CoV-Based Metrics for Quantifying the Regularity of Hard-Core Point Processes for Modeling Base Station Locations

Faraj Lagum, Sebastian S. Szyszkowicz, and Halim Yanikomeroglu

***Abstract*—Base station locations in wireless networks can be modeled via repulsive random point processes with an amount of regularity that is tunable between that of a triangular lattice and that of a homogeneous Poisson point process. However, it is currently difficult to quantify this regularity, or compare different repulsive point processes. In this letter, we examine three regularity metrics based on the coefficient of variation (CoV) of geometric properties of point processes and identify the CoV of the nearest neighbour distance as the most sensitive metric. We also compare three hard-core point processes in terms of their regularity range and the density of the generated points.**

***Index Terms*—Stochastic geometry, hard-core point process, repulsive point process, regularity, second-order statistics.**

I. Introduction

RECENT years have seen an increasing interest in point processes (a subset of stochastic geometry) for more realistic modeling of base station (BS) locations in wireless networks. Traditionally, the triangular lattice (hexagonal cells) is used to model the spatial structure of the BSs, and, more recently, the homogeneous Poisson point process (PPP) is also proposed because of its simplicity and analytic tractability [1]–[5]. Both of these models are conceptually simple and can be characterized by a single parameter: the density. However, they are less accurate models and represent two extremes, while the real deployment of the BS locations falls somewhere in between [1]–[4]. Indeed, modeling BS locations using a triangular lattice gives over-optimistic network performance results, while modeling BS locations using the PPP gives the most pessimistic performance estimate [1].

Repulsive point processes (RPPs) have additional parameters apart from their density that can be tuned, with a resulting variation in the amount of regularity. Notably, hard-core point processes are characterized by the hard-core distance, which is the closest distance that two points of that process can ever be to each other. RPPs can also be of the soft-core variety, where points can appear arbitrarily close to each other with a certain probability. Having extra parameters increases the complexity of the models but provides control over of the amount of regularity.

In this letter, we examine three hard-core processes already proposed in wireless literature to model BS locations [6]–[10]. Other RPPs used in the literature include determinantal point process models [11], [12], and the family of Gibbs point processes [2], [4]. RPPs are also useful for modeling HetNets [3] and wireless sensor networks [13].

Motivation: Due to the dependence of the network performance on the BS locations [2], the amount of regularity of the spatial structure of BSs is an important characteristic of wireless cellular networks. Motivated by the lack of an adequate scalar metric to describe the spatial structure of these BSs, we propose using the coefficient of variation (CoV) of particular geometric properties as scalar metrics to quantify the regularity [5]. These scalar metrics are useful as network performance indicators as shown in Section V-B. Functional summary characteristics such as the nearest neighbour distribution function, the empty space function, and the Ripley K-function are widely used in the literature [2], [4], [11], [12] as metrics to capture the spatial structure of wireless networks. However, these metrics are functions, and therefore we still need to quantify the difference between them. Qualitative terms such as less (or more) repulsive [7], [10], [12], [13] are also used in the literature to compare the amount of regularity of RPPs. Therefore, finding a precise and meaningful scalar metric to quantify the regularity is a necessity [13].

Contributions: We evaluate these CoV-based metrics for three common hard-core point processes in wireless literature. These point processes have only two tuning parameters: the density and the hard-core distance. This letter's contributions are as follows: (i) We evaluate the use of three CoV-based metrics for quantifying the amount of regularity, (ii) we show that the CoV-based metrics are capable of measuring the amount of regularity of RPPs and that the CoV of the nearest neighbor distance is the most sensitive metric among them, and (iii) we compare the hard-core point processes as models for BS locations in terms of the achievable range of regularity.

The rest of this letter is organized as follows: In Section II, three hard-core point processes are introduced. In Section III, three CoV-based metrics are presented. In Section IV, the CoV-based metrics are evaluated for quantifying the regularity of simulated realizations of the hard-core point processes. Finally, we compare one CoV-based metric with function-based metrics in Section V, before drawing conclusions in Section VI.

II. Hard-Core Point Processes

A hard-core point processes X is a RPP where two points are strictly prohibited from being closer than a predefined hard-core distance $r > 0$ apart [6], [7], [14].

