Arbitrary Constellations with Coded Maximum Ratio Transmission over Downlink Nakagami-*m* Fading Channels

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***Abstract*—There has been a rejuvenated interest in the use of arbitrary constellations (non-equally spaced) after promising performance results were demonstrated compared to conventional grid and circular constellations. To enable the development of such constellations, in this contribution we derive an upper bound bit-error-rate (BER) expressions for coded transmit maximum ratio combining (TMRC) systems operating in Nakagami-*m* fading environments with arbitrary constellations and encoder types. Unlike most existing works on error performance analysis for coded systems, a chosen pair of constellation and encoder need not be quasi-regular. Furthermore, the system model includes a provision for multiple orthogonal transmission phases with different number of transmit antennas. Simulation results demonstrate the accuracy of the derived analytical results for a range of system scenarios.**

I. Introduction

Due to the fundamental nature of the bit-error-rate (BER) as a performance metric, numerous results exist in the literature, for both coded and uncoded systems. Considering the more general Nakagami-*m* fading model, examples include [1], [2], and references listed therein. The results, however, are limited to specific, classical constellations. In contrast, with a recent revival of constellation design through optimization techniques [3], [4], the derivation of error performance expressions for completely *arbitrarily shaped*[2] constellations remains largely an open problem.

Most BER literature for coded scenarios is applicable only to quasi-regular (QR) systems, where all performance analysis is independent of transmitted sequences [5]. While facilitating analysis, quasi-regularity is not associated with improved performance [6]. Furthermore, many systems are found to be non-QR, especially when encoders are paired with arbitrary constellations [7].

Fundamental to evaluating the BER performance of multiple antenna systems in Nakagami-*m* fading is the distribution of the sum of Gamma random variables (RVs), which dates back to [8], where the authors derive the PDF of the sum of independent but not necessarily identically distributed (i.n.i.d) Gamma RVs in the form of an infinite series. Closed form results were first developed by [9] where PDF and CDF expressions were derived for sums of squared Nakagami-*m* RVs with integer fading parameters. The results were expressed as weighted sums of Erlang PDFs and CDFs. More recently, [10] has shown that the PDF of the sum of Gamma RVs can be expressed in terms of the Fox $\bar{H}$-function (for non-integer-valued Nakagami-*m* fading parameters) and the Meijer G-function (for integer-valued Nakagami-*m* fading parameter).

In [7], we used the product state matrix technique [11] in the calculation of the generating function to enable the derivation of error bounds for coded systems with any signal constellations. In addition to enabling the computation of the error bound for non-QR codes, the system model in [7] allowed for two stages of transmissions, encompassing systems such as CoMP, HARQ and relaying.

This paper extends the results of [7] to a multiple antenna system, specifically coded transmit maximum ratio combining (TMRC) [12], [13]. Using new results for the statistics of sum of Gamma RVs [10], we develop compact expressions for the generating function in such systems. Furthermore, we consider a very flexible system model, one which allows for an arbitrary number of orthogonal transmission phases with

- different number of transmit antennas employed in each transmission phase,
- arbitrary constellations, different for each phase,
- Nakagami-*m* fading with integer and non-integer fading parameters, different for each phase[3], and
- different path loss in each transmission phase.

The remainder of this paper is organized as follows. In Section II, we present a TMRC downlink system model. The error performance analysis including the generating function calculation via product state matrix technique in Nakagami-*m* fading environment is given in Section III. Section IV

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[2]We use the term *arbitrary* in reference to the randomness of symbol point locations, rather than simply the size of a classical M-ary constellation.

[3]Due to space limitations, the analysis for spatially correlated fading is left as a future study item.

presents numerical results comparing the derived BER bound expressions with Monte Carlo simulations for a range of fading and signalling level scenarios. Section V gives concluding remarks.

