BER Analysis of Three-Phase XOR-and-Forward Relaying Using Alamouti STBC

Rajab M. Legnain, Rosdy H. M. Hafez, and Ian D. Marsland

Abstract—In this paper we investigate the performance of two-way relaying that uses Alamouti space-time block code (STBC) and network coding. In this scheme, two nodes use Alamouti STBC to transmit their symbols to the relay consecutively in two phases. The relay detects and demodulates the transmitted symbols from the nodes, and then uses the bitwise-XOR operation to combine the detected bits. Then the relay modulates and broadcasts the XORed bits to the nodes using Alamouti STBC. In this paper, the bit error rate of the proposed scheme is derived for an uncorrelated Rayleigh fading channel with perfect channel estimation at receivers.

Index Terms—Two-way relaying, network coding, STBC.

I. INTRODUCTION

TWO-WAY relaying schemes that use network coding (NC) have attracted the attention of many researchers, because these schemes can improve the spectral efficiency [1]–[3]. The NC can be divided into digital NC (DNC) schemes [4], [5] and analog NC (ANC) schemes. In the DNC schemes, the relay detects the information bits that are transmitted from the nodes, and then combines the detected bits using bitwise-XOR operation. While the ANC schemes can be further divided into two schemes: non-generative schemes and generative schemes. In the non-generative schemes [1] the relay amplifies the received signal, and then re-transmits it to the nodes. In the generative scheme [6], [7] the relay estimates the transmitted symbols from the nodes, combines them linearly, and then broadcasts the combined symbols to the nodes.

Two-way relaying using NC can be divided into two protocols: three-phase relay protocol [2], [8] and two-phase protocol [1]. In the three-phase protocol the two nodes require three phases to exchange their signals. In phase one, \(A_1\) sends its signal to the relay, and in phase two, \(A_2\) sends its signals to the relay. The relay combines the two transmitted signals from the nodes, and then broadcasts the combined signal in the third phase. Since each node knows its own transmitted signal, it can use this knowledge to extract the signal of the other node. In the two-phase protocol the two nodes require two phases to exchange their signals. In the first phase, \(A_1\) and \(A_2\) simultaneously transmit their signals to the relay. The relay processes the received signals, and then broadcasts the combined signal in the second phase. Several two-way relaying schemes using NC have been proposed and investigated [9]–[11]

In [9], the authors investigate the performance of two-phase relaying protocol using DNC. Where in the first phase, two nodes (each has one antenna) transmit their symbols simultaneously to the relay. The relay (which has two antennas) uses maximum likelihood detector to estimate the transmitted symbols, and then uses bitwise-XOR to combine them. The relay then, in the second phase, broadcasts the XORed symbols using either Alamouti space-time block code (STBC) or antenna selection.

In this paper we investigate the performance of the network coded three-phase relaying scheme when the nodes and the relay use the Alamouti STBC technique to transmit their symbols. We derive an approximate closed-form expression of the bit error rate (BER) for the proposed scheme, when the nodes and the relay have two antennas and use gray coded square \(M\)-QAM (i.e., \(M = 4, 16, 64\)) to map the bits. The rest of this paper is organized as follows. The system model of the proposed scheme is described in Section II. In Section III we derive the approximate closed-form expression of the BER for the proposed scheme. The simulation results and the conclusions are presented in Sections IV and V, respectively.

Notation: Throughout the paper, the following notations are used. Bold lowercase and bold uppercase letters denote vectors and matrices, respectively. \([\cdot]^*, [\cdot]^T, [\cdot]^H\) and \(\|\cdot\|_F\) denote complex conjugate, transpose and Hermitian operations, and Frobenius norm of a vector or a matrix, respectively. \(E[\cdot]\) denote the expectation operation. \(\mathbf{I}_n\) is used to denote \(n \times n\) identity matrix. \(Q(x)\) denote Gaussian Q-function.

II. SYSTEM MODEL

We consider two nodes, \(A_1\) and \(A_2\), communicating through a relay as shown in Fig. 1. The nodes and the relay have \(N = 2\) antennas and operate in a half-duplex mode, i.e., they cannot transmit and receive simultaneously. The channels between the nodes and the relay are assumed to be uncorrelated slow Rayleigh fading channel. The communication between \(A_1\) and \(A_2\) is done in three phases: the first two phases for multiple access (MA), and the third for broadcasting (BC). In the first and second phase \(A_1\) and \(A_2\) transmit their symbols to the
relay, respectively. In the third phase the relay processes the received signal and broadcasts it to the nodes.

