Asymptotic BER Analysis of Threshold Digital Relaying Schemes in Cooperative Wireless Systems

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***Abstract*— In uncoded cooperative wireless networks, error propagation due to the detection errors at the relay limits the performance of cooperative digital relaying. Threshold relaying is a simple and effective technique to mitigate error propagation without any reliance on the channel coding. This paper analyzes the asymptotic end-to-end (e2e) bit error rate (BER) of threshold digital relaying in a network with a single relay and links experiencing independent Rayleigh fading. It is shown that the optimal threshold that minimizes the e2e BER increases as $\log(\mathsf{SNR})$ as the average link signal-to-noise ratios are increased simultaneously. The resulting e2e BER decreases as $\log(\mathsf{SNR})/\mathsf{SNR}^2$, thereby achieving dual diversity.**

I. Introduction

In wireless networks, cooperative relaying creates independent paths from a source to its destination. The destination can combine signals from these independent paths using traditional combining methods such as Maximal Ratio Combining (MRC) and improve the e2e reliability. Several relaying protocols to achieve cooperative diversity are proposed in [1], [2]. According to the signal processing performed by the relay, cooperative relaying protocols can be classified into two categories: analog and digital. In analog relaying, the relay amplifies the signal without detection. In digital relaying, the relay first detects and then remodulates the signal. The focus of the present paper is digital relaying.

Consider a source-destination pair and a single digital relay. if the detection at the relay is error-free, the destination is provided with two diversity branches: one from the relay and the other from the source. By combining these branches, the error probability at the destination is reduced significantly. However, retransmitting erroneous symbols from the relay causes the post-combining SNR at the destination to be very small and a symbol error at the destination becomes very likely. This latter event is usually called *error propagation*. If the relay retransmits all the symbols regardless of the reliability of its detection, i.e., simple relaying, the e2e diversity order is only one [3]. Hence, error propagation limits the e2e performance of digital relaying.

There are different ways to mitigate error propagation. In systems with embedded error detection codes, the relay can transmit only if no errors are detected. In this case, the relay can also correct some of the symbol errors by decoding and re-encoding the received data block. However, some coding schemes such as LDPC and turbo coding increases both the relay processing and delay. Moreover, in heterogeneous networks including nodes with limited capabilities, such as sensors, it may be preferable to use relaying schemes that are transparent to coding. When error detection is either unavailable or inefficient, an alternate approach is to use the instantaneous SNR of the source-relay link as a measure of the reliability of the relay detection. If the source-relay SNR is larger than a threshold, the probability of an error at the relay is small and hence the relay retransmits the signal. Otherwise, the relay remains silent. These kind of schemes are called threshold digital relaying (TDR) [4]–[6].

In [5] Herhold et al. proposed a TDR scheme, where the threshold is selected jointly with the power fraction used by the relay and the source. They determine these parameters through numerical optimization. The BER of TDR schemes in a network with multi-antenna relays using a similar threshold function was studied by Adinoyi and Yanikomeroglu in [4]. In [6] the optimal threshold functions that minimize e2e BER have been derived analytically. It is shown that if the threshold is selected properly, TDR can improve the BER performance significantly compared to the simple relaying, where the relay always forwards the received data. In [7] Ponnaluri and Wilson considered a system with two parallel relays and equal gain combining at the destination. They showed that a threshold of the form $c\mathsf{SNR}^{\epsilon/2}$ achieves diversity order of $d = 3 - \epsilon$. With this threshold, despite the asymptotic diversity gain, the e2e BER performance does not improve significantly in the practical ranges of link SNRs, especially for low ϵ values.

Wang et al. proposed a new digital relaying scheme, called

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link adaptive relaying (LAR), that aims to reduce error propagation by adapting the relay transmit power to the link SNRs, rather than transmitting with full power all the time [8]. This idea is a generalization of TDR if the relay can control its power continuously and reduces to TDR if the relay can only perform on-off power control. In [8], the authors show that with their proposed power control scheme continuous version of LAR achieves full diversity while the on-off version of LAR has no diversity gain. In the present paper we prove that this result is not due to the limitation of on-off power control, but due to the specific threshold used in [8]. An optimal choice of threshold still achieves diversity order 2.

Another line of research for improving the e2e BER of digital cooperative relaying focused on the enhancements that can be applied at the destination [9], [10]. Chen and Laneman derived the maximum-likelihood (ML) receiver and a piecewise linear receiver approaching the performance of the ML, that considers the possibility of symbol errors at the source-relay link [9]. In particular, the destination receiver makes use of the average BER at each relay during the first hop, which is sent to the destination by the relay. The authors show that in digital relaying systems these receivers can achieve diversity order d of $(M+1)/2 \leq d \leq M/2+1$ for M even and $d=(M+1)/2$ for M odd, where $M-1$ is the number of relays. In [10], a novel combining scheme to be employed at the destination is proposed. This scheme, which is called Cooperative MRC (C-MRC), exploits the instantaneous BER at source-relay links. The C-MRC achieves full diversity in uncoded digital relaying systems.

