Analysis of Interference from Large Clusters as Modeled by the Sum of Many Correlated Lognormals

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***Abstract*—We examine the statistical distribution of the interference produced by a cluster of very many co-channel interferers, e.g., a sensor network, or a city full of active wireless devices and access points. We consider an arbitrary statistical interferer layout and consider the interference as experienced at a given point outside (and not immediately near to) the interferer area. We model the paths as experiencing power law attenuation and lognormal correlated shadowing. It has been shown in literature that adding correlation to the shadowing model can give qualitatively different (and probably more realistic) results.**

Our results are mostly analytical, with a small amount of numerical integration required. Whereas simulations of very many correlated interferers are very computationally heavy, our method's complexity is independent of the number of interferers, and its precision in fact improves when increasing the number of terms.

***Index Terms*—sum of lognormals, correlated shadowing, interference analysis, sensor network.**

I. Introduction

We study the distribution of the total co-channel interference power coming from a cluster of very many interferers. Examples could include a large sensor network, with thousands or even millions of nodes, or an urban area with very many access points, wireless terminals, or other interference sources.

At this point of our research, we want to develop an analytical tool that will accurately approximate the total interference power distribution without the need for Monte Carlo simulations. The common approach in treating such problems has been to apply the Central Limit Theorem when the number of interferers became large [1]–[5]. However, we want to argue against such an approach. Field measurements show (and intuition supports the fact) that shadowing along different paths is correlated, with the correlation being particularly significant for closely-located sources. Simulations show that one can obtain substantially different results for the probability of outage in interference-limited systems, whether one uses an independent or a correlated shadowing model [6]. Because the simulation of many correlated interferers can be computationally prohibitive, we wish to develop an analytical and numerical method that will not increase in complexity as the number of interferers N increases.

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In order to analyse the problem, we use another limit theorem that we developed in [7], where we analyse the sum of identically distributed, equally and (strictly) positively correlated jointly lognormal (SIDEPCJLN) random variables, and show that this sum converges to a lognormal distribution as the number of terms $N \to \infty$. A central idea of this paper is that, although it may not be immediately apparent, this problem is very similar to the scenraio we consider here. It is true that the received powers from various interferers are not identically distributed, nor are all pairs equally correlated. However, we argue that if the positions of the interferers are not known *a priori*, but are random, independent and identically distributed, then, by symmetry, the individual interference powers can be considered as identically distributed and equally correlated. We will explain this in more detail in Section IV. This rather indirect approach to solving such a problem is novel: although the approximate distribution is lognormal, we do not use direct moment-matching as in [8], [9], which would yield complicated and unseparable integrals, unlike our solution in Section III.

We assume that the cluster of interferers is not located too close to the point of interest (receiver), which implies that the interferers are in the far-field of the receiver. Also the ratio between the distance from the farthest and the nearest point of the region of interferers to the receiver should not be too large, so that it is certain that the distribution of the interference coming from one randomly-located interferer can be considered approximately lognormal. It is wise to verify this by simulation for just one interferer before performing the analysis. We observe that as the ratio between the distance to the farthest and to the nearest point decreases, the interference becomes more and more lognormal-like. This has been observed in many of our simulations and can be intuitively explained. The distance from the interferer to the receiver is a random variable which we may call r_i. Now the received power from the interferer is proportional to $\exp(S_i - \beta \ln r_i)$, where S_i is a Gaussian random variable representing shadowing, and is independent of r_i. It can be said that the Gaussian distribution is an "attracting distribution", in the sense that it remains (approximately) Gaussian if anything independent is added to it. It can be shown that this assumption will be valid if the variance of r_i is not too large, which in turn means that the distance should not have too large a span.

The weakness of our argument is that while the individual interference powers are (approximately) lognormal, identically distributed, and equally correlated, there is no evidence to show that they are *jointly* lognormal. Still, for the scenario analysed in Section V, the theory matched the analytical results very well. However, the consequences of having non-jointly lognormal terms need to be further studied. We provide some initial thoughts on this in Section VI.

