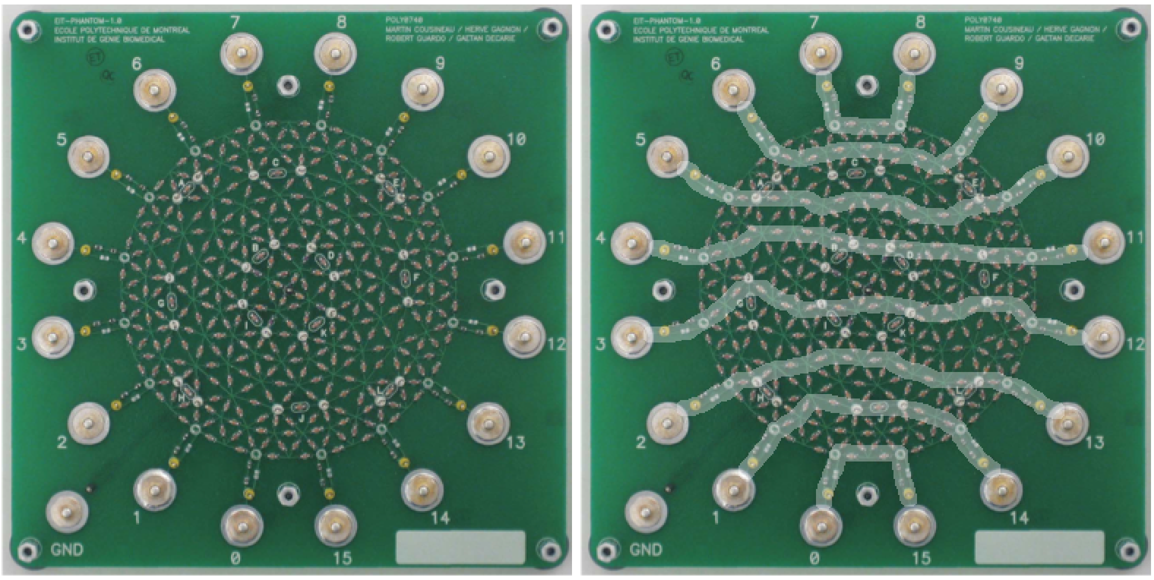

Figure 2: *Left*: A resistor phantom from Gagnon *et al*[7] with 350 resistors and 16 electrodes. A minimal equivalent network would have $16(16-1)/2 = 120$ resistors but they would not be symmetrically arranged and they would not necessarily be standard resistor values. *Right*: Illustrating that this network is well connected where P is the first 8 electrodes and Q the remaining 8 electrodes

guarantee that the transfer conductance on a electrodes on a single plane satisfy (4). It is clear however that for a given pair drive current there are locations of electrodes, specifically on the equipotential on the surface, that have the same voltage and so for some arrangement of electrodes (4) fails. As a consequence non-planar resistor networks are needed to test EIT systems for more general arrangements of electrodes. The example in Fig 3 shows that this can happen even in the plane of electrodes.

4 Which resistor networks correspond to some FEM?

It is well known that the system matrix for an isotropic first order (two or three dimensional) FEM model with non-obtuse elements is the Ohm-Kirchhoff matrix of a resistor network with the same topology as the FE mesh and resistors given by a cotangent formula (see eg [10]). It is interesting in this context to ask the converse question of which resistor networks (with the topology of a FE mesh) have an assignment of node positions and conductivities that give rise to a system matrix equal to the Ohm-Kirchoff matrix. A partial answer to this was given by [9] who showed that for a fairly general two dimensional family of layered meshes an open set of resistances resulted in an equivalent isotropic planar FE model. Further progress has been made in [11].

References

[1] I. Cederbaum, Applications of matrix algebra to network theory, IRE Trans. Circuit Theory, vol. CT-3, 179-182,1956

[2] E.B. Curtis, D. Ingerman and J.A. Morrow, Circular planar graphs and resistor networks,, Linear algebra and its applications,283,p115–150,1998.

[3] D Ingerman, J.A. Morrow, On a characterization of the kernel of the Dirichlet-to-Neumann map for a planar region, SIMA Vol. 29 Number 1 pp. 106-115 1998

[4] Y. Colin de Verdière, Réseaux électriques planaires I, Publ. Inst. Fourier, V 225, p1-20, 1992.