In this section, we describe three hard-core point processes. In general, generation of these three hard-core point processes

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The authors are with the Department of Systems and Computer Engineering, Carleton University, Ottawa, ON K1S 5B6, Canada (e-mail: faraj.lagum@sce.carleton.ca; sz@sce.carleton.ca; halim@sce.carleton.ca).

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begins with generating a PPP, and then removing points that violate the hard-core condition. Different ways of removing points lead to different RPPs with different densities.

A. Matérn Hard-Core Process of Type I (MHC-I)

The MHC-I is generated as follows: From a PPP Φ with density λ_{P}, simultaneously remove all points that are closer than r from each other. The density of the MHC-I is $\lambda = \lambda_{\mathrm{P}} e^{-\lambda_{\mathrm{P}} \pi r^2}$ [7], and its normalization by λ_{P} is

$$\frac{\lambda}{\lambda_{\mathrm{P}}} = e^{-\pi \tilde{r}^2}, \tag{1}$$

where $\tilde{r} = r\sqrt{\lambda_{\mathrm{P}}}$ is the normalized hard-core distance.

B. Matérn Hard-Core Process of Type II (MHC-II)

The MHC-II is generated by following three steps: First, generate a PPP Φ with density λ_{P}. Second, associate a mark U_i which is an independent uniform random variable on [0,1] to each point $x_i \in \Phi$. Then, simultaneously remove all points that have higher marks than their neighbours within a distance r. The density of the MHC-II is $\lambda = \frac{1-e^{-\lambda_{\mathrm{P}} \pi r^2}}{\pi r^2}$ [7], and its normalized density is

$$\frac{\lambda}{\lambda_{\mathrm{P}}} = \frac{1 - e^{-\lambda_{\mathrm{P}} \pi r^2}}{-\lambda_{\mathrm{P}} \pi r^2} = \frac{1 - e^{-\pi \tilde{r}^2}}{\pi \tilde{r}^2}. \tag{2}$$

C. Simple Sequential Inhibition (SSI)

Given the required density of points λ_{P} in a domain, candidate points are generated sequentially using a PPP. Points are discarded if they are within a distance r from any previously accepted point. The process terminates when the required density is attained or when adding more point becomes impossible [7], [15]. The density of the accepted points is λ. We are not aware of any closed-form expression for the SSI density, but we find a good fit:

$$\frac{\lambda}{\lambda_{\mathrm{P}}} = \min\left\{1, \quad 0.61 \cdot \tilde{r}^{-1.79}\right\}. \tag{3}$$

The density ratios as a function of $\tilde{r}$ for MHC-I, MHC-II, and SSI, given by (1)–(3), are shown in Fig. 1.

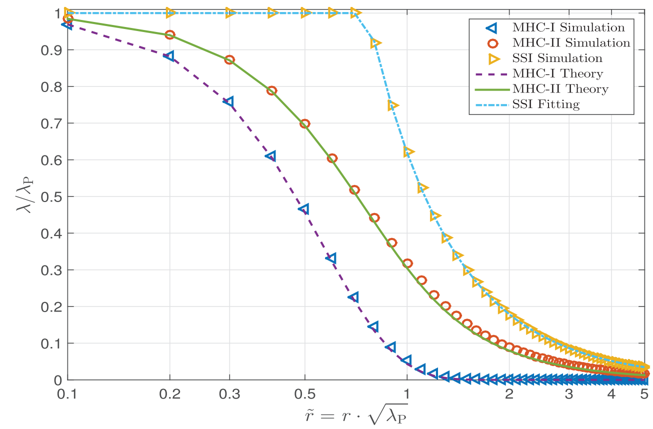

Fig. 1. The density ratio of MHC-I, MHC-II, and SSI as a function of $\tilde{r}$.

III. CoV-Based Metrics

In statistics, the CoV of a random quantity is defined as the ratio of its standard deviation to its mean. The CoVs of three geometric properties have been introduced in [5] to measure the clustering of mobile user locations. We propose using these metrics to measure the amount of regularity of the BS locations. Each metric is normalized by a constant factor so that the CoV of the PPP is always 1.

