The Statistics of Normalized Data in Electrical Impedance Tomography.

[1] **P. Hashemzadeh,** [2] **A. Adler and** [1]**R. Bayford**

[1] Department of Natural Science, Middlesex University, NW4 4BT, London, UK. [2] Department of Systems and Computer Engineering, Carleton University.

E-mail: hashemzadeh1354@gmail.com

Abstract. This study focuses on the noise statistics of the normalized data, in time-difference imaging. The commonly practiced normalization procedure, represents the normalized data as the ratio of two random variables. We provide closed form expressions for computing estimates of the first, second order moments, and the covariance matrix of normalized data. The expressions are derived using Taylor series expansion method for the function of random variables (propagation of uncertainty, delta method), and accurate provided that the signal to noise ratio of the reference measurement is larger than 16 dB. The analysis presented here, provides an explanation as well as insight, into the potential cause and presence of spikes in normalized data. Furthermore, we demonstrate that the statistics of normalized data, is significantly different to that of the raw measurements. We validate the closed form expressions using a Monte-Carlo method. We also conclude, that a minimum SNR of 13 dB is required for the reference measurement, in order to guarantee the existence of moments, and to ensure the accuracy of reconstructed images.

1. Introduction

Electrical impedance tomography is a low frequency imaging modality [1, 2]. It involves estimating the unknown electromagnetic properties, namely the permittivity and conductivity in a given domain from a finite number of voltage measurements made on it's boundary [3]. It is therefore a parameter estimation problem, and is the subject of much interest in inverse problems and signal processing [4, 5].
This paper focuses on the noise statistics of normalized data in time-difference imaging. The normalization procedure implemented in the freely distributed software package (EIDORS) [6], represents the normalized data as the ratio of two random variables. Moreover, at times, large unexpected spikes are observed in the normalized data. If the measurement noise is an additive Gaussian random process, then the normalized data will have a Cauchy distribution [7]. It is well known that the Cauchy distribution is an example of a distribution which has no mean, variance or higher order moments defined [7, 8]. However if the signal to noise ratio (SNR) of the reference measurement is above 13 dB, then estimates of the mean and variance may be obtained numerically [7, 9, 8].

Under the stated condition, one can then obtain an estimate of the covariance matrix of the normalized data. This matrix features explicitly in the Gauss-Newton solver, Kalman filter, and the Cramér Rao Bound analysis [10, 3, 5, 11].
The outline of the paper is follows : In section 2, we formulate the problem and describe the normalization procedure in time-difference imaging. In section 3, we provide closed form expressions for approximating the first, and second order moments of normalized data. Moreover, we outline a method for computing the non-diagonal elements of the covariance matrix. In section 4, we validate the closed form expressions using a Monte-Carlo method. Finally, the conclusion of this study is discussed in section 5.

2. Formulation of the Problem

Time-difference imaging is a linearized solution to the nonlinear estimation problem [12, 13, 10, 3]. We consider, an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$[8]. Where, Ω denotes the sample space, $\mathcal{F}$ is a sigma field on Ω, and $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ is the probability measure. The voltage measurements represent a discrete time stochastic process. This can be expressed succintly by $\mathbf{v} : \Omega \times \mathbb{N}^{+} \rightarrow \mathbb{R}^{\mathrm{M}}$ ($\forall \mathrm{k} \in \mathbb{N}^{+}, \mathbf{v}[\mathrm{k}] : \Omega \rightarrow \mathbb{R}^{\mathrm{M}}$). Where, M denotes the dimension of the measurement vector. In practice, only a finite number of samples can be recorded. We denote the total number of discrete observations by N_s. The normalization procedure involves selecting a reference time index, denoted by $\mathrm{k}_0 \leq \mathrm{N}_s$. The voltages measured at time index k_0 are used to form difference voltages. The *normalized* difference voltages are computed using the expression below

$$\delta v_i[\mathrm{k}] = \frac{v_i[\mathrm{k}]}{v_i[\mathrm{k}_0]} - 1; \qquad 1 \leq i \leq \mathrm{M} \tag{1}$$

It is clear from equation (1), that the normalization procedure results in the ratio of two random variables. As a result, $\delta\mathbf{v}[\mathrm{k}]$ may not have first and second order moments (i.e $\mathbb{E}(\delta v_i) = \pm\infty$, and $\mathrm{VAR}(\delta v_i) = \infty$). It is by this, that we mean that the first and second order moments do not exist. If they exist, then the objective is to compute the covariance matrix of the normalized voltages denoted here, by $\mathrm{C}_{\delta\mathrm{v}}$.