The schemes proposed in [9] and [10] place the computing burden on the destination while keeping the relays relatively simple whereas TDR simplifies the destination processing by making the relay smarter. In TDR the destination performs MRC only (if the relay retransmits) and it does not require any additional channel state information. However, [9] and [10] both require source-relay SNR to be known at the destination - average SNR for [9] and instantaneous SNR for [10].

In this paper, we study the asymptotic e2e BER of the TDR in relation to the optimal threshold. After describing the system model in Section II, we summarize the prior results on the optimal threshold and the performance of the TDR in Section III. Section IV is dedicated to the analysis of the asymptotic BER of the optimal TDR in the high SNR regime. In this section we prove that the e2e BER of the optimal TDR decreases as $\log(\mathsf{SNR})/\mathsf{SNR}^2$. Hence, the diversity order of the optimal TDR is 2. In Section V, we verify our results and compare the performance of the optimal TDR to several similar schemes proposed in the literature. Section VI summarizes the findings of the paper.

II. System Model

This paper studies a simple network as shown in Fig 1, where a single relay R is used to assist the communication from a source node S to the destination node D. For clarity of exposition, it is assumed that both S and R use BPSK modulation. A two phase digital relaying protocol is considered. In the first phase S transmits a data block while R and D listens, and in the second phase R either retransmits the data or remains silent. If R chooses to remain silent, the second phase is idle. At the cost of increased complexity, R can send a short message to S about its decision and in the second phase S can retransmit the same data or continue with a new block of data. However, since we assume slow fading as explained below, the retransmission by S does not provide any diversity gain. If S transmits the next data block, we can potentially reduce the average bandwidth. As we will show in Section IV, this gain is insignificant at high SNR regime; the probability that R remains silent goes to zero as SNR increases, i.e., almost all the data blocks are transmitted twice.

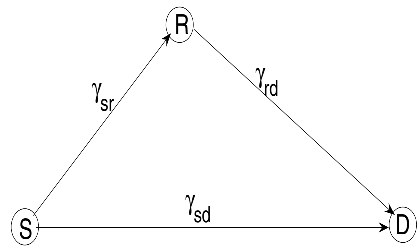

Fig. 1. The system model.

Our system does not rely on any kind of coding and can process the data symbol by symbol as well. For simplicity, we assume that all the links use BPSK modulation. All the links experience independent Rayleigh fading. The instantaneous received SNR per bit for a link from node i to node j is denoted by γ_{ij} and is given by $\gamma_{ij} = X_{ij}^2 \sigma_{ij}^2$, where X_{ij}^2 is an exponential random variable with unit mean and σ_{ij}^2 is the average SNR. We represent the average SNR as $\sigma_{ij}^2 = \lambda_{ij}\,\mathsf{SNR}$, where SNR is a reference signal-to-noise ratio, and λ_{ij} is a scaling factor with respect to the reference SNR representing non-identical distance dependent loss and shadowing for different links. Link SNRs vary in time following independent block fading: γ_{ij} is assumed to be constant for two blocks, precluding retransmit diversity, and a new independent realization is used for the next two blocks. The scalars X_{ij}, and thus γ_{ij}, are independent from link to link.

The instantaneous received SNR per bit for S-R, R-D, and S-D links are denoted by γ_{sr}, γ_{rd}, and γ_{sd}, respectively. Their average values are denoted by σ_{sr}^2, σ_{rd}^2, and σ_{sd}^2. An error event in the point-to-point link between node i and node j is represented by $\mathcal{E}_{ij}$. The error event that occurs after the destination combines the incorrectly regenerated relay signal and the source signal is referred to as *error propagation* and is denoted by $\mathcal{E}_{prop}$. We use the term *cooperative error* for the event that an error occurs after the destination combines the correctly regenerated relay signal and the source signal, and denote this event by $\mathcal{E}_{coop}$.

We assume that R has $\{\gamma_{sr}, \sigma_{rd}^2, \sigma_{sd}^2\}$ available in order to make relaying decisions. The relaying decisions are made by comparing the instantaneous S-R SNR γ_{sr} to a threshold γ_t, i.e., R transmits only when $\gamma_{sr} > \gamma_t$. There are several ways of calculating the threshold γ_t considered in the literature. Our work focuses on the performance with the optimal threshold value that minimizes e2e BER, which has been derived in [6]. We denote this threshold by γ_t^* and we refer to the TDR using γ_t^* as the optimal TDR. We note that the optimal threshold can also be calculated for the case where the relay has the instantaneous R-D and S-D SNRs, γ_{rd}, γ_{sd}. This additional information can improve e2e BER. However, the diversity order cannot be improved any further as the current model achieves full diversity.