[5] Y. Colin de Verdière, I. Gitler and D. Vertigan, Réseaux électriques planaires II, Comment. Math. Helvetici 71, 144-167, 1996 [10] W. R. B. Lionheart and K. Paridis, Finite elements and anisotropic EIT reconstruction, Journal of Physics: Conference Series, vol. 224, no. 1, p. 012022, 2010

[6] H Griffiths, A phantom for electrical impedance tomography, Clin. Phys. Physiol. Meas., 9, 15-20, 1988.

[7] H. Gagnon *et al* A resistive mesh phantom for assessing the performance of EIT systems, IEEE T Biomed. Eng., 57:2257?2266, 2010.

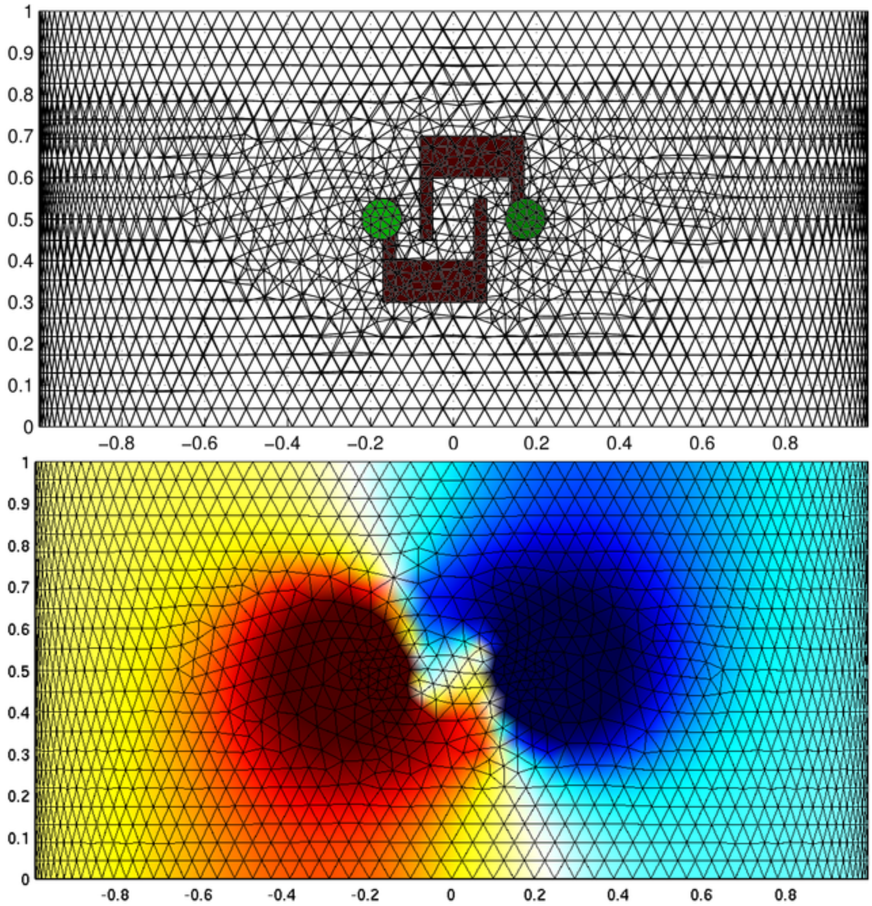

Figure 3: *Top*: An asymmetrical conductivity anomaly in cylindrical domain created using EIDORS and Netgen. Electrodes in green. *Bottom*: The equipotential lines on the surface resulting from driving current between the two circular electrodes . Note that in the plane through the electrodes that voltage is not monotonically decreasing from source to sink, see for example the isopotential between the yellow and white shading.

[8] J. Just *et al*, Constructing resistive mesh phantoms by an equivalent 2D resistance distribution of a 3D cylindrical object. Proceedings EIT Conference, Bath, 4-6 May, 2011.

[9] A. Al Humaidi. Resistor networks and finite element models. PhD thesis, University of Manchester, Manchester, UK, 2011.

[10] W.R.B. Lionheart and K. Paridis, Finite elements and anisotropic EIT reconstruction. Journal of Physics: Conference Series, 224, 2010.

[11] W.R.B. Lionheart and K. Paridis, Determination of an embedding consistent with discrete Laplacian on a triangular graph, in preparation.