II. Problem Statement

Let there be N interferers distributed randomly on an area, according to a density function $g(x,y)$. The density is located in such a way that our point of interest, or receiver (where the interference is measured) is located at the origin. We assume each interferer transmits with equal constant power. Also, let $\vec{r}_i$ be the vector from the origin to the location of an interferer i, $r_i = \|\vec{r}_i\|$, and let the power law pathloss exponent be β. Let S_i be the shadowing in logarithmic scale along the path $\vec{r}_i$, such that the interference power received from one interferer is

$$I_i = c r_i^{-\beta} e^{S_i}, \tag{1}$$

where c is a constant accounting for multiplicative factors such as antenna gains, absolute distances, and transmit power. Without loss of generality, we set $c = 1$. The interfering signals from the various interferers add incoherently, therefore the total interference power at the origin is:

$$I = \sum_{i=1}^{N} I_i = \sum_{i=1}^{N} r_i^{-\beta} e^{S_i}. \tag{2}$$

It is usually assumed [6] that the shadowing vector $\vec{S} = [S_i]_{i=1}^N$ is jointly Gaussian, and that the correlation coefficient between each pair is $h_{i,j} = h\left(\left|\angle\left(\vec{r}_i, \vec{r}_j\right)\right|, \left|\ln r_i - \ln r_j\right|\right)$, i.e., a function of the angle between the two paths, and the ratio of their lengths.

In order to generate $\vec{S}$ for Monte-Carlo simulations, it is first necessary to generate the correlation matrix $\mathbf{H}_{N\times N}$ with entries $h_{i,j}$. We then need to find its "square root" , such that:

$$\mathbf{H} = \mathbf{C}^T\mathbf{C}. \tag{3}$$

This is usually done by Cholesky factorisation [6]. Note that this is a computationally-intensive operation for large matrices [10]. Finally, we need to generate a vector $\vec{Z} = [Z_i]_{i=1}^N$ of independent standard Gaussian random variables. We can then generate $\vec{S}$ as

$$\vec{S} = \vec{Z}\mathbf{C}. \tag{4}$$

We wish to find the distribution of I for any realistic functions g and h, pathloss β, shadowing variance σ_s^2 and a large number of interferers N. Note that the shadowing variance needs to be converted from logarithmic to natural units: $\sigma_\mathrm{s}[\mathrm{nat}] = \sigma_\mathrm{s}[\mathrm{dB}] \times 0.1 \ln 10$. Thus a shadowing of 6 dB becomes $\sigma_\mathrm{s} = 1.382$.

III. Solution

The distribution of the total interference I can be well approximated by a lognormal with parameters $\left(m, s^2\right)$:

$$\begin{aligned} m &= \tfrac{3}{2}\ln N - \beta G_1 - \tfrac{1}{2}k, \\ s^2 &= \sigma_\mathrm{s}^2 - \ln N + \beta^2\left(G_2 - G_1^2\right) + k, \\ k &= \ln\left(1 + (N-1)e^{\left(\sigma_\mathrm{s}^2(G_\mathrm{cor}-1) - \beta^2\left(G_2 - G_1^2\right)\right)}\right), \end{aligned} \tag{5}$$

for both $N = 1$ and large N, and somewhat well for intermediate N. For large N, (5) simplifies to

$$\begin{aligned} m &\approx \ln N + \tfrac{1}{2}\left(\beta^2 G_2 + 1 - \left(\beta G_1 + 1\right)^2 + \sigma_\mathrm{s}^2\left(1 - G_\mathrm{cor}\right)\right), \\ s^2 &\approx \sigma_\mathrm{s}^2 G_\mathrm{cor}. \end{aligned} \tag{6}$$

G_n and G_cor are to be found by numerical integration of the functions g and h:

$$\begin{aligned} G_n &= \iint_{\mathbb{R}^2} \left(\ln^n r_1\right) g\left(\vec{r}_1\right) d\vec{r}_1, \\ G_\mathrm{cor} &= \iint_{\mathbb{R}^2}\iint_{\mathbb{R}^2} h\left(\left|\angle\left(\vec{r}_1, \vec{r}_2\right)\right|, \left|\ln\frac{r_1}{r_2}\right|\right) \\ &\quad \times g\left(\vec{r}_1\right) g\left(\vec{r}_2\right) d\vec{r}_1 d\vec{r}_2. \end{aligned} \tag{7}$$

These integrals are well-behaved and can be well approximated by a Riemann sum (trapezoidal rule) with a moderate number of terms. Of course the region of integration is not infinite, since it will be determined by the domain of g. G_n and G_cor are termed "geometric coefficients", and are an idea borrowed from [8], [11].

IV. Proof of our Method

The proof of our method will be indirect. We will recall a result we developed in [7] on a very particular case of sums of correlated lognormals. We will then explain how this particular problem is very similar to the more complex scenario we are considering here. We will then be able to match the statistics between the two methods in order to obtian the results in Section III.