III. Performance of TDR and the Optimal Threshold

A. The e2e BER of TDR as a function of γ_t

The average e2e BER for a TDR protocol using threshold γ_t can be expressed using the law of total probability:

$$\text{BER}_{e2e}^{TDR}(\gamma_t) = \mathbb{P}\{\gamma_{sr} > \gamma_t\}\Big[\mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t\}\mathbb{P}\{\mathcal{E}_{prop}\} + (1 - \mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t\})\mathbb{P}\{\mathcal{E}_{coop}\}\Big] + \mathbb{P}\{\gamma_{sr} \leq \gamma_t\}\mathbb{P}\{\mathcal{E}_{sd}\}. \quad (1)$$

Next, we derive the terms in (1) as functions of average link SNRs. If the SNR of the S-R link γ_{sr} is below γ_t, the protocol falls back to direct transmission and has error probability equal to [11]

$$\mathbb{P}\{\mathcal{E}_{sd}\} = \frac{1}{2}\left(1 - \sqrt{\frac{\sigma_{sd}^2}{1 + \sigma_{sd}^2}}\right). \quad (2)$$

Since γ_{sr} is an exponential random variable with mean $\sigma_{sr}^2 = \lambda_{sr}\mathsf{SNR}$, the probability that $\gamma_{sr} \leq \gamma_t$ is equal to

$$\mathbb{P}\{\gamma_{sr} \leq \gamma_t\} = 1 - \exp(-\gamma_t/\sigma_{sr}^2). \quad (3)$$

If $\gamma_{sr} > \gamma_t$, the probability of bit error at the S-R link decreases, but it remains nonzero regardless of the value of γ_t. The probability of bit error at the S-R link given that $\gamma_{sr} > \gamma_t$ is equal to

$$\mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t\} = \int \frac{1}{2}\operatorname{erfc}(\sqrt{\gamma_{sr}})\, p_{\gamma_{sr}|\gamma_{sr}>\gamma_t}(\gamma_{sr})\mathrm{d}\gamma_{sr}, \quad (4)$$

where erfc denotes the complementary error function. The term $p_{\gamma_{sr}|\gamma_{sr}>\gamma_t}$ is the conditional pdf of γ_{sr} and is given by

$$p_{\gamma_{sr}|\gamma_{sr}>\gamma_t}(\gamma_{sr}) = \exp(\gamma_t/\sigma_{sr}^2)\frac{1}{\sigma_{sr}^2}\exp(-\gamma_{sr}/\sigma_{sr}^2),$$

where $\gamma_{sr} > \gamma_t$. Hence, the term $\mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t\}$ can be represented by

$$\mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t\} = \frac{\exp(\gamma_t/\sigma_{sr}^2)}{2\sigma_{sr}^2} \times \int_{\gamma_t}^{\infty} \operatorname{erfc}(\sqrt{\gamma_{sr}})\, \exp(-\gamma_{sr}/\sigma_{sr}^2)\, \mathrm{d}\gamma_{sr}, \quad (5)$$

which can be calculated using integration by part [5]:

$$\mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t\} = \frac{1}{2}\operatorname{erfc}(\sqrt{\gamma_t}) - \frac{1}{2}e^{\gamma_t/\sigma_{sr}^2}\sqrt{\frac{\sigma_{sr}^2}{1 + \sigma_{sr}^2}}\ \operatorname{erfc}\left(\sqrt{\gamma_t(1 + \frac{1}{\sigma_{sr}^2})}\right). \quad (6)$$

When $\gamma_{sr} > \gamma_t$ and there is no bit error on the S-R link, then the probability of bit error at D after MRC is equal to [11, pp. 846-847]

$$\mathbb{P}\{\mathcal{E}_{coop}\} = \begin{cases} \frac{1}{2}\left(1 - \sqrt{\frac{\sigma_{rd}^2}{1+\sigma_{rd}^2}}\right)^2\left(1 + \frac{1}{2}\sqrt{\frac{\sigma_{rd}^2}{1+\sigma_{rd}^2}}\right), & \sigma_{rd}^2 = \sigma_{sd}^2; \\ \frac{1}{2}\left[1 - \frac{1}{\sigma_{sd}^2-\sigma_{rd}^2}\left(\sigma_{sd}^2\sqrt{\frac{\sigma_{sd}^2}{1+\sigma_{sd}^2}} - \sigma_{rd}^2\sqrt{\frac{\sigma_{rd}^2}{1+\sigma_{rd}^2}}\right)\right], & \text{else.} \end{cases} \quad (7)$$

At high SNR ($\sigma_{rd}^2, \sigma_{sd}^2 >> 1$), the probability of error propagation can be approximated by the probability that $\gamma_{rd} > \gamma_{sd}$ [6]:

$$\begin{aligned} \mathbb{P}\{\mathcal{E}_{prop}\} &\approx \mathbb{P}\{\gamma_{rd} > \gamma_{sd}\} \\ &= \int_0^{\infty} p_{\gamma_{sd}}(\gamma_{sd}) \int_{\gamma_{sd}}^{\infty} p_{\gamma_{rd}}(\gamma_{rd})\mathrm{d}\gamma_{rd}\mathrm{d}\gamma_{sd} \\ &= \frac{\sigma_{rd}^2}{\sigma_{rd}^2 + \sigma_{sd}^2}. \end{aligned} \quad (8)$$

Hence, the average e2e BER for a given threshold value can be calculated analytically by substituting (2), (3), and (6)-(8) in equation (1).