II. TMRC Downlink System

We consider a TMRC system consisting of K orthogonal transmission phases as shown in Fig. 1. Originally proposed in [14], [15], the orthogonal transmission phases offer a flexible representation of complex systems, such as CoMP, HARQ and relaying. Each transmission phase, k, employs N_k transmit antennas, and one receive antenna. The channels between the transmit antennas and the receive antenna are modelled by frequency non-selective Nakagami-*m* slow fading, characterized by an $N_k \times 1$ vector, $\mathbf{h}_k$. Each entry of $\mathbf{h}_k$, $h_{k,i}$, corresponding to the channel between ith transmit and the receive antenna in the kth orthogonal phase $(i \in \{1, \ldots, N_k\}, k \in \{1, \ldots, K\})$, is a complex Nakagami-*m* random variable with a shaping parameter $m_{k,i}$ and an average fading power $\Omega_{k,i}$. The envelope of the channel gain, $|h_{k,i}|$, is modelled by the Nakagami-*m* distribution. The phase coefficient, $\theta_{k,i}$, is generated using the model given in [16][4].

In the k-th transmission phase, the information symbols are convolutionally encoded and the resulting bits are assigned an output symbol, $s^{(k)}$, based on a specific constellation $\chi^{(k)}$. We assume that perfect channel state information (CSI) is available at the transmitter and the receiver. Each transmitter employs a precoder $\mathbf{w}_k$, matched to the normalized channel vector $\mathbf{h}_k$,

$$\mathbf{w}_k = \frac{\mathbf{h}_k^H}{\|\mathbf{h}_k\|}, \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean norm. The received signal during kth transmission phase is given by

$$\begin{aligned} y_k &= \mathbf{h}_k \mathbf{w}_k s^{(k)} + n_k \\ &= \alpha_k s^{(k)} + n_k, \end{aligned} \tag{2}$$

where $\alpha_k = \dfrac{\mathbf{h}_k{}^H \mathbf{h}_k}{\|\mathbf{h}_k\|} = \|\mathbf{h}_k\|$ and n_k is the additive white Gaussian noise (AWGN) sample with zero-mean and $N_0/2$ variance per dimension. We assume independent fading between the K orthogonal transmission phases with possibly different shaping parameters $m_{k,i}$.

III. Performance Analysis

A. BER bound methodology

We begin with the outline of the BER methodology based on the product state matrix technique [7], [17]. The probability of decoding an erroneous codeword vector $\hat{\mathbf{s}} = [\hat{s}^{(1)}\hat{s}^{(2)} \ldots \hat{s}^{(K)}]$ at the receiver in place of the transmitted codeword $\mathbf{s} = [s^{(1)}s^{(2)} \ldots s^{(K)}]$ where $\exists k,\ s^{(k)} \neq \hat{s}^{(k)}$ can be expressed in terms of the probability of an erroneous decoder transition $(v \to \bar{v})$ for an actual

[4] It is shown in [16] that the uniform phase distribution assumption does not hold for $m_k \neq 1$.

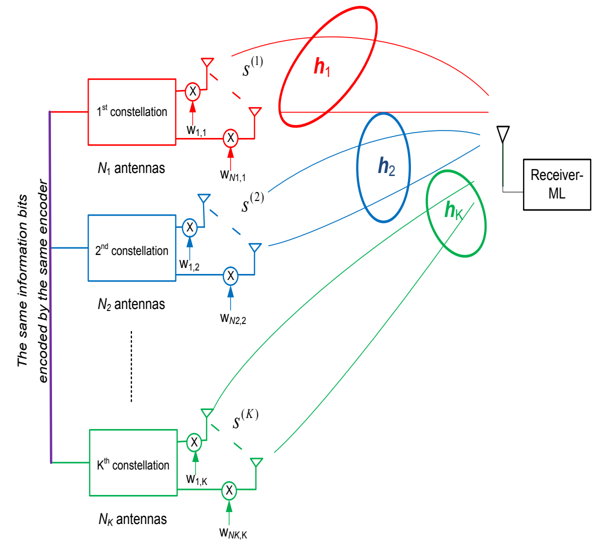

Fig. 1: TMRC system model.