Let us first consider a set of identically distributed equally correlated jointly lognormal random variables. Let $\vec{W} = [W_i]_{i=1}^N$ be a vector of N jointly Gaussian random variables, each with the same mean μ, same variance σ^2, and each pair with the same correlation coefficient ρ. Their correlation matrix can thus be written as

$$\mathbf{K}_{N\times N} = \sigma^2 \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{pmatrix}. \tag{8}$$

Let

$$Y_i = e^{W_i}. \tag{9}$$

We then say that $\vec{Y} = [Y_i]_{i=1}^N$ is a jointly lognormal vector, characterised by the same parameters μ, σ^2, ρ, $\mathbf{K}$.

Now consider

$$X = \sum_{i=1}^{N} Y_i. \tag{10}$$

We say that X follows a SIDEPCJLN distribution with parameters μ, σ, ρ, N. In [7] we proved the following theorem: Let X be defined as in (10) with parameters μ, $\sigma > 0$, $\rho > 0$, N. Then, as $N \to \infty$, the quantity X/N tends in distribution to a lognormal random variable. Using moment-matching [9] for the first and second moments, we calculated in [7] that the distribution of X can be approximated by a lognormal with parameters $\left(m_{\mathrm{X}}, s_{\mathrm{X}}^2\right)$:

$$\begin{aligned} m_{\mathrm{X}} &= \mu + \tfrac{3}{2}\ln N - \tfrac{1}{2}\ln\left(1 + (N-1)e^{(\rho-1)\sigma^2}\right), \\ s_{\mathrm{X}}^2 &= \sigma^2 - \ln N + \ln\left(1 + (N-1)e^{(\rho-1)\sigma^2}\right). \end{aligned} \tag{11}$$

For high N, these expressions simplify to

$$\begin{aligned} m_{\mathrm{X}} &\approx \mu + \ln N + \tfrac{1}{2}(1-\rho)\sigma^2, \\ s_{\mathrm{X}}^2 &\approx \rho\sigma^2, N \to \infty. \end{aligned} \tag{12}$$

On the other hand, consider the interference I_i produced by one interferer i. It is often assumed ([8], [12]) that I_i is approximately lognormal, however it is better to verify this assumption by simulation for particular cases. Now let us look at the marginal distibution of each I_i. Since no interferer is privileged against another since the positions $\vec{r}_i$ are all identically distributed, it follows that all I_i are interchangable, and thus they all have the same distribution. Let us also consider each pair $(I_i, I_j), i \neq j$. Again, no pair is privileged against any other pair, and all pariwise joint distibutions are the same. Now notice that in the SIDEPCJLN problem, this is also the case: the summands all have the same marginal lognormal distributions and the same pairwise joint distributions. The only substantial difference between these two scenarios, is that in the SIDEPCJLN case, all terms are *jointly* lognormal, whereas it is not evident if this is the case for all I_i's. Nevertheless, because of the strong similarity between the two models, we proceed to match the following statistics:

$$\begin{aligned} \mathbb{E}\{\ln I_i\} &= \mathbb{E}\{W_i\} = \mu, \\ \mathbb{E}\left\{\ln^2 I_i\right\} &= \mathbb{E}\left\{W_i^2\right\} = \sigma^2 + \mu^2, \\ \mathbb{E}\{\ln I_i \ln I_j\} &= \mathbb{E}\{W_i W_j\} = \rho\sigma^2 + \mu^2, i \neq j. \end{aligned} \tag{13}$$

Now the moments of the interference terms can be found:

$$\begin{aligned} \mathbb{E}\{\ln I_i\} &= \mathbb{E}\{S_i - \beta \ln r_i\} = -\beta G_1, \\ \mathbb{E}\left\{\ln^2 I_i\right\} &= \mathbb{E}\left\{S_i^2 + \beta^2 \ln^2 r_i - 2\beta S_i \ln r_i\right\} \\ &= \sigma_{\mathrm{s}}^2 + \beta^2 G_2, \\ \mathbb{E}\{\ln I_i \ln I_j\} &= \mathbb{E}\left\{S_i S_j + \beta^2 \ln r_i \ln r_j\right\} \\ -\beta\mathbb{E}\{S_j \ln r_i + S_i \ln r_j\} &= \sigma_{\mathrm{s}}^2 G_{\mathrm{cor}} + \beta^2 G_1^2, i \neq j, \end{aligned} \tag{14}$$

where

$$\begin{aligned} G_n &= \mathbb{E}\{\ln^n r_i\}, \\ G_{\mathrm{cor}} &= \frac{\mathbb{E}\{S_i S_j\}}{\sigma_{\mathrm{s}}^2}, i \neq j, \end{aligned} \tag{15}$$

and can be calculated by numerical integration (7).