B. The optimal threshold γ_t^*

The threshold value γ_t^* that minimizes the average e2e BER given in (1) is derived in [6]:

$$\gamma_t^* = \begin{cases} \left(\operatorname{erfc}^{-1}(2\delta)\right)^2, & \delta < 0.5; \\ 0, & \text{otherwise,} \end{cases} \quad (9)$$

where δ is equal to

$$\delta = \frac{\mathbb{P}\{\mathcal{E}_{sd}\} - \mathbb{P}\{\mathcal{E}_{coop}\}}{\mathbb{P}\{\mathcal{E}_{prop}\} - \mathbb{P}\{\mathcal{E}_{coop}\}}. \quad (10)$$

By substituting (2), (7), and (8) in (10), δ is obtained as

$$\delta(\sigma_{rd}^2, \sigma_{sd}^2) \approx \frac{\frac{1}{2}\left[\frac{1}{\sigma_{sd}^2-\sigma_{rd}^2}\left(\sigma_{sd}^2\sqrt{\frac{\sigma_{sd}^2}{1+\sigma_{sd}^2}} - \sigma_{rd}^2\sqrt{\frac{\sigma_{rd}^2}{1+\sigma_{rd}^2}}\right) - \sqrt{\frac{\sigma_{sd}^2}{1+\sigma_{sd}^2}}\right]}{\frac{\sigma_{rd}^2}{\sigma_{rd}^2+\sigma_{sd}^2} - \frac{1}{2}\left[1 - \frac{1}{\sigma_{sd}^2-\sigma_{rd}^2}\left(\sigma_{sd}^2\sqrt{\frac{\sigma_{sd}^2}{1+\sigma_{sd}^2}} - \sigma_{rd}^2\sqrt{\frac{\sigma_{rd}^2}{1+\sigma_{rd}^2}}\right)\right]}.$$

IV. Asymptotic Performance of Optimal TDR

In order to study the high SNR behavior of e2e BER, following an approach similar to the one in [12], we fix the parameters λ_{sr}, λ_{rd}, and λ_{sd}, and analyze the e2e BER as SNR goes to ∞. Diversity order is a useful measure in quantifying the diversity benefit of any scheme in high SNR regime [12]. We adopt the following definition of diversity order given in [12][1]: $d = -\lim_{\mathsf{SNR}\to\infty} \frac{\log(\mathrm{BER})}{\log(\mathsf{SNR})}$.

We also use the following asymptotic notation:

Definition 1: Let $f(x)$ and $g(x)$ be two positive functions defined on real numbers. We say $f(x) = O(g(x))$, if $\limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty$.

Definition 2: Two functions $f(x)$ and $g(x)$ are called asymptotically equivalent, $f(x) \sim g(x)$, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

In rest of this section, we first show that the optimal threshold function γ_t^* given in (9) increases logarithmically with SNR (Lemma 1). Then, using this result, we prove that if the relay sets its threshold to γ_t^*, the probability that it remains silent decreases as $\log(\mathsf{SNR})/\mathsf{SNR}$ and the probability that the relay has a detection error decreases at least as fast as $1/\mathsf{SNR}^2$ (Lemmas 2 and 3). Finally, we use all these results to show that the asymptotic e2e BER of the optimal TDR decreases as $\log(\mathsf{SNR})/\mathsf{SNR}^2$ and the optimal TDR achieves diversity order 2 (Proposition 1).

We note that the BER of TDR is larger by a factor of $\log(\mathsf{SNR})$ than the BER of a traditional diversity system, where BER decreases as $1/\mathsf{SNR}^2$ at large SNR. Since the diversity order represents the relation of the BER and SNR up to an exponential factor, the diversity order of TDR is still 2. Hence, we conclude that it is possible to achieve maximum diversity order in a single relay network using threshold relaying with the optimal threshold selection.

A. Asymptotic behavior of γ_t^*, $\mathbb{P}\{\gamma_{sr} \le \gamma_t^*\}$, and $\mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t^*\}$

The results on the asymptotic behavior of γ_t^*, $\mathbb{P}\{\gamma_{sr} \le \gamma_t^*\}$, and $\mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t^*\}$ are given in Lemmas 1, 2, and 3, respectively. See the Appendix for the proofs.

Lemma 1 (Asymptotic behavior of γ_t^*): The optimal threshold γ_t^*, given in (9), is upper and lower-bounded by two log functions for sufficiently large SNR. That is, there exist $c_1, c_2 > 0$ such that

$$c_1 \log(\mathsf{SNR}) < \gamma_t^*(\mathsf{SNR}) < c_2 \log(\mathsf{SNR}), \quad \text{as } \mathsf{SNR} \to \infty. \tag{11}$$

Lemma 2 (Asymptotic behavior of $\mathbb{P}\{\gamma_{sr} \le \gamma_t^*\}$): For sufficiently large SNR, $\mathbb{P}\{\gamma_{sr} \le \gamma_t^*\}$ can be upper and lower bounded as follows. There exists $c_1', c_2' > 0$ such that

$$c_1' \frac{\log(\mathsf{SNR})}{\mathsf{SNR}} < \mathbb{P}\{\gamma_{sr} < \gamma_t^*\} < c_2' \frac{\log(\mathsf{SNR})}{\mathsf{SNR}}, \text{ as } \mathsf{SNR} \to \infty. \tag{12}$$

Lemma 3 (Asymptotic behavior of $\mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t^*\}$): If the relay uses the optimal threshold γ_t^*, then $\mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t^*\}$ can be upper bounded as follows. There exists a $c > 0$ such that

$$\mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t^*\} < c\,\frac{1}{\mathsf{SNR}^2}, \quad \text{as } \mathsf{SNR} \to \infty. \tag{13}$$

[1]Throughout this paper all the logarithms are in the natural base.