3. Covariance Matrix

The covariance matrix of normalized data features explicitly in the expressions for image reconstruction and lower bound analysis [14, 15, 10, 3, 5, 11]. One must therefore, ensure that is it is computed accurately. The elements of this matrix at time index k, can be computed below by

$$[\mathrm{C}_{\delta\mathrm{v}}[\mathrm{k}]]_{i,j} = \mathbb{E}\Big[\Big(\frac{v_i[k]}{v_i[k_0]}\frac{v_j[k]}{v_j[k_0]}\Big)\Big] - \mathbb{E}\Big[\frac{v_i[k]}{v_i[k_0]}\Big]\mathbb{E}\Big[\frac{v_j[k]}{v_j[k_0]}\Big]; \tag{2}$$

The closed form expressions presented here, are based on a Taylor series expansion method (propagation of uncertainty, delta method) [9], and accurate provided that the SNR of the reference measurements is $\frac{S_R}{N_R} > 16$ dB. We introduce the following notation for convenient $Z = v_i[\mathrm{k}]$, $W = v_i[\mathrm{k}_0]$. The mean (first order moment) reads as

$$\mathbb{E}(\delta v_i) = \mathbb{E}\Big[\frac{Z}{W}\Big] - 1 \simeq \frac{\mu_z}{\mu_w} - \frac{\sigma_z \sigma_w \rho_{z,w}}{\mu_w^2} + \frac{\mu_z}{\mu_w^3}\sigma_w^2 - 1 \tag{3}$$

The variance (second order moment) of the difference voltage is given by

$$\mathrm{VAR}(\delta v_i) = \mathrm{VAR}\Big(\frac{Z}{W} - 1\Big) \simeq \frac{\sigma_z^2}{\mu_w^2} - 2\Big(\frac{\mu_z \sigma_z \sigma_w \rho_{z,w}}{\mu_w^3}\Big) + \frac{\mu_z^2 \sigma_w^2}{\mu_w^4} \tag{4}$$

Where μ_z, and σ_z, respectively denote the mean and variance of Z, and μ_w, σ_w respectively denote the mean and variance of W. Here, $\rho_{z,w}$ is the correlation coefficient of the pair (Z, W). It is clear from the expressions (3), and (4), that if $\mu_w \to 0$, then $\mathbb{E}(\delta v_i) \to \pm\infty$, and $\mathrm{VAR}(\delta v_i) \to \infty$. This is the potential explanation behind the presence of spikes in normalized data. The spikes will occur, if $\mu_w \simeq 0$, or $\mu_w < 4\sigma_w$. However, If $\mu_w > 6\sigma_w$ ($\frac{S_R}{N_R} > 16$ dB), then $\mathrm{VAR}(\delta v_i) < \infty$, and we have obtained an an accurate estimate for diagonals of the covariance matrix $\mathrm{C}_{\delta \mathrm{v}}$. It should be noted that expressions (3), and (4) do not require the assumption that the measurement noise is additive Gaussian.

We now focus on computing, the non-diagonal elements of the covariance matrix $\mathrm{C}_{\delta \mathrm{v}}$. In order to simplify the analytical expressions, for the non-diagonal elements, we make the assumption, that the raw measurements are corrupted by a Gaussian random process. It can be seen from expression (2), that $(v_i[k]v_j[k], v_i[k_0]v_j[k_0])$ form a product pair. We must, first compute the mean, variance and the correlation coefficient of this product pair. Once, these values are obtained, we can simply use expression (3) to compute the expected value of this ratio. The remaining terms $\mathbb{E}\Big[\frac{v_i[k]}{v_i[k_0]}\Big]\mathbb{E}\Big[\frac{v_j[k]}{v_j[k_0]}\Big]$, can be computed directly using expression (3).

We introduce the following set of notations for convenience: $X_1 = v_i[k], X_2 = v_j[k]$, $X_3 = v_i[k_0]$, $X_4 = v_j[k_0]$, $\sigma_{i,j} = \mathrm{Cov}(\mathrm{X}_i, \mathrm{X}_j)$, $\sigma_i^2 = \mathrm{Var}(X_i)$, $Z = X_1 X_2$, and $W = X_3 X_4$. It is trivial to show that $\mu_z = \mathbb{E}Z = \sigma_{1,2} + \mu_1\mu_2$, and $\mu_w = \mathbb{E}W = \sigma_{3,4} + \mu_3\mu_4$. The expression for $\mathrm{Var}(Z)$, is given below by

$$\sigma_z^2 = (\mu_1\sigma_2)^2 + 2\mu_1\mu_2\sigma_{1,2} + (\mu_2\sigma_1)^2 + (\sigma_1\sigma_2)^2 + \sigma_{1,2}^2 \tag{5}$$

By inspection, $\mathrm{Var}(W)$ reads as

$$\sigma_w^2 = (\mu_3\sigma_4)^2 + 2\mu_3\mu_4\sigma_{3,4} + (\mu_4\sigma_3)^2 + (\sigma_3\sigma_4)^2 + \sigma_{3,4}^2 \tag{6}$$