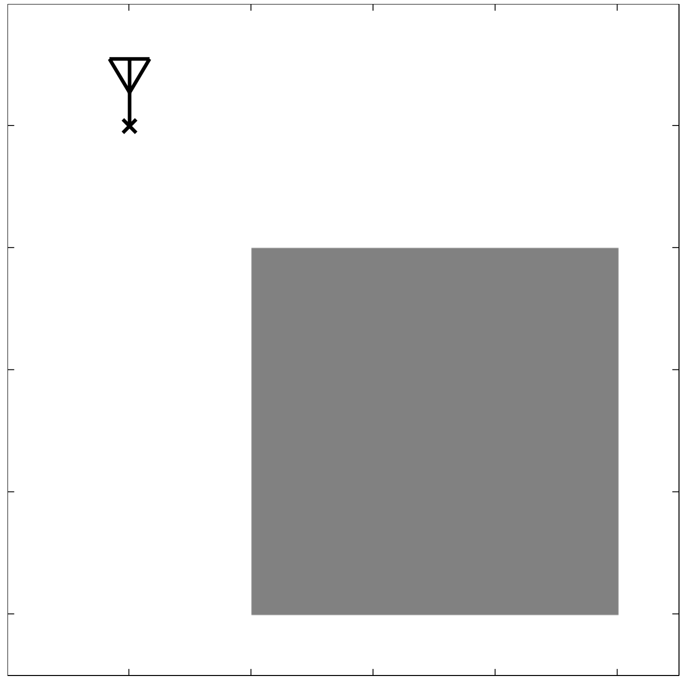

Fig. 1. Interferers' layout: the receiver is located at the base of the drawn antenna, while the interferers are located with a random uniform distribution within the gray square.

Equating (13) with (14) gives:

$$\begin{aligned} \mu &= -\beta G_1, \\ \sigma^2 &= \beta^2\left(G_2 - G_1^2\right) + \sigma_{\mathrm{s}}^2, \\ \rho &= \frac{\sigma_{\mathrm{s}}^2}{\sigma^2} G_{\mathrm{cor}}. \end{aligned} \tag{16}$$

We substitute these into (11) and (12) and, by equating $\left(m, s^2\right) = \left(m_{\mathrm{X}}, s_{\mathrm{X}}^2\right)$, we obtain (5) and (6) respectively. ■

In the next section, we will observe that this analysis does lead to a good prediction of the simulated distributions.

V. Simulations

We choose a simple layout for the interferers, a uniform distribution over a square region as illustrated in Fig. 1:

$$g(x,y) = \begin{cases} \frac{1}{9}, & (x,y) \in [1,4]^2, \\ 0, & (x,y) \notin [1,4]^2. \end{cases} \tag{17}$$

We may now generate the positions $\vec{r}_i$ of the interferers.

We also choose a shadowing correlation function: we use the model "1.0/0.0 R6" from [6], illustrated in Fig. 2. This is a fairly conservative correlation model in the sense that it gives rather low correlation coefficients compared to say the "1.0/0.4 R6" model [6]. Because we observed [7] (and it is intuitively confirmed) that convergence to the lognormal distribution is faster for higher ρ, we can safely assume that if our analysis holds for our conservative models, it will also hold for the