A. CoV of the Areas of Voronoi Tessellation Cells

Considering the areas of the cells of the Voronoi tessellation [7] of a set of points, the CoV-based metric is

$$C_{\mathrm{V}} = \frac{1}{k_{\mathrm{V}}} \cdot \frac{\sigma_{\mathrm{V}}}{\mu_{\mathrm{V}}}, \qquad k_{\mathrm{V}} \cong 0.529, \tag{4}$$

where μ_{V} is the mean and σ_{V} is the standard deviation of the Voronoi cell areas, and k_{V} is a normalization factor [5].

B. CoV of the Lengths of Delaunay Triangulation Edges

Taking the edges of the Delaunay triangulation [7] of a set of points, the CoV-based metric is

$$C_{\mathrm{D}} = \frac{1}{k_{\mathrm{D}}} \cdot \frac{\sigma_{\mathrm{D}}}{\mu_{\mathrm{D}}}, \qquad k_{\mathrm{D}} \cong 0.492, \tag{5}$$

where μ_{D} is the mean and σ_{D} is the standard deviation of the Delaunay edge lengths, and k_{D} is a normalization factor [5].

C. CoV of the Distances to the Nearest Neighbour

Taking the distance from every point to its nearest neighbour [16], the CoV-based metric is

$$C_{\mathrm{N}} = \frac{1}{k_{\mathrm{N}}} \cdot \frac{\sigma_{\mathrm{N}}}{\mu_{\mathrm{N}}}, \qquad k_{\mathrm{N}} = \sqrt{\frac{4-\pi}{\pi}} \cong 0.5227, \tag{6}$$

where μ_{N} is the mean and σ_{N} is the standard deviation of the nearest neighbour distances; and k_{N} is a normalization factor derived[1] from [16].

The CoV-based metrics take the value of 0 for a triangular lattice and 1 for a PPP. Values between 0 and 1 are found for RPPs, as will be seen in the next section, while values above 1 are found for processes with clustering [5]. Practical models for the BS locations should have a CoV between 0 and 1. The metrics C_{V}, C_{D}, and C_{N} are unit-less quantities and are invariant under scaling of the measured point process, and can thus be adjusted independently of its density λ.

The three geometric properties are also meaningful in the context of BS locations: the Voronoi tessellation represents the cell area associated with each BS under the assumption the users always connect to the nearest BS [14], while the Delaunay triangulation connects each BS to its strongest interfering BSs, and the nearest neighbour characterizes the dominating interfering BS and has been of interest in measuring the regularity of RPPs [13].

[1] In [5], the value for k_{N} is erroneously given as 0.653.

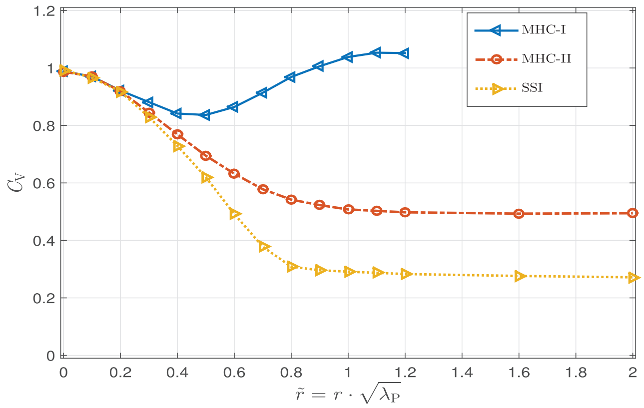

Fig. 2. The CoV of the areas of the Voronoi tessellation cells as a function of the normalized hard-core distance for hard-core point processes.

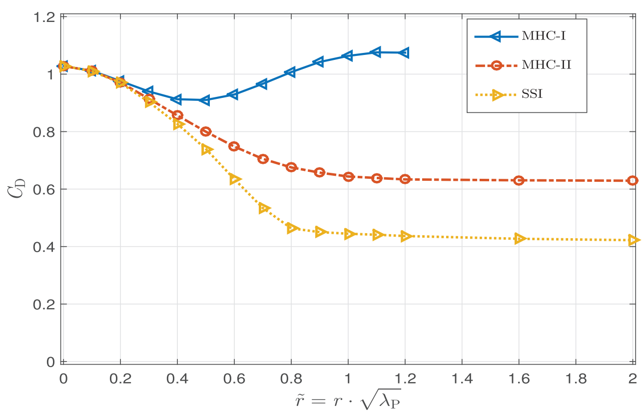

Fig. 3. The CoV of the lengths of the Delaunay triangulation edges as a function of the normalized hard-core distance for hard-core point processes.

