# On the Statistics of the Sum of Correlated Generalized-*K* RVs

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# Presentation Outline

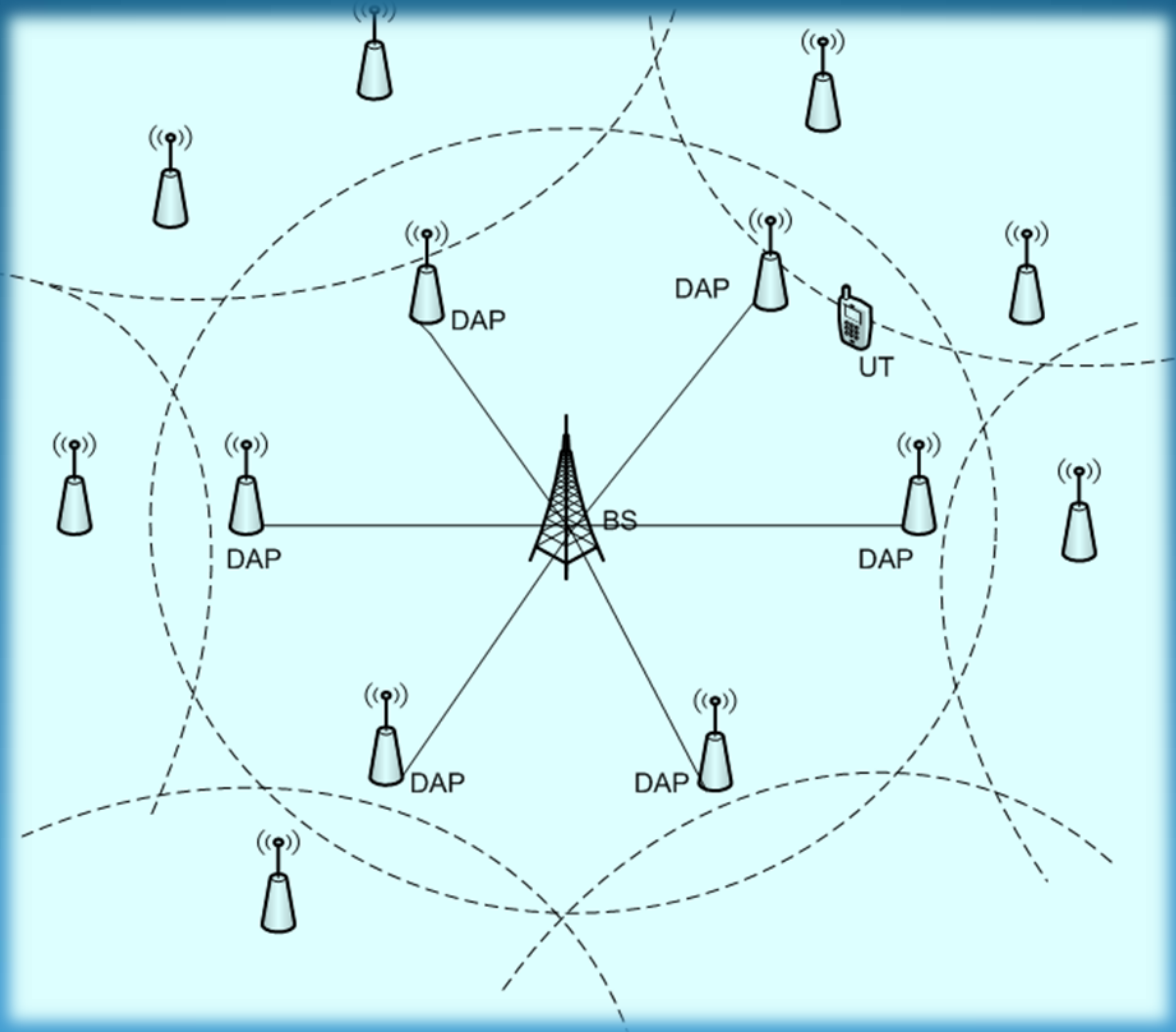
- Introduction
- Generalized-*K* Composite Fading Model
- Related Work on the Statistics of Correlated Generalized-*K* Random Variables (RVs)
- Amount of Fading (AF) Expressions
- Results
- Conclusions

# Introduction

- Composite channel models take place in wireless communications (multi-path fading plus shadowing), radar cross section scattering of targets, reverberation in sonar, etc.
- Modeling such a phenomenon plays an important role in the design and performance analysis for such channels.
- In wireless communications, emerging systems such as coordinated multipoint transmission and reception (CoMP) systems (network MIMO) including distributed antenna systems, and cooperative relay networks require such modeling.