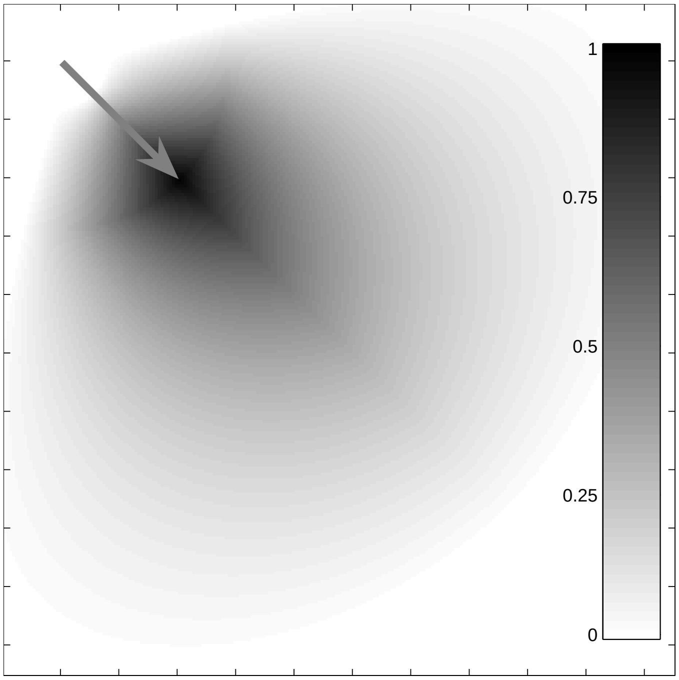

Fig. 2. Shadowing correlation function: the figure shows the correlation coefficient between two shadowed signals received at the base of the arrow. One transmitter is located at the tip of the arrow, while the other is located anywhere on the plane.

higher-correlated models. We define

$$\theta = |\angle(\vec{r}_i, \vec{r}_j)|$$

$$h_\Theta(\theta) = \begin{cases} 1 - \dfrac{3\theta}{\pi}, & 0 \le \theta < \dfrac{\pi}{3}, \\ 0, & \theta \ge \dfrac{\pi}{3}, \end{cases} \tag{18}$$

and

$$R = |\ln r_i - \ln r_j|$$

$$h_\mathrm{R}(R) = \begin{cases} 1 - \dfrac{R}{0.6 \ln 10}, & 0 \le R < 0.6 \ln 10, \\ 0, & R \ge 0.6 \ln 10. \end{cases} \tag{19}$$

Then the correlation coefficient between paths $\vec{r}_i$ and $\vec{r}_j$ is given by

$$h_{i,j} = h(\theta, R) = h_\Theta(\theta) h_\mathrm{R}(R). \tag{20}$$

We can now generate the shadowing correlation matrix $\mathbf{H}$.

Subsequently, we perform a Cholesky factorisation [6] of the matrix $\mathbf{H}$ and obtain samples of the total interference power I from (4) and (2). Note that because the Cholesky factorisation is the bottleneck in terms of simulation time, we do not regenerate the interferer positions at every trial. For example, for $N = 1000$ we performed 7,000,000 trials of I, but only generated the set of $\vec{r}_i$ and calculated $\mathbf{H}$ 700 times, and for each realisation of the interferers' positions we then generated 10,000 realisations of the shadowing values.

Now the numerical integrations (7) for this scenario give

$G_1 = 1.2607$,

$G_2 = 1.6566$,

$G_\mathrm{cor} = 0.5767$.

We use these in (5) and (6) to obtain the analytical lognormal approximations to the *cdf* of I. We plot both simulation and analytical curves in Figs. 3 and 4 on lognormal paper [13], such that all lognormal *cdf*'s appear as straight lines.

We first observe that the distribution of I_i (i.e., $N = 1$) is well approximated by a lognormal, which is a necessary first criterion for our method to work. Then, for $N = 10$, the lognormal approximation is no longer very good, as has been observed in [9], [14]–[17] for sums of small numbers (< 25) of correlated terms. What is interesting is that for $N = 100$, the approximation does not deteriorate (as one might extrapolate) but in fact improves, and is very good for $N = 1000$. We observe however for $N = 1000$ that the far upper tail diverges from the lognormal distribution. This has been observed and explained in [7], and its effect is expected to diminish with increasing N.

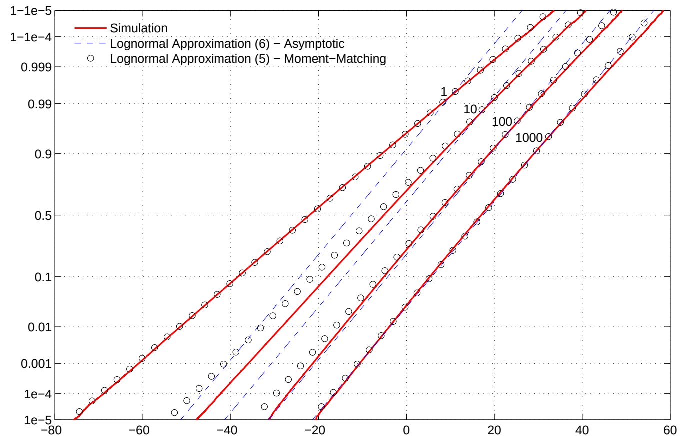

Fig. 3. Total interference power *cdf* on lognormal paper, $\beta = 4$, $\sigma_\mathrm{s} = 12\,\mathrm{dB} = 2.763$.

