Artifacts due to Conformal Deformations in Electrical Impedance Tomography

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Introduction

Artifacts in the images created using Electrical Impedance Tomography (EIT) due to movement of the boundary in difference imaging have been an issue, particularly in pulmonary EIT where chest movement due to breathing and posture change is a regular event.[1] With the recent development of algorithms to detect some types of boundary movement directly from the EIT measurements, it has become possible to correct for many of these boundary distortions by assuming an isotropic medium.[2][3] The further classification of boundary movement into two types, conformal movements and those that are not, provides the opportunity for further refinement of this algorithm.[4] In this paper, we discuss the conformal movements and their properties and show, through the governing conductivity equation for EIT, that conformal movement of an isotropic conductivity domain results in a new isotropic conductivity where the change in conductivity is directly related to the conformal movement.

Conformal Vector Fields

Let X be a vector field, assumed to be small in magnitude. A domain Ω is distorted by the map $x \mapsto x + X\epsilon$. This vector field X indicates the velocity of change in shape over time everywhere over the domain, where multiplying the vector field by some small time ϵ will give a map that is a new geometry for the domain.

A *conformal map* is one that preserves the angles but not necessarily lengths between vectors on the domain. If the map $x \mapsto x + X\epsilon$ is a conformal map, then the vector field X is referred to as an infinitesimal conformal map, known classically as an infinitesimal conformal motion, conformal Killing field or more simply a conformal vector field.

A conformal vector field is defined by the fact that, if the distorted domain is to have an isotropic field (e.g. conductivity in EIT) consistent with the boundary conditions, then for a change in the boundary of the domain Ω, the vector field X must be conformal and sufficiently smooth. Therefore, X is a conformal vector field if and only if the conformal Killing field equation is satisfied (the symmetrized derivative of X is a multiple of the identity) (1). [5, §3.7] [6, §1.4]

$$\frac{\partial X_i}{\partial x_j} + \frac{\partial X_j}{\partial x_i} = \alpha \delta_{ij} \tag{1}$$

where α is a scalar on the domain Ω.[5, (3.7.3)] In two-dimensions, summing over i and j, we see from taking the trace of (1) that α must be the divergence of X.

$$\frac{\partial X_i}{\partial x_j} + \frac{\partial X_j}{\partial x_i} = \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} \right) \delta_{ij} \tag{2}$$

Now, setting $i = j = 1$, gives the first Cauchy-Riemann equation, and on the other hand, setting $i = 1$, $j = 2$ gives the second Cauchy-Riemann equation,

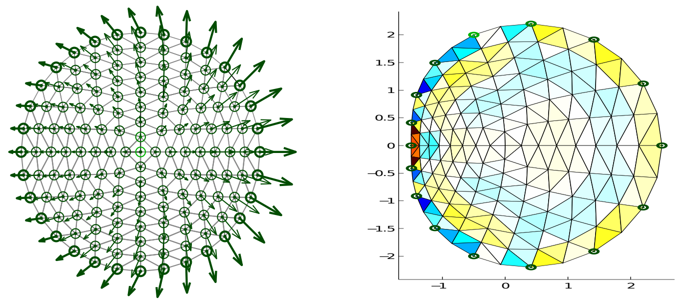
$$\frac{\partial X_1}{\partial x_1} - \frac{\partial X_2}{\partial x_2} = 0 \qquad\qquad \frac{\partial X_1}{\partial x_2} + \frac{\partial X_2}{\partial x_1} = 0 \tag{3}$$

Thus, if a function X is differentiable, its derivative is continuous, and it satisfies the Cauchy-Riemann equations, then it is complex analytic on the part of the plane that satisfies the Cauchy-Riemann equations. With any complex analytic function, the real and imaginary parts are harmonic conjugate. [7] Specifically, since the components of a conformal vector field $X_1 + \mathrm{i}X_2$ are complex analytic and satisfy $\nabla X_1 \cdot \nabla X_2 = 0$, $\|\nabla X_1\|^2 = \|\nabla X_2\|^2$, and Laplace's equation $\nabla^2 X_1 = \nabla^2 X_2 = 0$, the components of the vector field (X_1, X_2) are perpendicular, but furthermore, ∇X_2 is 90° anti-clockwise from ∇X_1 and equal in magnitude. (3)