B. Asymptotic e2e BER and diversity order of the optimal TDR

Proposition 1: The e2e BER of TDR, given in (1), using the threshold γ_t^* satisfies $\mathrm{BER}_{e2e}^{TDR}(\gamma_t^*) = O(\log(\mathsf{SNR})/\mathsf{SNR}^2)$ and hence achieves the maximum diversity order of 2.

Proof: Let us denote the first term of (1) as P_1 and the second term as P_2.

$$P_1 = \mathbb{P}\{\gamma_{sr} > \gamma_t^*\}\Big[\mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t^*\}\,\mathbb{P}\{\mathcal{E}_{prop}\} + \mathbb{P}\{\mathcal{E}_{coop}\} - \mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t^*\}\,\mathbb{P}\{\mathcal{E}_{coop}\}\Big]$$
$$P_2 = \mathbb{P}\{\gamma_{sr} \le \gamma_t^*\}\,\mathbb{P}\{\mathcal{E}_{sd}\}.$$

In P_1, the term $\mathbb{P}\{\gamma_{sr} > \gamma_t^*\} = \exp(-\gamma_t/(\lambda_{sr}\mathsf{SNR}))$ goes to 1 as $\mathsf{SNR} \to \infty$. Since $\mathbb{P}\{\mathcal{E}_{prop}\} \sim \frac{\lambda_{rd}}{\lambda_{sd}+\lambda_{rd}}$ and $\mathbb{P}\{\mathcal{E}_{coop}\} \sim \frac{3}{16\lambda_{rd}\lambda_{sd}}\frac{1}{\mathsf{SNR}^2}$, the decay rate of P_1 is equal to the minimum of the rates of $\mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t^*\}$ and $\mathbb{P}\{\mathcal{E}_{coop}\}$. From Lemma 3, $\mathbb{P}\{\mathcal{E}_{sr}|\gamma_{sr} > \gamma_t^*\} = O(1/\mathsf{SNR}^2)$. Hence, $P_1 = O(1/\mathsf{SNR}^2)$.

Since $\mathbb{P}\{\mathcal{E}_{sd}\} \sim \frac{1}{4\lambda_{sd}}\frac{1}{\mathsf{SNR}}$ and $\mathbb{P}\{\gamma_{sr} \le \gamma_t^*\} = O(\log(\mathsf{SNR})/\mathsf{SNR})$ (from Lemma 2), $P_2 = O(\log(\mathsf{SNR})/\mathsf{SNR}^2)$. The e2e BER of the optimal TDR satisfies

$$\mathrm{BER}_{e2e}^{TDR}(\gamma_t^*) = P_1 + P_2 = O(1/\mathsf{SNR}^2) + O(\log(\mathsf{SNR})/\mathsf{SNR}^2) = O(\log(\mathsf{SNR})/\mathsf{SNR}^2). \tag{14}$$

Thus, $\mathrm{BER}_{e2e}^{TDR}(\gamma_t^*)$ is limited by the second term P_2, which corresponds to the event that the SNR of the S-R link is below threshold.

The diversity order of the optimal TDR is equal to

$$d^{TDR} = -\lim_{\mathsf{SNR}\to\infty} \frac{\log(\log(\mathsf{SNR})/\mathsf{SNR}^2)}{\log(\mathsf{SNR})} = -\lim_{\mathsf{SNR}\to\infty} \frac{\log(\log(\mathsf{SNR}))}{\log(\mathsf{SNR}^2)} + \lim_{\mathsf{SNR}\to\infty} \frac{\log(\mathsf{SNR}^2)}{\log(\mathsf{SNR})} = 2,$$

since the term $\frac{\log(\log(\mathsf{SNR}))}{\log(\mathsf{SNR})} \to 0$ as $\mathsf{SNR} \to \infty$. ■

V. Results

In this section, we compare the average BER of the optimal TDR to several schemes which are described below. As the two schemes that are simpler than TDR, we consider direct transmission and *simple relaying*. In simple relaying the relay always retransmits in the second phase. The average BER of simple relaying is equal to

$$\mathrm{BER}_{e2e}^{simple} = \mathbb{P}\{\mathcal{E}_{sr}\}\mathbb{P}\{\mathcal{E}_{prop}\} + (1 - \mathbb{P}\{\mathcal{E}_{sr}\})\mathbb{P}\{\mathcal{E}_{coop}\},$$

which can be calculated analytically by substituting (2), (7), and (8). The *genie-aided relaying* and *receive diversity* schemes are also considered as performance upper bounds. The genie-aided relaying is based on the hypothetical assumption that the relay has perfect error detection for each symbol. Hence, in phase 2 the relay retransmits only those symbols received correctly in phase 1. The e2e BER of the genie-aided digital relaying is equal to

$$\mathrm{BER}_{e2e}^{genie} = \mathbb{P}\{\mathcal{E}_{sr}\}\mathbb{P}\{\mathcal{E}_{sd}\} + (1 - \mathbb{P}\{\mathcal{E}_{sr}\})\mathbb{P}\{\mathcal{E}_{coop}\}, \tag{15}$$

which can be calculated by substituting (2) and (7) in (15). The second upper bound, receive diversity, is obtained by assuming