transition state at encoder $(u \to \bar{u})$ and it is denoted by $D_{(u,v),(\bar{u},\bar{v})}$. For a given channel coefficient set, $\mathbf{h}_1, ..., \mathbf{h}_K$, the conditional $D_{(u,v),(\bar{u},\bar{v})}$ can be expressed as [7]

$$\begin{aligned} D_{(u,v),(\bar{u},\bar{v})|\mathbf{h}_1 \ldots \mathbf{h}_K} = \Pr\Bigg(\sum_{k=1}^{K} \left| y_k - \alpha_k s^{(k)} \right|^2 \\ - \left| y_k - \alpha_k \hat{s}^{(k)} \right|^2 \geq 0 \,|\mathbf{h}_1 \ldots \mathbf{h}_K \Bigg). \end{aligned} \tag{3}$$

At this point, 'good' ($\mathcal{G}$) and 'bad' ($\mathcal{B}$) states are defined based on comparing the initial and the final states of possible pair of transitions occurring at the encoder and the decoder [7], [17]. Using this classification and suitably ordering the product states, $\mathbf{S}$ can be written in the form of [11]

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{\mathcal{GG}} & \mathbf{S}_{\mathcal{GB}} \\ \mathbf{S}_{\mathcal{BG}} & \mathbf{S}_{\mathcal{BB}} \end{bmatrix}. \tag{4}$$

A particular entry of $\mathbf{S}$, $\mathbf{S}_{(u,v),(\bar{u},\bar{v})}$ can be expressed by

$$\begin{aligned} \mathbf{S}_{(u,v),(\bar{u},\bar{v})} &= \Pr\left(u \to \bar{u} | u\right) \\ &\times \sum_n p_n I^{\mathrm{W}(u \to \bar{u}) \oplus \mathrm{W}(v \to \bar{v})} D_{(u,v),(\bar{u},\bar{v})}, \end{aligned} \tag{5}$$

where the summation in (5) is over possible n parallel transitions depending on a given encoder, p_n denotes the probability of nth parallel transition between $(u \to \bar{u})$ if it exists, otherwise $p_n = 1$. $\Pr\left(u \to \bar{u}|u\right)$ is the conditional probability of a transition from state u to state $\bar{u}$ given state u and $\mathrm{W}\left(i \to j\right)$ denotes the Hamming weight of information sequence for the transition from i to j where $i \in \{u, v\}$ and $j \in \{\bar{u}, \bar{v}\}$ [17].
After some mathematical manipulations, (3) can be rewritten as

$$D_{(u,v),(\bar{u},\bar{v})|\mathbf{h}_1,\ldots,\mathbf{h}_K} = e^{-\sum_{k=1}^{K} d_k X_k}, \tag{6}$$

where $d_k = \dfrac{|s^{(k)} - \hat{s}^{(k)}|^2}{4N_0}$ and $X_k = \alpha_k^2$ is a sum of squared envelope Nakagami-*m* faded coefficients or, equivalently, the sum of Gamma variables.

Averaging (6) over the channel statistics yields the unconditional probability of erroneous transition, $D_{(u,v),(\bar{u},\bar{v})}$, which is required by (5). After obtaining each entry of $\mathbf{S}$, the generating function, $T(I)$, can be computed by [11]

$$T(I) = \mathbf{1}^T \mathbf{S}_{\mathcal{G}\mathcal{G}} \mathbf{1} + \left(\mathbf{1}^T \mathbf{S}_{\mathcal{G}\mathcal{B}}\right)^T [\mathbf{I} - \mathbf{S}_{\mathcal{B}\mathcal{B}}]^{-1} \mathbf{S}_{\mathcal{B}\mathcal{G}} \mathbf{1}, \quad (7)$$

which in turn is used to compute the upper bound BER using [18]

$$P_b \leq \frac{1}{l} \frac{\partial T(I)}{\partial I}\bigg|_{I=1}, \quad (8)$$

where l denotes the number of information bits per output symbol. In (7), $\mathbf{1}$ and $\mathbf{I}$ denote the unity and identity matrices, respectively.

We now derive the closed form expression for $D_{(u,v),(\bar{u},\bar{v})}$, subsequently used to obtain the upper bound BER using (5)-(8).