- Shadowing correlations are typical in such wireless geographically distributed systems.

# Ex. A multi-cell DAS



# Models of Composite Fading

| Multipath | Shadowing | Composite | Complexity | Appeared |
|---|---|---|---|---|
| Rayleigh | lognormal | Suzuki | No closed-form expression | Suzuki (1979) |
| Nakagami | lognormal | Gamma-lognormal | No-closed form expression | Lewinsky (1983) Stuber (2001) |
| Rayleigh | Gamma | Exponential-Gamma | Closed-form but limited | Jakeman (1976) Abdi (1998) |
| Nakagami | Gamma | Gamma-Gamma | Closed-form | Lewinsky(1983) Shankar (04) |

# The Generalized-*K* Model

- Using the Nakagami model for multipath fading and the Gamma model for shadowing results in the Gamma-Gamma (generalized-*K*) model

Multipath fading

$$p_{\gamma/\Omega}(x)=\frac{1}{\Gamma(m_m)}\left(\frac{m_m}{\Omega}\right)^{m_m}x^{m_m-1}\exp\left\{-\frac{m_m x}{\Omega}\right\},\ x>0, m_m\geq 0.5$$

and

shadowing

$$p_{\Omega}(y)=\frac{1}{\Gamma(m_s)}\left(\frac{m_s}{\Omega_0}\right)^{m_s}y^{m_s-1}\exp\left\{-\frac{m_s}{\Omega_0}y\right\},\ y\geq 0,\ m_s>0$$

After averaging,

$$p_{\gamma}(x)=\int_0^{\infty}p(x/\Omega)p_{\Omega}(y)dy$$

$$p_{\gamma}(x)=\frac{2}{\Gamma(m_s)\Gamma(m_m)}(b)^{m_s+m_m}x^{\left(\frac{m_s+m_m}{2}\right)-1}K_{m_s-m_m}\left(b\sqrt{x}\right),\qquad x\geq 0, m_m\geq 0.5, m_s>0$$

Where $m_m$, $m_s$ are the multipath fading and shadowing parameters,

$K_{m_s-m_m}(\cdot)$ is the Bessel function of the second type and $b=2\sqrt{\frac{m_m m_s}{\Omega_0}}$ where $\Omega_0$ is the mean of the local power.

# The Generalized-*K* Model (Contd)

- The generalized-K (GK) PDF, being the PDF of the product of two independent Gamma RVs, belongs to the family of Fox *H*-function PDF family.
- The PDF of the product of *N* independent *H*-function RVs is another *H*-function PDF as well as the PDF of the quotient of two independent *H*-function RVs [Springer, 1979]. However, not for the sum.
- However, no closed-form expression for the PDF of the product, quotient, or the sum of correlated *H*-function RVs is known. Otherwise, the PDF of the sum of correlated Rayleigh, Nakagami, etc. would have been known.

# Related Work On the Statistics of Correlated GK RVs

- The simple case where the shadowing components are fully correlated (collocated antenna systems) was considered in [Shankar, 2006] where it was shown that the sum follows another GK distribution with $m_{m,sum} = Nm_m$ *and* $m_{s,sum} = m_s$.

- Infinite series expressions for the PDF, the CDF, and the CHF of the joint distribution were derived in [P. Bithas, et.al.] and the performance of maximal ratio combining (MRC) and equal gain combining for a dual diversity combiner are studied.

# Total correlation coefficient between two GK RVs

The sum of $N$ correlated generalized-$K$ RVs can be expressed as

$$\xi = z_1 w_1 + z_2 w_2 + \ldots\ldots + z_N w_N$$

where $z_i$ and $w_i$ denote the multipath fading and shadowing RVs, respectively. In general, while the multipath and shadowing components are independent, correlation among the $z_i$'s and correlation among the $w_i$'s may exist in certain propagation scenarios. So,

$$\rho_{i,j} = \frac{E[z_i w_i z_j w_j] - E[z_i w_i]E[z_j w_j]}{\sigma_{z_i w_i}\sigma_{z_j w_j}}, \quad i, j = 1, \ldots\ldots, N.$$

However, we may write $E[z_i w_i z_j w_j] = E[z_i z_i]E[w_i w_j]$

Since multipath fading is independent from shadowing

Furthermore, using the composite fading multiplicative model, the correlation coefficient can be expressed as

$$\rho_{i,j} = \frac{\rho_{z_i,z_j}\sqrt{m_{s,i}m_{s,j}} + \rho_{w_i,w_j}\sqrt{m_{m,i}m_{m,j}} + \rho_{z_i,z_j}\rho_{w_i,w_j}}{\sqrt{m_{m,i}+m_{s,i}+1}\sqrt{m_{m,j}+m_{s,j}+1}}$$