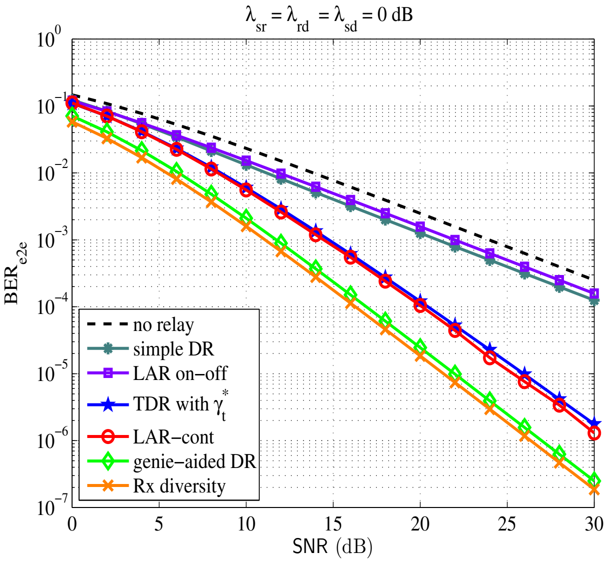

Fig. 2. The e2e BERs for different schemes as a function of SNR in a symmetric network, where $\lambda_{sr} = \lambda_{rd} = \lambda_{sd} = 0$ dB.

that the S-R link is error-free, i.e., $\text{BER}_{e2e}^{Rx2} = \mathbb{P}\{\mathcal{E}_{coop}\}$, where $\mathbb{P}\{\mathcal{E}_{coop}\}$ is given in (7).

Finally, we compare the BER of the optimal TDR to other similar schemes proposed in the literature. The Link Adaptive Relaying (LAR) proposed in [8] aims to reduce error propagation by adapting the relay transmit power to the link SNRs rather than transmitting with full power P_{full} all the time. If R is able to adapt its transmit power continuously, [8] proposes a scheme in which R transmits with $\alpha \times P_{full}$, where the scaling factor $\alpha = \frac{\min(\gamma_{sr}, \sigma_{rd}^2)}{\sigma_{rd}^2}$. We call this scheme as *LAR-cont*. If R can perform only on-off power adaptation, then it can be seen that LAR simplifies to TDR with threshold function $\gamma_{t,LAR} = \sigma_{rd}^2$. We call this scheme as *LAR-on/off*. For all threshold based relaying schemes, the e2e BER can be calculated analytically by plugging in the appropriate threshold value as γ_t in (1). We resort to Monte Carlo simulations to obtain the e2e BER of *LAR-cont* only.

In Fig. 2 we plot the average BER as a function of SNR in a symmetric network where $\lambda_{sr} = \lambda_{sd} = \lambda_{rd} = 0$ dB. The on-off version of LAR has poor performance; its BER is larger than even the BER of the simple relaying. Since $\gamma_{t,LAR}$ increases linearly with SNR, from (20) we observe that $\mathbb{P}\{\gamma_{sr} \leq \gamma_{t,LAR}\}$ will be a constant independent of SNR. Then, in the e2e BER expression given in (1) the second term, $\mathbb{P}\{\gamma_{sr} \leq \gamma_{t,LAR}\} \times \mathbb{P}\{\mathcal{E}_{sd}\}$, decreases only as fast as the BER of the S-D link. Hence, the diversity order of on-off LAR is 1, which has also been reported in [8]. This argument applies to any TDR scheme that uses a threshold increasing linearly with the SNR[2]. The continuous version of LAR, which is shown to achieve full diversity in [8], performs better than TDR by a small margin. We observed that this result depends on the average link SNRs; in some non-symmetrical scenarios TDR outperforms LAR-cont, but the performance gap is again very small.

In Fig. 2 the performance of genie-aided relaying is close to that of receive diversity. This shows that the main problem that limits the performance is the inability of R to detect errors, rather than the occurrence of the errors in the S-R link. Also, although they all have the same diversity order, there is a considerable gap between the performances of the optimal TDR, and the genie-aided relaying and receive diversity. This result suggests that there might be room for improvement by incorporating TDR with smarter combining techniques at the destination such as those proposed in [9] and [10].

[2]More discussion on other thresholds that do achieve diversity can be found in [13].