B. $D_{(u,v),(\bar{u},\bar{v})}$ for general m parameter

The closed form expression for the PDF of X_k for the non-integer fading parameters can be written in terms of the Fox's $\bar{H}$ function [10], that is

$$f_{X_k}(X_k) = \prod_{i=1}^{N_k} \left(\frac{m_{k,i}}{\Omega_{k,i}}\right)^{m_{k,i}} \bar{\mathrm{H}}^{0,N_k}_{N_k,N_k} \left[e^{X_k} \,\middle|\, \begin{matrix} \Xi^{(1)}_{N_k} \\ \Xi^{(2)}_{N_k} \end{matrix} \right], \quad (9)$$

where the coefficient sets $\Xi^{(1)}_{N_k}$ and $\Xi^{(2)}_{N_k}$ are defined as [10]

$$\Xi^{(1)}_{N_k} = \overbrace{\left(1 - \frac{m_{k,1}}{\Omega_{k,1}}, 1, m_{k,1}\right), \cdots, \left(1 - \frac{m_{k,N_k}}{\Omega_{k,N_k}}, 1, m_{k,N_k}\right)}^{N_k-\text{bracketed terms}}$$
$$\Xi^{(2)}_{N_k} = \overbrace{\left(-\frac{m_{k,1}}{\Omega_{k,1}}, 1, m_{k,1}\right), \cdots, \left(-\frac{m_{k,N_k}}{\Omega_{k,N_k}}, 1, m_{k,N_k}\right)}^{N_k-\text{bracketed terms}}. \quad (10)$$

Averaging (6) over the PDFs of X_k in (9) yields

$$D_{(u,v),(\bar{u},\bar{v})} = \prod_{k=1}^{K} \prod_{i=1}^{N_k} \left(\frac{m_{k,i}}{\Omega_{k,i}}\right)^{m_{k,i}} Z_k, \quad (11)$$

where Z_k is defined by

$$Z_k = \int_0^\infty e^{-d_k X_k} \bar{\mathrm{H}}^{0,N_k}_{N_k,N_k} \left[e^{X_k} \,\middle|\, \begin{matrix} \Xi^{(1)}_{N_k} \\ \Xi^{(2)}_{N_k} \end{matrix} \right] dX_k. \quad (12)$$

Using the explicit definition of Fox-$\bar{H}$ function given in [10, (A.2)], (12) can be rewritten in the form of Mellin-Barnes contour integral which is

$$Z_k = \int_0^\infty e^{-d_k X_k} \frac{1}{2\pi i} \oint_C \frac{\prod_{j=1}^{N_k} \Gamma(1-\alpha_j + A_j)^{a_j}}{\prod_{j=1}^{N_k} \Gamma(1-\beta_j + B_j)^{b_j}} e^{s X_k} ds \; dX_k. \quad (13)$$

Here, (α_j, A_j, a_j) and (β_j, B_j, b_j) correspond to the elements of j-th coefficient in (10). Then, interchanging the order of integrals in (13) and utilizing the Gamma function definition [19, 8.331.1], we obtain the following expression for the inner integral in (13)

$$\int_0^\infty e^{-d_k X_k} e^{s X_k} dX_k = \frac{1}{d_k - s} = -\frac{\Gamma(s-d_k)}{\Gamma(s-d_k+1)}. \quad (14)$$

Utilizing (14) and [10, (A.2)], Z_k can be found as

$$\begin{aligned} Z_k &= -\oint_C \frac{\prod_{j=1}^{N_k} \Gamma(1-\alpha_j + A_j)^{a_j}}{\prod_{j=1}^{N_k} \Gamma(1-\beta_j + B_j)^{b_j}} \frac{\Gamma(s-d_k)}{\Gamma(s-d_k+1)} ds \\ &= -\bar{\mathrm{H}}^{0,N_k+1}_{N_k+1,N_k+1} \left[1 \,\middle|\, \begin{matrix} \Xi^{(1)}_{N_{k,k}}, (1+d_k,1,1) \\ \Xi^{(2)}_{N_{k,k}}, (d_k,1,1) \end{matrix} \right], \end{aligned} \quad (15)$$