The expression simplifies for the identically distributed case to (i.d.) Case to

$$\rho_{i,j} = \frac{\rho_{z_i,z_j}m_s + \rho_{w_i,w_j}m_m + \rho_{z_i,z_j}\rho_{w_i,w_j}}{m_m + m_s + 1}$$

# The Amount of Fading (AF) expressions

The amount of fading (AF) is defined as the ratio of the variance to the squared mean

$$AF = \frac{\mathrm{var}(\xi)}{[E(\xi)]^2}$$

The AF was introduces to quantify the fading severity; however, it has been shown later that it can be related to some performance measures like the symbol error rate [B. Holter, and E. Oien, 2005] and the ergodic capacity [Y. Li and Kishore, 2008].

The AF for the sum of correlated GK RVs can be expressed as

$$AF_\xi = \frac{\sum_{i=1}^{N} AF_i \Omega_{0,i}^{\;2} + \sum_{i=1} \sum_{j=1, i\neq j} \rho_{i,j} \sqrt{AF_i}\sqrt{AF_j}\,\Omega_{0,i}\Omega_{0,j}}{\left(\sum_{i=1}^{N} \Omega_{0,i}\right)^2}$$

For i.d. case and a constant correlation model

$$AF_\xi^{\,i.d.e.c} = \frac{1-\rho_m}{Nm_m} + \frac{1-\rho_s}{Nm_s} + \frac{1-\rho_m\rho_s}{Nm_m m_s} + \frac{\rho_m}{m_m} + \frac{\rho_s}{m_s} + \frac{\rho_m\rho_s}{m_m m_s}$$

# Cont.....

Typically, for geographically spaced antennas, only shadowing correlations are significant; hence the AF expression reduces to

$$AF^{i.d.e.c} = \frac{1}{Nm_m} + \frac{1-\rho_s}{Nm_s} + \frac{1}{Nm_m m_s} + \frac{\rho_s}{m_s}$$

Clearly $AF^{i.d.e.c}_{N\to\infty} = \frac{\rho_s}{m_s}$ as diversity does not combat the correlated part.

Negative correlation may take place in some propagation [K. Butterworth, et.al. 1997, E. Perhia, et.al, 2001].

For $N$=2, the effect of shadowing correlation vanishes for $\rho_s$=-1. In general, it can be shown that the effect of shadowing correlation vanishes for $\rho_s$=-1/($N$-1) for $N$>1.

# Application to Capacity of DAS Systems

The effect of shadowing correlations on macrodiversity (DAS) can be studied using the following approximation [Y. Li and Kishore, 2008]

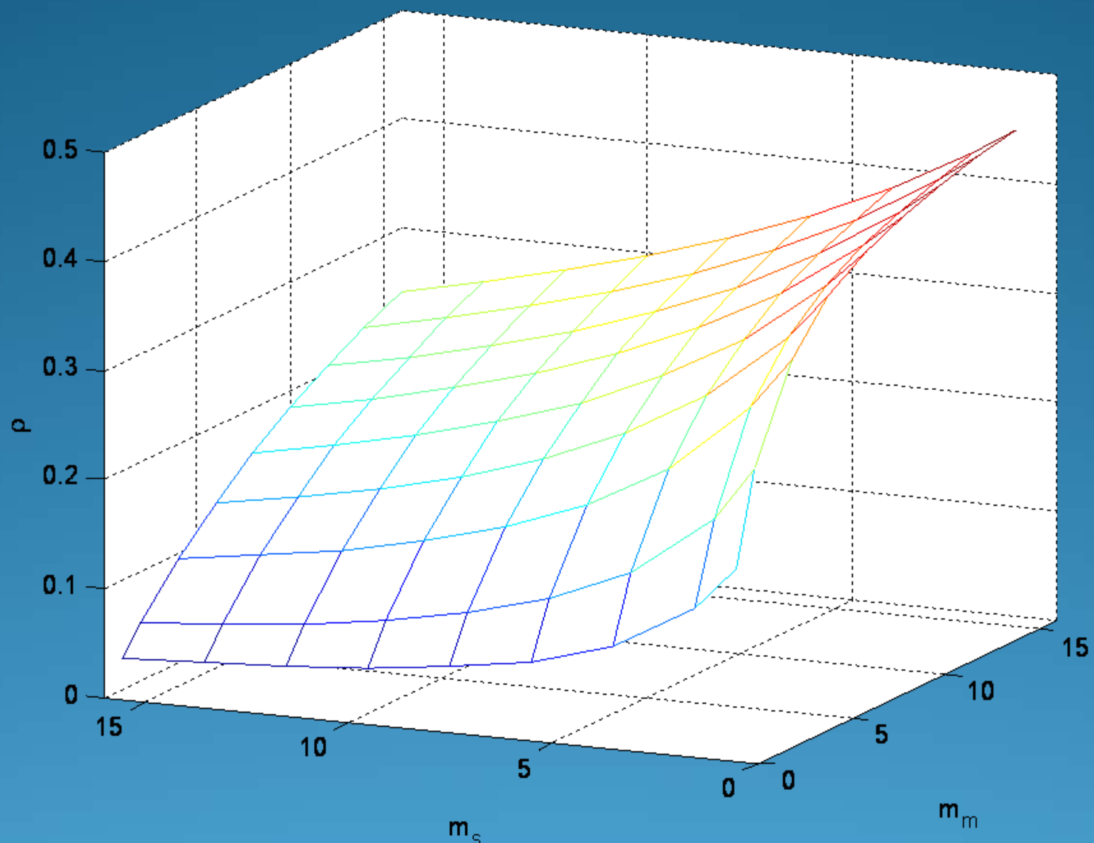
The ergodic capacity, after MRC, can be approximated, (for $AF$<0.5), as