IV. Evaluation of CoV-Based Metrics for Hard-Core Point Processes

We generate spatial patterns of BS locations using the point processes defined in Section II and measure their amount of regularity using the metrics defined in Section III.

The density λ is fixed to be 100 points in a 1 km^2 square domain. $\tilde{r}$ is swept over a wide range to change the regularity of the RPP, which is captured using the CoV-based metrics. For each RPP and metric combination, a Monte-Carlo simulation is performed with 1000 realizations. The ensemble mean of the CoV-based metrics of the resulting points as a function of $\tilde{r}$ is shown in Figs. 2, 3, and 4.

We observe that the SSI process has the widest CoV ranges[2] and achieves the highest density ratio, making it the most attractive RPP. Conversely, the MHC-I process is the least desirable among the investigated RPPs. It has the lowest density ratio, making it inefficient in generating a given number of points, and its CoV values fluctuate in a narrow range around 1. We interpret this behaviour as being caused by the nature of the MHC-I process itself: Removing all points that violate the hard-core condition creates large holes in the generated pattern when the hard-core distance is large, causing some of the remaining points to cluster, relatively speaking. Since cluster processes were shown [5] to have CoVs greater than 1, this clustering increases the CoV value of MHC-I.

[2]Indeed, in [10], SSI is observed to be more regular than MHC-II.

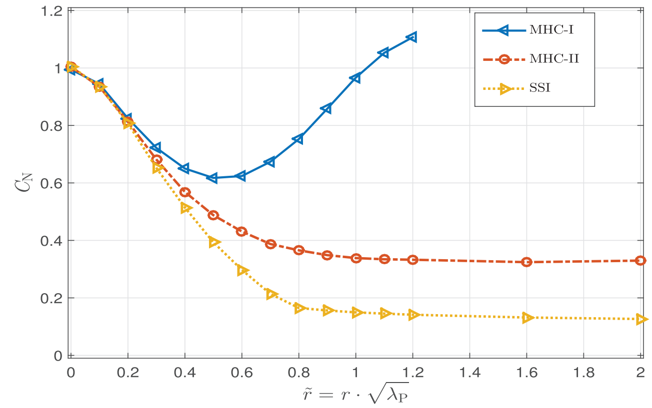

Fig. 4. The CoV of the distance to the nearest neighbour as a function of the normalized hard-core distance for hard-core point processes.

TABLE I
CoV-Based Metrics Floor for Hard-Core Point Processes

		C_V	C_D	C_N
CoV floor	MHC-I	0.84	0.91	0.62
	MHC-II	0.49	0.63	0.33
	SSI	0.27	0.42	0.13

Our results also show that (i) the amount of regularity of hard-core point processes is tunable and can be quantified using CoV-based metrics and (ii) the useful tuning range of these RPPs is $\tilde{r} < 1$, i.e., where the CoVs are sensitive to changes in $\tilde{r}$.

The ranges of the CoV-based metrics are summarized in Table I. The C_{N} metric provides the widest value range, making it the most sensitive to changes in the amount of regularity of the RPP.

V. Comparison of C_{N} With Two Function-Based Metrics

In this section, we show the relation between C_{N} and two function-based metrics, including network performance.

A. *Ripley's K-Function*

Ripley's K-function[3] $K(r)$ is defined as the ratio of the mean number of extra points within distance r from a typical point (not included in the counting) to the density of the spatial pattern [6], [13]. It can characterize the regularity or clustering of a point process. The L-function, $L(r) = \sqrt{K(r)/\pi}$, is a normalized form of $K(r)$. While $L(r) = r$ for a PPP, a spatial pattern with $L(r) < r$ is repulsive.

As shown in Fig. 5, RPPs with the same C_{N} value have similar L-function and hard-core distance r, apart from the MHC-I in the second regime (in which the CoV value increases with $\tilde{r}$). This indicates that matching RPPs using Ripley's functions is not always possible as the curves can have very different shapes.