A. Multiple Access Phases

Each node, \( A_k \) \((k = 1, 2)\), transmits a pair of symbols (drawn from a Gray coded \( M \)-QAM) denoted by \( x_k = [x_{k,1}, x_{k,2}]^T \). The covariance matrix of \( x_k \) is \( E[x_kx_k^H] = \frac{1}{2}I_2 \), where \( \hat{E}_k \) is the average transmitted energy per time slot at \( A_k \). Each node encodes its pairs of symbols in space-time as follows [12].

\[
X_k = \begin{bmatrix} x_{k,1} & -x_{k,2} \\ x_{k,2} & x_{k,1} \end{bmatrix}
\]

Node \( A_k \) transmits \( X_k \) to the relay over two consecutive time slots in the \( p^{th} \) phase \((p = 1, 2)\), where the first and second columns are transmitted in the first and second time slots, respectively, and the first and second rows are transmitted over the first and second transmit antennas, respectively. We assume \( p = k \), i.e., \( A_1 \) and \( A_2 \) transmit in the first and second phase, respectively.

The channel matrix between node \( A_k \) and the relay in the MA phase is given by

\[
H_k = \begin{bmatrix} h_{k,1,1} & h_{k,1,2} \\ h_{k,2,1} & h_{k,2,2} \end{bmatrix}
\]

where \( h_{k,i,j} \) represents the channel coefficient between the \( j^{th} \) transmit antenna of node \( A_k \) and the \( i^{th} \) receive antenna of the relay. Each element of the channel matrix is modeled as a complex zero-mean Gaussian (CZMG) random variable with variance \( \sigma_k^2 \), where \( \sigma_k^2 \) is the path-loss between \( A_k \) and the relay.

At the relay, the received sample can be written as

\[
Y_R^{(p)} = \begin{bmatrix} y_{R,1,1}^{(p)} & y_{R,1,2}^{(p)} \\ y_{R,2,1}^{(p)} & y_{R,2,2}^{(p)} \end{bmatrix} = H_kX_k + N_R^{(p)},
\]

where \( y_{R,i,t}^{(p)} \), \( i = 1, 2 \) and \( t = 1, 2 \), represents the received sample on the \( i^{th} \) receive antenna in the \( t^{th} \) time slot. \( N_R \) is the \( 2 \times 2 \) noise matrix, where the element \( n_{i,t} \) represents the noise on the \( i^{th} \) receive antenna in the \( t^{th} \) time slot, which is modeled as a CZMG random variable with variance \( \sigma_R^2 \). The receiver rearranges the received samples in (3) as

\[
\begin{split}
\begin{bmatrix} y_{R,1,1}^{(p)} \\ y_{R,1,2}^{(p)} \\ y_{R,2,1}^{(p)} \\ y_{R,2,2}^{(p)} \end{bmatrix} &= \begin{bmatrix} h_{k,1,1} & h_{k,1,2} \\ h_{k,2,1} & h_{k,2,2} \\ h_{k,1,1} & -h_{k,2,1} \\ h_{k,1,2} & -h_{k,2,2} \end{bmatrix} \begin{bmatrix} x_{k,1} \\ x_{k,2} \end{bmatrix} + \begin{bmatrix} n_{1,1}^{(p)} \\ n_{1,2}^{(p)} \\ n_{2,1}^{(p)} \\ n_{2,2}^{(p)} \end{bmatrix} \\
&= \mathcal{H}_k \begin{bmatrix} x_{k,1} \\ x_{k,2} \end{bmatrix} + \mathcal{N}_R^{(p)},
\end{split}
\]

where \( \mathcal{H}_k \) is the effective channel matrix. This matrix has orthogonal columns (i.e., \( \mathcal{H}_k^H \mathcal{H}_k = ||\mathcal{H}_k||^2_F I_2 \)).

The relay processes the received sample vectors to estimate the transmitted symbols coming from the nodes. Since the signals coming from the nodes do not interfere with each other (i.e., because there are transmitted on different phase), the relay detects the symbols as follows

\[
\hat{x}_k = Q \left( \frac{\mathcal{H}_k^H Y_R^{(p)}}{||\mathcal{H}_k||^2_F} \right),
\]

where \( Q(\cdot) \) denotes the hard estimation.