giving the result for $D_{(u,v),(\bar{u},\bar{v})}$ of

$$\begin{aligned} D_{(u,v),(\bar{u},\bar{v})} &= \prod_{k=1}^{K} \prod_{i=1}^{N_k} \left(\frac{m_{k,i}}{\Omega_{k,i}}\right)^{m_{k,i}} \\ &\times -\bar{\mathrm{H}}^{0,N_k+1}_{N_k+1,N_k+1} \left[1 \,\middle|\, \begin{matrix} \Xi^{(1)}_{N_{k,k}}, (1+d_k,1,1) \\ \Xi^{(2)}_{N_{k,k}}, (d_k,1,1) \end{matrix} \right]. \end{aligned} \quad (16)$$

C. $D_{(u,v),(\bar{u},\bar{v})}$ for integer m parameter

While (9) gives the distribution characteristics of X_k for any value of $m_{k,i}$, it includes the Fox-$\bar{H}$ function which currently has limited availability in the standard mathematical packages. Considering integer valued m parameters for Nakagami-*m* fading channels simplifies to the analysis in a significant way, where the results can be expressed as fractional products.

In the case of integer valued $m_{k,i}$, (9) simplifies to a closed form PDF expression given by [20]

$$f_{X_k}(X_k) = \prod_{i=1}^{N_k} \left(\frac{m_{k,i}}{\Omega_{k,i}}\right)^{m_{k,i}} G^{\kappa,0}_{\kappa,\kappa} \left(e^{-X_k} \,\middle|\, \begin{matrix} \psi^{(1)}_\kappa \\ \psi^{(2)}_\kappa \end{matrix} \right). \quad (17)$$

Here, $G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} \psi^{(1)}_\kappa \\ \psi^{(2)}_\kappa \end{matrix} \right)$ denotes the Meijer-G function [19], $\kappa = \sum_{i=1}^{N_k} m_{k,i}$ is an integer, and the coefficient sets $\psi^{(1)}_\kappa$ and $\psi^{(2)}_\kappa$ are defined as [20]

$$\psi^{(1)}_\kappa = \overbrace{\underbrace{\left(1 + \frac{m_{k,1}}{\Omega_{k,1}}\right), \cdots}_{}, \ldots, \left(1 + \frac{m_{k,N_k}}{\Omega_{k,N_k}}\right), \cdots}^{\kappa-\text{bracketed terms}}$$
$$\psi^{(2)}_\kappa = \overbrace{\left(\frac{m_{k,1}}{\Omega_{k,1}}\right), \cdots, \ldots, \left(\frac{m_{k,N_k}}{\Omega_{k,N_k}}\right), \cdots}^{\kappa-\text{bracketed terms}}. \quad (18)$$

where in each of $\psi^{(1)}_\kappa$ and $\psi^{(2)}_\kappa$ the first term is repeated $m_{k,1}-\text{times}$ and the last term $m_{k,N_k}-\text{times}$.

Averaging (6) over the PDFs of X_k in (17), $D_{(u,v),(\bar{u},\bar{v})}$ can be expressed by

$$D_{(u,v),(\bar{u},\bar{v})} = \prod_{k=1}^{K} \prod_{i=1}^{N_k} \left(\frac{m_{k,i}}{\Omega_{k,i}}\right)^{m_{k,i}} Z_k, \quad (19)$$

where Z_k is given by

$$Z_k = \int_0^\infty e^{-d_k X_k} G_{\kappa,\kappa}^{\kappa,0}\left(e^{-X_k}\left|\begin{matrix}\psi_\kappa^{(1)}\\ \psi_\kappa^{(2)}\end{matrix}\right.\right) dX_k \,. \tag{20}$$

Using the solution for (20) derived in Appendix A and rearranging the terms, the final expression for $D_{(u,v),(\bar{u},\bar{v})}$[5] is

$$D_{(u,v),(\bar{u},\bar{v})} = \prod_{k=1}^{K}\prod_{i=1}^{N_k}\left(1 + d_k \frac{\Omega_{k,i}}{m_{k,i}}\right)^{-m_{k,i}}. \tag{21}$$

IV. Numerical Results

In this section, we present simulation results to validate the upper bound BER expressions derived in Section III. In all cases considered herein, it is assumed that the wireless channel coefficients stay constant during one symbol duration, and the natural bit-to-symbol mapping rule is selected to generate output symbols from encoded bits[6].