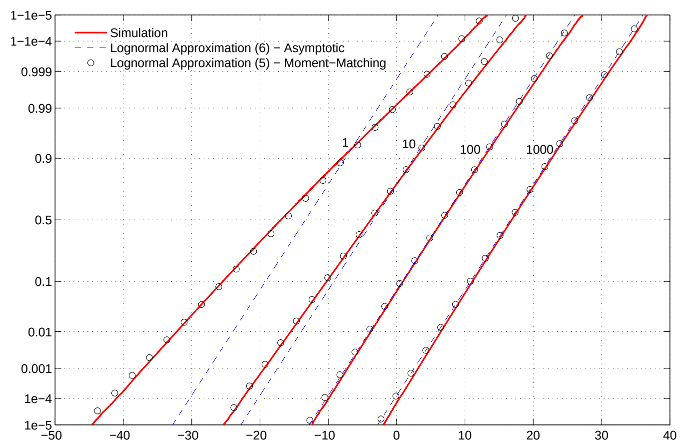

Fig. 4. Total interference power *cdf* on lognormal paper, $\beta = 3$, $\sigma_\mathrm{s} = 6\,\mathrm{dB} = 1.382$.

VI. Conclusion

In this work, we successfully reproduced simulation results for $N = 1000$ interferers using only simple numerical integration. The geometric coefficients G_1 and G_2 were computed almost instantaneously, $G_{\rm cor}$ took under a minute, and they are all independent of β, $\sigma_{\rm s}$ and N. On the other hand, the Monte Carlo simulations took roughly a full day per figure to perform, and were of course dependent on all the parameters. Moreover, if we were interested in the interference I produced by $N = 10,000$ interferers, it is clear from (6) that the approximate *cdf* of I would simply shift by 10 dB to the right, and the approximation would probably be even better, as suggested by the results in [7]. On the other hand, simulating the system for 10,000 or more interferers is computationally prohibitive on most of today's computers, notably because of the size of the matrix $\mathbf{H}$ and its required Cholesky factorisation, which has complexity $\mathcal{O}\left(N^3\right)$ [10]. It is a fortunate turn of events that the more computationally prohibitive the problem (with increasing N), the better our method works.

In order for our analysis to hold several conditions need to apply, most notably on the interferer location *pdf*. These have not yet been fully explored, but some of our preliminary simulations (not shown here) lead us to suggest a few tentative guidelines for the applicability of our method:

1) The bulk of the mass of the *pdf* g should not be too close to the receiver (origin). More precisely, the ratio of distances from the origin between the farthest and nearest point (in (17), this ratio is 4) must not be too large. A ratio of 10 is certainly too large for a uniform square distribution. If g is to close in this sense to the receiver, simulations show that I_i is not at all lognormal, and nor does I become lognormal for high N.
2) In [7] we have observed that the convergence of the SIDEPCJLN distribution to the lognormal is faster for higher ρ and lower σ. These are approximately equal to $G_{\rm cor}$ and $\sigma_{\rm s}$ respectively. Thus, for higher shadowing variance, or lower average correlation coefficient, it will take a higher N for our method to converge.
3) We do yet have any definite criteria on how large N has to be for any given $(G_{\rm cor}, \sigma_{\rm s})$ before the results are accurate enough, according to some metric.
4) If we take h as given in Fig. 2 to be a typical (or worst-case) correlation function, it becomes clear that as the angle at which the receiver sees the interferers (in (17), this angle is $\approx 62^\circ$) increases, the equivalent correlation coefficient ρ decreases. We already mentioned why this is detrimental to our method. However another problem arises: because it is not proven that the vector $[I_i]$ is *jointly* lognormal, it may be that our theorem [7] does not apply. Our guess is that for lower ρ, the jointness assumption becomes weaker. We have simulations that suggest that our method will not work for low (< 0.25) $G_{\rm cor}$, even for high N. Thus we recommend that the whole *pdf* g be located within a sector of not much more than 60°.
5) Although for simplicity we only used a uniform interferer *pdf* g, this is not at all a constraint. Our method should also apply to non-uniform, and also disjoint *pdf*'s, as long as the angle of arrival spread is not much greater than 60°, and the maximal ratio of distances is not much greater than 4. In particular, any weighted subset of the area in Fig. 1 fulfills these criteria.

These questions require further study, but suggest that our approach can be used to analyse at least a subset of the many realistic interference scenarios.

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