With Respect to EIT

Recall the governing conductivity equation for EIT

$$\nabla \cdot \sigma \nabla \Phi = \begin{cases} 0 & \text{inside} \\ J_n & \text{on the boundary} \end{cases} \tag{4}$$



(a) (b)

Figure 1: Conformal deformation artifacts. (a) the green arrows represent the conformal vector field $X : z \to z + z^2/2$ (b) an isotropic conductivity resulting from the conformal deformation [8]

Since the boundary measurements for a conformal movement can also be explained by a change in conductivity on the domain, (4) can be written as

$$\nabla \cdot \sigma_c \nabla \Phi_c(x_1, x_2) = \nabla \cdot \sigma_m \nabla \Phi_m(x_1 + X_1, x_2 + X_2) \tag{5}$$

where the subscripts indicate conductivity change c and conformal movement m and $\Phi_c(x_1, x_2) = \Phi_m(x_1 + X_1, x_2 + X_2)$ since there is no change in the measured voltages at the electrodes on the boundary. Conversion between the two coordinate systems gives

$$\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial(x_1+X_1)}{\partial x_1} & \frac{\partial(x_1+X_1)}{\partial x_2} \\ \frac{\partial(x_2+X_2)}{\partial x_1} & \frac{\partial(x_2+X_2)}{\partial x_2} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial(x_1+X_1)} \\ \frac{\partial}{\partial(x_2+X_2)} \end{bmatrix} \tag{6}$$

where iff X is conformal then from the Cauchy-Riemann equations (3)

$$\frac{\partial X_1}{\partial x_1} = \frac{\partial X_2}{\partial x_2} = A - 1 \qquad\qquad \frac{\partial X_1}{\partial x_2} = -\frac{\partial X_2}{\partial x_1} = B \tag{7}$$

and taking the inverse of (6) gives

$$\begin{bmatrix} \frac{\partial}{\partial(x_1+X_1)} \\ \frac{\partial}{\partial(x_2+X_2)} \end{bmatrix} = \underbrace{\frac{1}{A^2 + B^2} \begin{bmatrix} A & -B \\ B & A \end{bmatrix}}_{T} \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{bmatrix} \tag{8}$$

Assuming an isotropic conductivity, substituting (8) into (5) gives $\sigma_c = TT^{\mathrm{T}}\sigma_m$ where $TT^{\mathrm{T}} = 1/(A^2+B^2)$ such that the conductivities are the same, divided by a scalar. As expected, the conductivities are equal if there is no movement. An example of a uniform isotropic conductivity deformed by a conformal map can be seen in Figure 1.

Discussion

In EIT, simultaneous reconstruction of an accurate isotropic conductivity and conformal deformation are not possible when only the measurements are known because the deformation does not change the resulting isotropy of the reconstruction. A conformal deformation does, however, result in specific changes in the image's conductivity and this has application in algorithms [3][8] that reconstruct the boundary movement.

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References

[1] Adler A Guardo R Berthiaume Y 1996 "Impedance imaging of lung ventilation: Do we need to account for chest expansion?" *IEEE Trans. Biomed. Eng.* 43 414–20
[2] Lionheart WRB 1998 "Boundary Shape and Electrical Impedance Tomography" *Inverse Problems* 14 139–47
[3] Soleimani M Gómez-Laberge C Adler A 2006 "Imaging of conductivity changes and electrode movement in EIT" *Physiol. Meas.* 27 S103–S113
[4] Boyle A Lionheart WRB Gómez-Laberge C Adler A 2008 "Evaluating Deformation Corrections in Electrical Impedance Tomography" *EIT Conf. Proceedings 2008*
[5] Goldberg S 1982 *Curvature and Homology* Dover Publications
[6] Schottenloher M 2008 *A Mathematical Introduction to Conformal Field Theory* Springer
[7] Marsden J Hoffman M 1998 *Basic Complex Analysis* 3rd Ed W.H. Freeman & Company
[8] Adler A Lionheart WRB 2006 "Uses and abuses of EIDORS: An extensible software base for EIT" *Physiol. Meas.* 27 S25–S42