B. Broadcasting Phase

Once the symbols are estimated, the relay demodulates them to obtain the estimated information bits. Then the relay uses the bitwise-XOR operation to combine the estimated information bits as

\[
b_R = \hat{b}_1 \oplus \hat{b}_2,
\]

where \( \hat{b}_k \) is the estimate of the information bits from node \( A_k \).

The channel matrix between the relay and node \( A_k \) in the BC phase is given by

\[
G_k = \begin{bmatrix} g_{k,1,1} & g_{k,1,2} \\ g_{k,2,1} & g_{k,2,2} \end{bmatrix}
\]

where \( g_{k,i,j} \) represents the channel coefficient between the \( j^{th} \) transmit antenna of the relay and \( i^{th} \) receive antenna of node \( A_k \). Each element of the channel matrix is modeled as a CZMG random variable with variance \( \sigma_k^2 \).

The relay modulates \( b_R \) and then broadcasts it to the nodes using Alamouti STBC in the third phase. The rearranged sample vector at node \( A_k \) is given by

\[
\begin{split}
\begin{bmatrix} g_{k,1,1} & g_{k,1,2} \\ g_{k,1,2} & -g_{k,1,1} \\ g_{k,2,1} & g_{k,2,2} \\ g_{k,2,2} & -g_{k,2,1} \end{bmatrix} &= \begin{bmatrix} x_{R,1} \\ x_{R,2} \end{bmatrix} + \begin{bmatrix} n_{1,1}^{(p)} \\ n_{1,2}^{(p)} \\ n_{2,1}^{(p)} \\ n_{2,2}^{(p)} \end{bmatrix} \\
&= \mathcal{G}_k \begin{bmatrix} x_{R,1} \\ x_{R,2} \end{bmatrix} + \mathcal{N}_R^{(p)},
\end{split}
\]

where \( \mathcal{G}_k \) is the effective channel matrix between the relay and node \( A_k \), and \( \mathcal{N}_R \) is the effective noise vector at node \( A_k \). \( X_R \) is the transmitted symbol vector from the relay, which has a covariance matrix of \( E[x_kx_k^H] = \frac{1}{2}I_2 \) where \( \hat{E}_R \) is the average transmitted energy per time slot at the relay.

At node \( A_k \), the receiver uses a linear detector to detect the transmitted symbols from the relay as

\[
\hat{x}_{R,k} = Q \left( \frac{\mathcal{G}_k^H Y_k}{||\mathcal{G}_k||_F} \right),
\]

where \( \hat{x}_{R,k} \) is the estimated transmitted symbol vector from relay at node \( A_k \).

The nodes demodulate the detected symbols, and then since the nodes know their transmitted information bits, they use this knowledge to extract the transmitted bit from the other node. This can be done as follows

\[
\hat{b}_2 = b_1 \oplus \hat{b}_{R,1},
\]

\[
\hat{b}_1 = b_2 \oplus \hat{b}_{R,2},
\]

where \( \hat{b}_1 \) and \( \hat{b}_2 \) are the estimated information bits of \( A_1 \) and \( A_2 \) at \( A_2 \) and \( A_1 \), respectively. \( b_{R,k} \) are the estimated information bits of relay at node \( A_k \).

III. PERFORMANCE ANALYSIS

In this section, we derive an approximate closed-form expression for the average bit error rate (BER) of the scheme when Gray coded square \( M \)-QAM (e.g., \( M = 4, 16, 64 \)) is used at the relay and the nodes.
A. MA Phases

Let $P_{s,k\to R}$ denotes the instantaneous symbol error rate (SER) of detecting the symbol of node $A_k$ at the relay. According to [13], $P_{s,k\to R}$ can be written as

$$P_{s,k\to R} = 4q Q \left( \sqrt{2\beta\gamma_{k,R}} \right) - 4q^2 Q^2 \left( \sqrt{2\beta\gamma_{k,R}} \right)$$

(12)

where $q = (1 - 1/\sqrt{M})$, and $\beta = 3/[2(M - 1)]$. $\gamma_{k,R}$ is the instantaneous SNR per symbol at the receiver output of the relay for the signal transmitted from $A_k$ and is given by [14]

$$\gamma_{k,R} = \frac{E_k}{2\sigma_N^2} \|H_k\|_F^2 = \frac{E_k}{2\sigma_N^2} \sum_{i,j=1}^{2} |h_{k,i,j}|^2.$$ 