In order to decode the same information bit along with different symbols assigned from different constellations, soft-decision Viterbi decoding is used once the ML rule metric is calculated from received signal within the orthogonal transmission phases. For all simulations, BER is plotted with respect to the average transmitted SNR per information bit, γ in a given orthogonal phase.

We consider four different scenarios with simulation parameters given in Table I. In the first two cases, a rate $R = 1/2$ convolutional encoder $[2,1]_8$ is used and the 2D constellations are generated by a random number generator for each orthogonal phase, given by $\chi^{(1)} = \{-1.33-0.50i, -0.16+1.88i, -0.37+0.82i, -1.25+0.17i\}$, $\chi^{(2)} = \{2.21-0.88i, -0.80+0.72i, -0.25-0.08i, -0.75+0.75i\}$, and $\chi^{(3)} = \{-1.96-0.31i, -0.78-1.09i, -0.57+1.15i, -1.09+0.79i\}$. In the last two scenarios, 64-ary signalling is used along with a rate $R = 1/3$ convolutional encoder $[133,171,165]_8$ specified in the 3GPP LTE standard for use in downlink control information (DCI) [21] is used. To generate arbitrary 64-ary constellations, we use the technique given in [18], originally proposed for generating non-uniform M-ary constellations. Specifically, a conventional 64-QAM constellation is first separated into two subsets, similar to the first step of the set partitioning process in conventional TCM design, then one of these subsets is rotated by a specific angle θ_k for each orthogonal phase. θ_k is chosen as $\theta_1 = 0$, $\theta_2 = \pi/7$ for Scenario-III and $\theta_1 = 0$, $\theta_2 = \pi/3$, $\theta_3 = \pi/6$ for Scenario-IV. Note that integer m values are only considering for the 64-ary signalling cases in order to reduce the simulation complexity using (21).

[5]Simplifying (21) for $N_1 = 1, K = 1$ reduces to a SISO case derived in [7]. Note that [7, (9)] contains a typo, where the second Ω in each factor should be replaced by "1".

[6]For instance, in a 64-ary signalling scheme, s_6 corresponds to the following 6 bits: 000110.

TABLE I: MONTE CARLO SIMULATION SCENARIOS FOR 4-ARY and 64-ARY SIGNALLING CASES

Scenarios	K	N_k	m_k	Ω_k
4-ary signalling Scenario-I	2	$N_1 = 2$ $N_2 = 3$	$m_1 = 2.5$ $m_2 = 3.5$	$\Omega_k = 1, \forall k$
4-ary signalling Scenario-II	3	$N_1 = 2$ $N_2 = 1$ $N_2 = 3$	$m_1 = 2.4$ $m_2 = 1.8$ $m_3 = 3.2$	$\Omega_1 = 0.9$ $\Omega_2 = 1.0$ $\Omega_3 = 0.8$
64-ary signalling Scenario-III	2	$N_1 = 2$ $N_2 = 1$	$m_1 = 2.0$ $m_2 = 3.0$	$\Omega_k = 1, \forall k$
64-ary signalling Scenario-IV	3	$N_1 = 2$ $N_2 = 1$ $N_3 = 1$	$m_1 = 2.0$ $m_2 = 1.0$ $m_3 = 1.0$	$\Omega_1 = 0.9$ $\Omega_2 = 1.0$ $\Omega_3 = 0.7$