VI. Conclusions

In this paper, we studied the asymptotic BER of threshold digital relaying in cooperative wireless networks. We showed that in a network with a single relay and independent Rayleigh fading links, in order to minimize e2e BER, the threshold used by the relay should be increased logarithmically with the average link SNR. It is proven that with increasing link SNR, using this optimal threshold the e2e BER of the threshold digital relaying decreases as $\log(\text{SNR})/\text{SNR}^2$, thereby achieving dual diversity.

Appendix

A. *Proof of Lemma 1 – Asymptotic behavior of γ_t^**

It can be easily verified that $\lim_{\text{SNR}\to\infty} \frac{\delta(\text{SNR})}{1/\text{SNR}} = \frac{1}{4}\frac{\lambda_{sd}+\lambda_{rd}}{\lambda_{sd}\lambda_{rd}}$. Hence, $\delta(\text{SNR})$ is asymptotically equivalent to

$$\delta(\text{SNR}) \sim \frac{1}{4}\frac{\lambda_{sd}+\lambda_{rd}}{\lambda_{sd}\lambda_{rd}}\frac{1}{\text{SNR}}. \tag{16}$$

For large SNR, $\delta(\text{SNR}) < 1/2$. Thus, we ignore the second case in (9) and assume that $\gamma_t^*(\text{SNR}) = \left(\text{erfc}^{-1}(2\delta(\text{SNR}))\right)^2$.

We make use of the following inequality given in [14]:

$$\sqrt{1-e^{-z^2}} < |\text{erf}(z)| < \sqrt{1-e^{-2z^2}}.$$

By replacing $\text{erfc}(z) = 1 - \text{erf}(z)$, the threshold value is equal to $\gamma_t^*(\text{SNR}) = \left(\text{erf}^{-1}(1-2\delta(\text{SNR}))\right)^2$. Since $\text{erf}(z) \geq 0$ for all $z \geq 0$, and $\sqrt{\gamma_t^*} \geq 0$, we can write

$$\sqrt{1-e^{-\gamma_t^*}} < \text{erf}(\sqrt{\gamma_t^*}) < \sqrt{1-e^{-2\gamma_t^*}}. \tag{17}$$

Substituting $\text{erf}(\sqrt{\gamma_t^*}) = 1 - 2\delta$ from (9), we obtain

$$\frac{1}{2}\log\left(\frac{1}{4\delta(1-\delta)}\right) < \gamma_t^* < \log\left(\frac{1}{4\delta(1-\delta)}\right). \tag{18}$$

It can be easily verified that

$$\lim_{\text{SNR}\to\infty} \log\left(\frac{1}{4\delta(1-\delta)}\right) / \log(\text{SNR}) = 1. \tag{19}$$

Thus, there exists constants $b_1, b_2 > 0$ such that

$$b_1 \log(\text{SNR}) < \log\left(\frac{1}{4\delta(1-\delta)}\right) < b_2 \log(\text{SNR}),$$

and (11) holds for $c_1 = b_1/2$ and $c_2 = b_2$, which concludes the proof. ■

B. Proof of Lemma 2 – Asymptotic behavior of $\mathbb{P}\{\gamma_{sr} \le \gamma_t^*\}$

Since γ_{sr} has mean $\sigma_{sr}^2 = \lambda_{sr}\mathsf{SNR}$, the probability that $\gamma_{sr} \le \gamma_t$, and hence relay remains silent, is equal to

$$\begin{aligned}\mathbb{P}\{\gamma_{sr} \le \gamma_t\} &= 1 - \exp\left(-\frac{1}{\lambda_{sr}}\frac{\gamma_t(\mathsf{SNR})}{\mathsf{SNR}}\right) \\ &= 1 - (\exp(-\gamma_t(\mathsf{SNR})))^{\frac{1}{\lambda_{sr}\mathsf{SNR}}}. \end{aligned} \tag{20}$$

By substituting the bounds derived in Lemma 1 in (20), we obtain

$$\begin{aligned} 1 - \left(e^{-c_2 \log(\mathsf{SNR})}\right)^{\frac{1}{\lambda_{sr}\mathsf{SNR}}} &< \mathbb{P}\{\gamma_{sr} \le \gamma_t^*\} \\ &< 1 - \left(e^{-c_1 \log(\mathsf{SNR})}\right)^{\frac{1}{\lambda_{sr}\mathsf{SNR}}} \\ \Rightarrow 1 - \left(\frac{1}{\mathsf{SNR}}\right)^{\frac{c_2}{\lambda_{sr}\mathsf{SNR}}} &< \mathbb{P}\{\gamma_{sr} \le \gamma_t^*\} \\ &< 1 - \left(\frac{1}{\mathsf{SNR}}\right)^{\frac{c_1}{\lambda_{sr}\mathsf{SNR}}}. \end{aligned}$$

We note that

$$\lim_{\mathsf{SNR}\to\infty} \frac{1 - \left(\frac{1}{\mathsf{SNR}}\right)^{\frac{c}{\lambda_{sr}\mathsf{SNR}}}}{\log(\mathsf{SNR})/\mathsf{SNR}} = \frac{c}{\lambda_{sr}},$$

and hence,

$$1 - \left(\frac{1}{\mathsf{SNR}}\right)^{\frac{c}{\lambda_{sr}\mathsf{SNR}}} \sim \frac{c}{\lambda_{sr}}\frac{\log(\mathsf{SNR})}{\mathsf{SNR}}. \tag{21}$$