B. *Coverage Probability*

The coverage probability $P(\gamma)$ is the probability that a typical user achieves a signal-to-interference ratio (SIR) higher

[3]The K-functions of MHC-I and MHC-II are known in a complicated integral form [6].

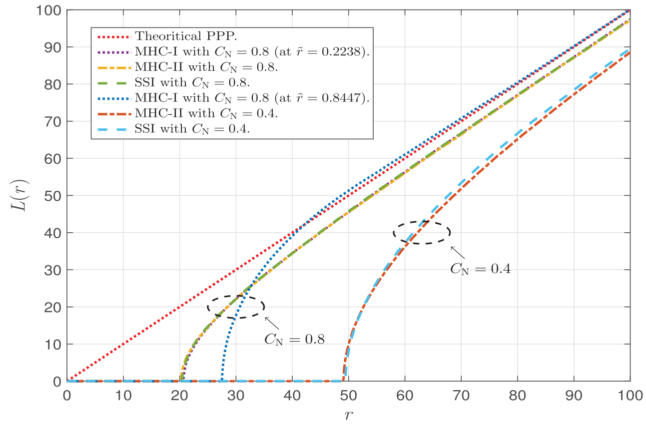

Fig. 5. Normalized Ripley's K-function for different RPPs and C_{N} values.

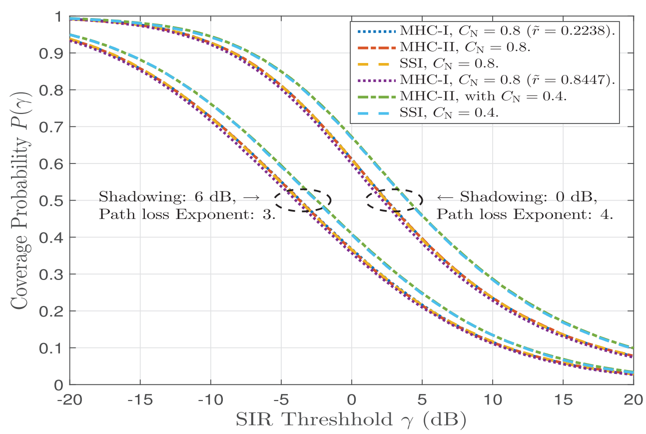

Fig. 6. The coverage probability for different hard-core models with constant density for different C_{N} values and channel environments.

than a given SIR threshold γ. We compare the downlink coverage probability where the BSs are deployed according to hard-core point processes introduced in section II with different amounts of regularity as measured using C_{N}. The following assumptions are used to evaluate the coverage probability: (i) the average density is 100 BSs in 1 km^2, (ii) all BSs transmit the same power. (iii) mobile users are uniformly distributed over the entire domain and each of them is associated to its nearest BS, (iv) the frequency reuse factor is 1, (v) all channels have Rayleigh fading, and (vi) the thermal noise is ignored. We also assumed two channel models: one with a path loss exponent of $\alpha = 3$ and 6 dB lognormal shadowing, and one with $\alpha = 4$ and no shadowing. Fig. 6 shows that different RPPs with the same C_{N} in different channel environments behave alike regarding coverage probability. This is true even for MHC-I in the second regime.

C_{N} is an additional factor that affects the network performance. A spatial pattern with a low C_{N} value has a better performance than one with a high C_{N} value. BSs deployed according to the same density and C_{N} have very similar SIR performance, regardless of the hard-core process chosen.

VI. Conclusion

We proposed three different CoV-based metrics to measure the amount of regularity of three spatial point processes used in cellular networks to model the locations of BSs. These metrics are also applicable to all stationary point processes, in any field of study. We found that C_{N} is the most sensitive to the regularity of RPPs. Different BS location models with the same density and C_{N} value have very similar SIR performance. Our results also show that the MHC-I process is undesirable for modeling points with regularity, whereas both the MHC-II and SSI processes are useful when their normalized hard-core distance is less than 1, SSI being the best in terms of the range of regularity and density ratio.

Given real deployments of BS locations, an interesting extension is to investigate whether the CoV-based metrics could work as a tool for fitting them to RPP models. Another extension could include investigating other RPPs as well as other scalar metrics such as the variance of nearest neighbour distribution and its noise figure [13].

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