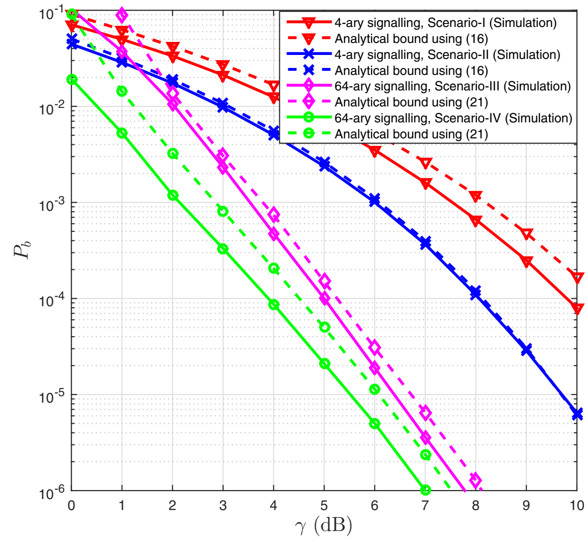

Fig. 2: Bit error probability of TMRC scheme for arbitrary 4-ary and 64-ary signalling.

Fig. 2 shows the BER values generated by Monte Carlo simulations and the derived upper bound BER expressions resulting from (11) for non-integer m values and (21) for integer ones. In addition to each orthogonal phase having different fading parameters, different Ω_k values enable us to represent the varying path-loss effects[7]. For instance, Scenario-II features around 1 dB SNR difference between the 2nd and the 3rd orthogonal transmission phases. Fig. 2 demonstrates that the developed upper bound BER expressions yield good agreement with simulations in the moderate and high SNR regions where P_b drops below 10^{-2} for both signalling and m parameter cases.

V. Concluding Remarks

We presented upper bound BER expressions for a generic, coded scheme which includes multiple orthogonal transmission phases and multiple transmit antennas. The proposed BER bound expressions based on the transfer function calculation from the product state matrix are compatible with any convolutional encoder and constellations where symbol locations can be completely arbitrary. This is contrast to previous methods which can only be used when the encoder and constellation satisfy the quasi-regularity

[7]The results (16) and (21) allow for different values of $m_{k,i}$ and $\Omega_{k,i}$ for each antenna in a given phase. This allows us to model a scenario of distributed antennas (in each phase). For simplicity, in this paper we consider only the case of co-located antennas which $m_k = m_{k,i}, \Omega_k = \Omega_{k,i}, \forall i$.

condition, likely to be violated in cases where optimized irregular constellations are used in the future systems.

Appendix A
Solution of (20)

Beginning with (20), we first perform the change of variable $y = e^{-X_k}$ giving

$$Z_k = \int_0^1 y^{d_k - 1} G^{\kappa,0}_{\kappa,\kappa}\left(y \,\middle|\, \begin{array}{c} \psi^{(1)}_\kappa \\ \psi^{(2)}_\kappa \end{array} \right) dy. \tag{22}$$

Using [19, (9.31.5)] and the definition of the Meijer-G function [22], changing the order of integrals allows us to write

$$\begin{aligned} Z_k &= \int_0^1 y^{d_k - 1} G^{\kappa,0}_{\kappa,\kappa}\left(y \,\middle|\, \begin{array}{c} \psi^{(1)}_\kappa \\ \psi^{(2)}_\kappa \end{array} \right) dy \\ &= \frac{1}{2\pi i} \oint_C \frac{\prod_{j=1}^{\kappa} \Gamma\left(\psi^{(2)}_{\kappa,j} - s\right)}{\prod_{j=1}^{\kappa} \Gamma\left(\psi^{(1)}_{\kappa,j} - s\right)} \frac{1}{s + d_k} ds. \end{aligned} \tag{23}$$

Using [19, 8.331.1] in (23) gives

$$Z_k = \frac{1}{2\pi i} \oint_C \frac{1}{\prod_{i=1}^{\kappa} \left(\frac{m_{k,i}}{\Omega_{k,i}} - s\right)} \frac{1}{s + d_k} ds, \tag{24}$$

and implementing the Cauchy's integral formula [23], which is

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz, \tag{25}$$

into (24) where

$$f(z) = \prod_{i=1}^{\kappa} \frac{1}{\frac{m_{k,i}}{\Omega_{k,i}} - z}, \quad a = -d_k, \tag{26}$$

results in

$$Z_k = \prod_{i=1}^{\kappa} \left(d_k + \frac{m_{k,i}}{\Omega_{k,i}} \right)^{-m_{k,i}}, \tag{27}$$

giving (21).

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