$$C \approx \log_2(1+SNR) - \frac{\log_2}{2} AF \left(\frac{SNR}{SNR+1}\right)^2$$

As, $\xrightarrow{N \to \infty}$ $C \approx \log_2(1+SNR) - \frac{\log_2}{2} \frac{\rho_s}{m_s} \left(\frac{SNR}{SNR+1}\right)^2$ $\xrightarrow{SNR \to \infty}$ $C_{loss} \approx \frac{\log_2}{2} \frac{\rho_s}{m_s} \quad b/s/Hz$

Hence, the asymptotic ergodic capacity loss due to correlated shadowing can be predicted for different values of $m_s$ and $\rho_s$ as far as $\frac{\rho_s}{m_s} \le 0.5$ .

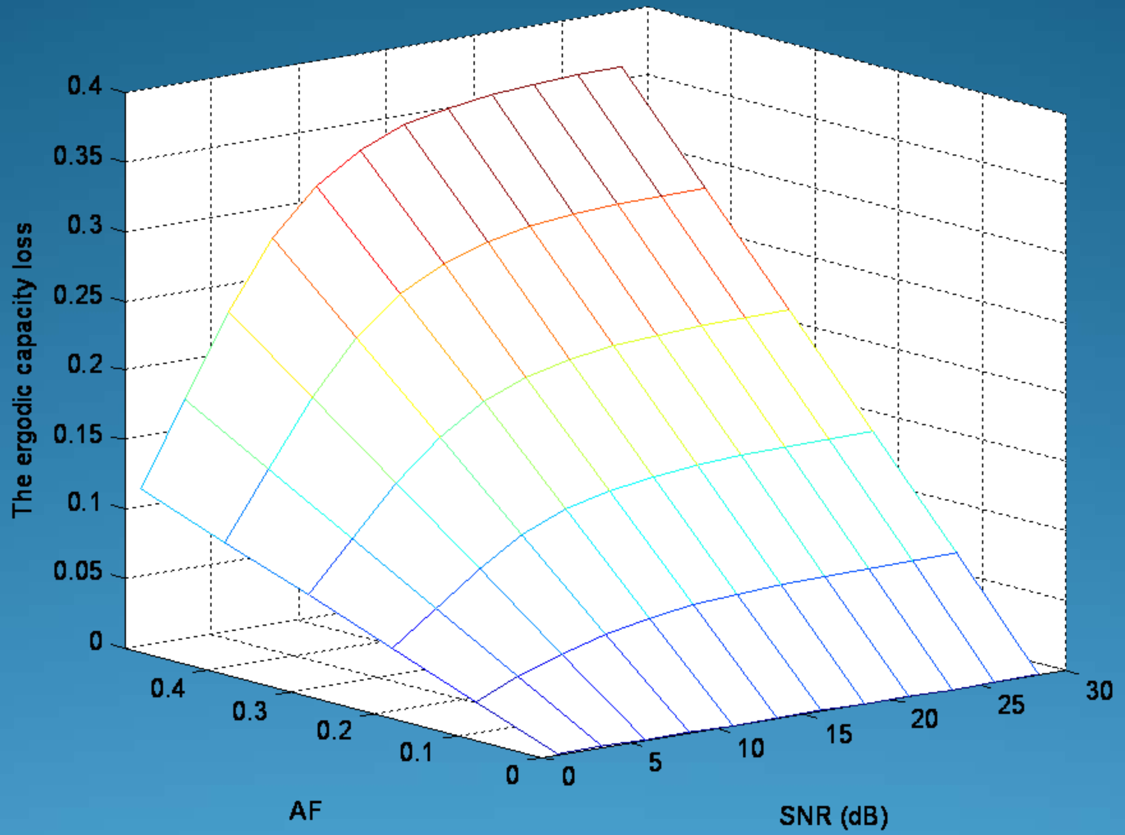
# Results




The plot of the correlation coefficient between two GK RVs for $\rho_s$=0.5.

The correlation coefficient decreases as m increases (less shadowing) and as m decreases where the multipath components dominate.

# Results




The plot for the ergodic capacity loss versus the *AF* and the SNR.

The asymptotic ergodic apacity loss for typical shadowing correlation scenarios (where $\rho_s$=0.5) is bounded by less than 0.4 b\s\Hz.

# Conclusions and Future Work

- The generalized-$K$ composite fading model is more tractable than lognormal-based model. However, some challenges are there.
- The correlation coefficient between two generalized-$K$ RVs is derived and subsequent expressions for the AF are presented.
- Moreover, the effect of negative correlations is highlighted.
- The effect of shadowing correlations on the performance of MRC receivers can be predicted.
- The formulation can be extended to MIMO scenario using the channel matrix Frobenius norm.

# References

[1] M. D. Springer, *The Algebra of Random Variables*, John Wiley Sons, Inc., 1979.

[2 ] P. M. Shankar, "Performance analysis of diversity combining algorithms in shadowed fading channels," *Wireless Personal Communications.*, v. 37, no. 2, Apr. 2006.

[3 ] Bithas, P.S. and Sagias, N.C. and Mathiopoulos, P.T., "The bivariate generalized-*K* KG distribution and its application to diversity receivers," *IEEE Transactions on Communications.,* v. 57, no. 9, Sep. 2009.