(13)

Where $E_k$ is the average transmitted energy per symbol per time slot at $A_k$. Since each channel coefficient, $h_{k,i,j}$, is a CZMG random variable, thus $\gamma_{k,R}$ is a chi-square random variable with eight degree of freedom, which results from the sum of their corresponding channel coefficients. The probability density function (PDF) of $\gamma_{k,R}$ is given by

$$Pr (\gamma_{k,R} | \gamma_{k,R}) = \frac{1}{(L-1)!} \gamma_{k,R}^{L-1} \exp (-\frac{\gamma_{k,R}}{\bar{\gamma}_{k,R}}),$$

(14)

where $L = 4$ denotes the number of independent channels $(h_{k,i,j})$, and $\bar{\gamma}_{k,R}$ is the average SNR per channel per symbol and can be written as

$$\bar{\gamma}_{k,R} = \frac{\sigma_N^2 E_k}{2\sigma_N^2}.$$ 

(15)

To find the average SER of detecting the symbol of node $A_k$ at the relay, we average (12) over the PDF of $\gamma_{k,R}$ as

$$\bar{P}_{s,k\to R} = \int_0^\infty 4q Q \left( \sqrt{2\beta\gamma_{k,R}} \right) Pr (\gamma_{k,R} | \gamma_{k,R}) d\gamma_{k,R} - \int_0^\infty 4q^2 Q^2 \left( \sqrt{2\beta\gamma_{k,R}} \right) Pr (\gamma_{k,R} | \gamma_{k,R}) d\gamma_{k,R}.$$ 

(16)

Substituting (18) in (17) and using the results in [15], the average SER of signal $A_k$ at the relay can be expressed as

$$\bar{P}_{s,k\to R} = 4q \left( \frac{1 - \mu}{2} \right) \sum_{l=0}^{L-1} \left( \frac{L - 1 + l}{1} \right) \left( \frac{1 + \mu}{2} \right)$$

$$- 4q^2 \left( \frac{1 - \mu}{4} \right) \left( \frac{1 - \mu}{2} \right) \left( \frac{1 + \mu}{2} \right) \sum_{l=0}^{L-1} \left( \frac{2l}{1} \right)$$

$$- \sin (D) \sum_{l=0}^{L-1} \sum_{n=1}^l T_{nl} \cos (D) (2l+1) \left( \frac{1 + \beta^2\gamma_{k,R}}{1 + \beta^2\gamma_{k,R}} \right),$$

where $\mu = \sqrt{\beta^2\gamma_{k,R}/(1 + \beta^2\gamma_{k,R})}$, $D = \tan^{-1} (\mu)$, and

$$T_{nl} = \left( \frac{2l}{1} \right) \left( \frac{2(l-n)}{l-n} \right) \left[ 4^n (2l + 1) \right].$$

Since the signal constellation points are arranged using Gray coding, the average BER of node $A_k$ at the relay can be approximated as

$$\bar{P}_{b,k\to R} \approx \frac{\bar{P}_{s,k\to R}}{\log_2 M}.$$ 

(20)

B. BC Phase

In similar way, the BER of the signal coming from the relay at node $A_k$ can be expressed as

$$\bar{P}_{b,R\to k} \approx \frac{\bar{P}_{s,R\to k}}{\log_2 M}.$$ 

(21)

where $\bar{P}_{s,R\to k}$ is the average SER of the signal coming from the relay at node $A_k$ and is given by

$$\bar{P}_{s,R\to k} = 4q \left( \frac{1 - \mu}{2} \right) \sum_{l=0}^{L-1} \left( \frac{L - 1 + l}{1} \right) \left( \frac{1 + \mu}{2} \right)$$

$$- 4q^2 \left( \frac{1 - \mu}{4} \right) \left( \frac{1 - \mu}{2} \right) \left( \frac{1 + \mu}{2} \right) \sum_{l=0}^{L-1} \left( \frac{2l}{1} \right)$$

$$- \sin (D) \sum_{l=0}^{L-1} \sum_{n=1}^l T_{nl} \cos (D) (2l+1) \left( \frac{1 + \beta^2\gamma_{R,k}}{1 + \beta^2\gamma_{R,k}} \right),$$