Then, there exist positive constants b_1' and b_2' such that

$$\frac{b_1' c_1}{\lambda_{sr}}\frac{\log(\mathsf{SNR})}{\mathsf{SNR}} < \mathbb{P}\{\gamma_{sr} \le \gamma_t^*\} < \frac{b_2' c_2}{\lambda_{sr}}\frac{\log(\mathsf{SNR})}{\mathsf{SNR}},$$

and (12) is satisfied for $c_1' = \frac{b_1' c_1}{\lambda_{sr}}$ and $c_2' = \frac{b_2' c_2}{\lambda_{sr}}$. ■

C. Proof of Lemma 3 – Asymptotic behavior of $\mathbb{P}\{\mathcal{E}_{sr} | \gamma_{sr} > \gamma_t^*\}$

Using Craig's formula given in [15], erfc can be represented as

$$\mathrm{erfc}(z) = \frac{2}{\pi}\int_0^{\pi/2} \exp(-z^2/\sin^2(\theta))\mathrm{d}\theta.$$

Substituting this alternate representation in (4) results in

$$\begin{aligned} &\mathbb{P}\{\mathcal{E}_{sr} | \gamma_{sr} > \gamma_t\} \\ &= \frac{\exp(\gamma_t/\sigma_{sr}^2)}{2\sigma_{sr}^2}\int_{\gamma_t}^{\infty} \mathrm{erfc}(\sqrt{\gamma_{sr}})\exp(-\gamma_{sr}/\sigma_{sr}^2)\,\mathrm{d}\gamma_{sr} \qquad (22) \\ &= \frac{\exp(\gamma_t/\sigma_{sr}^2)}{\pi\sigma_{sr}^2}\int_{\gamma_t}^{\infty} \exp\left(-\frac{\gamma_{sr}}{\sigma_{sr}^2}\right) \\ &\qquad \times \int_0^{\pi/2} \exp\left(-\frac{\gamma_{sr}}{\sin^2\theta}\right)\mathrm{d}\theta\,\mathrm{d}\gamma_{sr} \\ &= \frac{\exp(\gamma_t/\sigma_{sr}^2)}{\pi\sigma_{sr}^2}\int_0^{\pi/2}\int_{\gamma_t}^{\infty} \exp\left(-\gamma_{sr}\left(\frac{1}{\sin^2\theta} + \frac{1}{\sigma_{sr}^2}\right)\right) \\ &\qquad \times \mathrm{d}\gamma_{sr}\,\mathrm{d}\theta \\ &= \frac{1}{\pi}\int_0^{\pi/2} \frac{\sin^2\theta}{\sigma_{sr}^2 + \sin^2\theta}\exp(-\gamma_t/\sin^2\theta)\mathrm{d}\theta. \qquad (23) \end{aligned}$$

Since $\frac{\sin^2(\theta)}{\sigma_{sr}^2 + \sin^2(\theta)} < \frac{1}{\sigma_{sr}^2}$ for any $\sigma_{sr}^2 > 0$, $\mathbb{P}\{\mathcal{E}_{sr} | \gamma_{sr} > \gamma_t\}$ is upper bounded by

$$\begin{aligned} \mathbb{P}\{\mathcal{E}_{sr} | \gamma_{sr} > \gamma_t\} &< \frac{1}{\pi}\int_0^{\pi/2} \frac{1}{\sigma_{sr}^2}\exp(-\gamma_t/\sin^2\theta)\mathrm{d}\theta \\ &= \frac{1}{2\sigma_{sr}^2}\mathrm{erfc}(\sqrt{\gamma_t}). \end{aligned} \tag{24}$$

By substituting $\gamma_t = \gamma_t^*$, $\sigma_{sr}^2 = \lambda_{sr}\mathsf{SNR}$, and $\mathrm{erfc}(\sqrt{\gamma_t^*}) = 2\delta$ from (9), (24) is simplified to

$$\mathbb{P}\{\mathcal{E}_{sr} | \gamma_{sr} > \gamma_t^*\} < \frac{1}{\lambda_{sr}}\frac{\delta(\mathsf{SNR})}{\mathsf{SNR}}. \tag{25}$$

Since $\delta(\mathsf{SNR}) \sim \frac{1}{4}\frac{\lambda_{sd}+\lambda_{rd}}{\lambda_{sd}\lambda_{rd}}\frac{1}{\mathsf{SNR}}$, there exists a constant $c'' > 0$ such that $\delta(\mathsf{SNR}) < c''/\mathsf{SNR}$ for sufficiently large SNR. Hence, using (25) we obtain

$$\mathbb{P}\{\mathcal{E}_{sr} | \gamma_{sr} > \gamma_t\} < \frac{\delta(\mathsf{SNR})}{\lambda_{sr}\mathsf{SNR}} < \frac{c''}{\lambda_{sr}}\frac{1}{\mathsf{SNR}^2}, \tag{26}$$

and we conclude that (13) holds for $c = c''/\lambda_{sr}$. ■

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