[4] Li, Y. and Kishore, S., "Diversity factor-based capacity asymptotic approximations of MRC reception in Rayleigh fading channels," *IEEE Transactions on Communications.,* v. 56, no. 6, June 2008.

[5] Holter, B.; Oien, G.E.; , "On the amount of fading in MIMO diversity systems," *IEEE Transactions on Wireless Communications,* vol.4, no.5, pp. 2498- 2507, Sept. 2005.

# Appendix: the Fox *H*-function

*Definition*: The $H$-function can be defined as

$$\mathrm{H}_{p,q}^{m,n}\left[z\left|{}_{(b_1,\beta_1),\ldots,(b_q,\beta_q)}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right.\right] = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}\Gamma\left(b_j-\beta_j s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\alpha_j s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\beta_j s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-\alpha_j s\right)} z^s ds, \tag{A.1}$$

where $0 \le m \le q$, $0 \le n \le p$, $\alpha_j > 0$, $\beta_j > 0$. and $a_j$ $(j = 1, 2, \ldots.p)$ and $b_j$ $(j = 1, 2, \ldots.q)$ are complex numbers such that no pole of $\Gamma\left(b_j - \beta_j s\right)$ for $j = 1, 2, \ldots m$ coincides with any pole of $\Gamma\left(1 - a_j + \alpha_j s\right)$ for $j = 1, 2, \ldots., n$.

Many of the so-called special functions are special cases of the $H$-function, including Gauss and confluent hypergeometric functions, MacRobert's E-function, Meijer's function; and Bessel functions. The Meijer's function can be expressed as

$$\mathrm{G}_{p,q}^{m,n}\left[x\left|{}_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}\right.\right] = \mathrm{H}_{p,q}^{m,n}\left[z\left|{}_{(b_1,1),\ldots,(b_q,1)}^{(a_1,1),\ldots,(a_p,1)}\right.\right]. \tag{A.2}$$

*Definition*: The $H$-function distribution can be expressed as

$$p(x) = k\mathrm{H}_{p,q}^{m,n}\left[cx\left|{}_{(b_1,\beta_1),\ldots,(b_q,\beta_q)}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right.\right], x > 0, \tag{A.3}$$

where $k$ and $c$ are the parameters of the distribution such that $\int_0^\infty p(x)dx = 1$.

Questions