The expression for $\mathrm{Cov}(Z, W)$, is given below by

$$\sigma_{z,w} = \mu_1\mu_3\sigma_{2,4} + \mu_1\mu_4\sigma_{2,3} + \mu_2\mu_3\sigma_{1,4} + \mu_2\mu_4\sigma_{1,3} + \sigma_{1,3}\sigma_{2,4} + \sigma_{1,4}\sigma_{2,3} \tag{7}$$

We obtain the correlation coefficient, using the simple relation $\rho_{z,w} = \frac{\sigma_{z,w}}{\sigma_z\sigma_w}$. At this stage, we have all the terms required, to compute $\mathbb{E}\Big[\Big(\frac{v_i[k]}{v_i[k_0]}\frac{v_j[k]}{v_j[k_0]}\Big)\Big]$ using expression (3).

4. Results and Discussions

Here, we compare the analytical expressions presented in section 3, against a Monte-Carlo simulation. It is worth noting that $(\mathrm{VAR}(\frac{Z}{W}) < \infty) \Rightarrow (\mathbb{E}[\frac{Z}{W}] < \infty)$. The

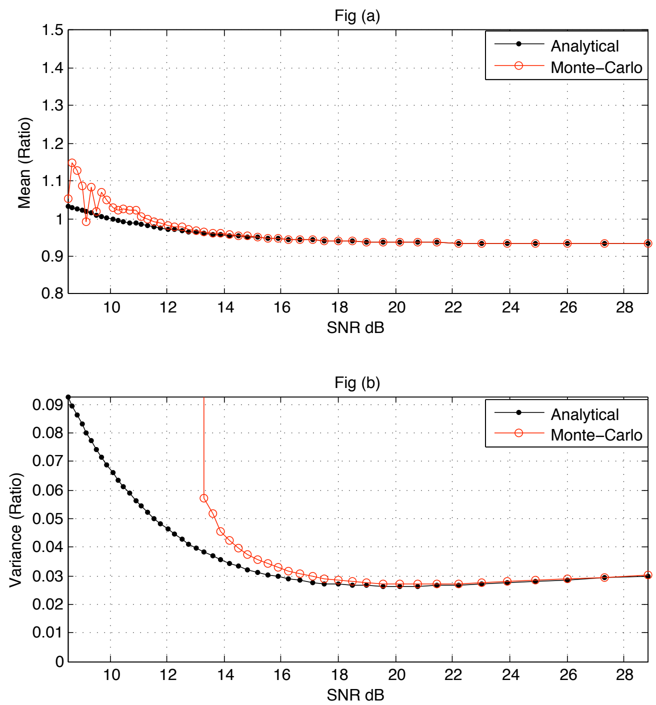

Figure 1. This figure shows the first and second order moments of the ratio Z/W as a function of $20\log\frac{\mu_w}{\sigma_w}$ on the x-axis. Black solid dotted line corresponds to the Taylor expansion method, and red solid line with circle is the Monte-Carlo method. The parameter are $\mu_z = 0.75, \mu_w = 0.8, \rho_{z,w} = 0.5, \sigma_z = 0.15$, and $\sigma_w \in [0.029, 0.24]$. Fig (a) is the plot of $\mathbb{E}[\frac{Z}{W}]$ (first order moment), Fig (b) is the plot of $\mathrm{VAR}(\frac{Z}{W})$ (second order moment).

results are show in Fig. 1. The numerical results, are as expected, and agree well with the theory. In Fig 1 (b), the Monte-Carlo simulations show, that when $|\mu_w| < 4\sigma_w$ ($20\log(\frac{\mu_w}{\sigma_w}) < 13$ dB), then $\mathrm{VAR}(\frac{Z}{W}) \to \infty$. This is a fundamental result in statistics, and under such condition, it becomes difficult to compute an accurate mean and variance. It is also clear that when, $|\mu_w| < 4\sigma_w$ ($20\log 10(\frac{|\mu_w|}{\sigma_w}) < 13$ dB), the mean and variance computed using the Taylor method is no longer accurate. However, when $|\mu_w| > 4\sigma_w$ ($20\log 10(\frac{|\mu_w|}{\sigma_w}) > 13$ dB), then the analytical approach begins to agree well with the Monte-Carlo simulation, and this is particularly true for $|\mu_w| > 6\sigma_w$ ($20\log 10(\frac{|\mu_w|}{\sigma_w}) > 16$ dB).

5. Conclusions

In this study, we have presented closed form expressions for computing the first and second order statistics of the normalized data, in time-difference imaging. These expressions are accurate provided that the signal to noise ratio of the reference measurement is above 16 dB. Moreover, they provide an explanation and insight, into the potential cause and presence of spikes in normalized data. We validate the closed form expressions using a Monte-Carlo procedure. It is also concluded that in order to guarantee accuracy in time-difference imaging, one must ensure that the minimum SNR of the reference measurement is above 13 dB.

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