(22)

where $\gamma_{R,k} = \frac{\sigma_N^2 E_d}{2\sigma_N^2}.$

C. End-to-End BER

Finally, we find the average end-to-end BER at $A_k$, which is defined as the average BER of detecting the signal of $A_t$ at node $A_k$, where $k,t \in \{1,2\}$ and $k \neq t$. The end-to-end BER at each nodes is effected by the BER at the relay, because the relay uses bitwise-XOR operation to combine the detected bits of both nodes. Thus, the average end-to-end BER at $A_k$ is

$$\bar{P}_{b,t\to k} = \bar{P}_{b,t\to R} (1 - \bar{P}_{b,k\to R}) (1 - \bar{P}_{b,R\to k})$$

$$+ \bar{P}_{b,t\to R} \bar{P}_{b,k\to R} (1 - \bar{P}_{b,R\to k}) + \bar{P}_{b,t\to R} (1 - \bar{P}_{b,k\to R}) \bar{P}_{b,R\to k}$$

$$+ \bar{P}_{b,t\to R} \bar{P}_{b,k\to R} \bar{P}_{b,R\to k}.$$ 

(23)
This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

For example, in the case of Alamouti STBC $(\text{A}_m)$, we observe from Fig. 3 that the signal that is transmitted over a higher SNR in MA phase will have a higher BER. For example, in Fig. 3 the BER of signal $A_2$ at $A_1$ is higher than the BER of signal $A_1$ at $A_2$ (i.e., $\bar{\gamma}_{2, R} > \bar{\gamma}_{1, R}$). This is because of that the signal of $A_2$ was transmitted over a higher SNR in the MA phase (i.e., $\gamma_{2, R} > \gamma_{1, R}$). Where two nodes use Alamouti STBC to exchange their messages through a relay. We investigated the performance of this scheme in slow uncorrelated Rayleigh fading channel. We derived approximate closed-form expression of the bit error rate for Gray coded squared $M$-QAM modulation. We also verified the analysis results with simulation results.

The BER analysis can be generalized to orthogonal STBC (OSTBC). This can be done by using $L = N_N N_R, \gamma_{k, R} = c_k \frac{\sigma^2_k E_k}{N_N N_R},$ and $\bar{\gamma}_{R, k} = c_R \frac{\sigma^2_k E_k}{N_N N_R}$. Where $N_N$ and $N_R$ are the number of antennas at the nodes and the relay, respectively. $c_k$ and $c_R$ are constants and they depend on the OSTBC type. For example, in the case of Alamouti STBC $c_k = 1$ [14].

**IV. SIMULATION RESULTS**

In this section, we provide simulation results for the end-to-end BER performance of the proposed scheme using Monte Carlo simulation. These results are compared to the analytical results, which obtained by using (23).

Fig. 2 shows the end-to-end BER for $M = 4, 16, 64$, when $\text{SNR}_2 = \text{SNR}_1$. As expected, the theoretical results are very close to the simulation results, and they match at high SNR.

Furthermore, the theoretical results approach to the simulation results as the modulation order ($M$) decreases.

In Fig. 3, we extend the simulation to the case when $\text{SNR}_2 = \text{SNR}_1 + 10$ dB. Again, we show that the theoretical results approaches to the simulation results as the SNR increases.

We also observe from Fig. 3 that the signal that is transmitted over a higher SNR in MA phase will have the higher end-to-end BER. For example, in Fig. 3 the BER of signal $A_2$ at $A_1$ is higher than the BER of signal $A_1$ at $A_2$ (i.e., $\bar{\gamma}_{2, R} > \bar{\gamma}_{1, R}$), despite the fact of that the signal of $A_2$ was transmitted over a higher SNR in the MA phase (i.e., $\gamma_{2, R} > \gamma_{1, R}$). This is because of that node $A_1$ uses its original transmitted bits to extract the bits of node $A_2$ from the XORed bits that were transmitted from the relay, where the detected bits of $A_1$ (at the relay) has higher BER, which will increase the BER of the signal $A_2$ at node $A_1$.

**V. CONCLUSIONS**

In this paper we considered three-phase relaying scheme, where two nodes use Alamouti STBC to exchange their messages through a relay. We investigated the performance of this scheme in slow uncorrelated Rayleigh fading channel. We derived approximate closed-form expression of the bit error rate for Gray coded squared $M$-QAM modulation. We also verified the analysis results with simulation results.

**REFERENCES**