Investigating the Validity of the Gaussian Approximation for the Distribution of the Aggregate Interference Power in Large Wireless Networks

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***Abstract*—The distribution of the aggregate interference power in large wireless networks has gained increasing attention with the emergence of different types of wireless networks such as ad-hoc networks, sensor networks, and cognitive radio networks. The interference in such networks is often characterized using a Poisson Point Process (PPP). As the number of interfering nodes increases, there might be a tendency to approximate the distribution of the aggregate interference power by a Gaussian random variable given that the individual interference signals are independent. However, some observations in literature suggest that this Gaussian approximation is not valid except under some specific scenarios. In this paper, we cast these observations in a single mathematical framework and express the conditions for which the Gaussian approximation will be valid for the aggregate interference power generated by a Poisson field of interferers. Furthermore, we discuss the effect of different system and channel parameters on the convergence of the distribution of the aggregate interference power to a Gaussian distribution.**

***Index Terms*—Co-channel Interference, Cumulants, Berry-Esseen Bound, Fading Distribution, Poisson Point Process.**

I. Introduction

The interest in characterizing the distribution of the aggregate interference in large wireless networks has increased with the emergence of different types of wireless networks. Examples of these networks include wireless sensor networks, ad-hoc networks, and cognitive radio networks [1]–[4].

It is common to characterize the interference in large wireless networks using a Poisson Point Process (PPP). Therefore, the aggregate interference can be considered as the sum of a large number of independent interference signals. Thus, there might be a tendency to approximate the aggregate interference power by a Gaussian random variable. However, this approximation is not valid, except under certain conditions. Authors in [5] consider the interference in a CDMA network and indicate that the distribution of the aggregate interference power from users in other cells is likely to be Gaussian if there is a large number of interfering users in the vicinity of the victim cell. Authors in [6] consider the aggregate interference in a CDMA network as well and show that the distribution of the aggregate interference power converges to a Gaussian distribution as the traffic measure (which can be related to the average number of interferers in a cell) goes to infinity. Authors in [2] indicate that the Central Limit Theorem (CLT) does not apply in the case where some of the interferers are dominant although the number of interferers may be large. It is indicated in [3] that the Gaussian distribution is a bad approximation for the distribution of the aggregate interference when the node density is low. Based on simulation results, [7] shows that the Gaussian approximation could be acceptable when there is a wide-enough exclusion (no interferers) region around the victim receiver.

Observing that the aggregate interference can be modeled as shot noise, discussions in [8] on the convergence of shot noise to a Gaussian random variable becomes relevant to our study. Authors in [8] proved that under certain conditions the shot noise converges in distribution to Gaussian when the intensity (density) of the underlying point process of the shot noise goes to infinity. However, no discussion has been given to the effect of the exclusion region on this convergence. In this paper, we apply some of the results obtained in [8] to the case of the aggregate interference power in large wireless networks. We incorporate in formulations the effect of the exclusion region and identify the rate of the Gaussian convergence with respect to the size of the exclusion region. Moreover, we discuss the effect of fading distributions, including the small-scale and large-scale fading, on this convergence.

The paper is organized as follows. Section II describes the system model. Then, Section III establishes the mathematical framework to quantify how far away the distribution of the aggregate interference power is from a Gaussian distribution. Also, the same section discusses some system and channel parameters affecting the convergence to a Gaussian distribution. The effect of the fading distributions is addressed in Section IV. The paper is concluded by some remarks in Section V.

II. System Model

In this paper, we model a wireless network by a random field of interferes (active nodes). These active nodes are

This work was funded by Saudi Aramco, Dhahran, Saudi Arabia.

distributed over a two dimensional Euclidean space according to a homogenous Poisson Point Process (PPP) with an intensity (density) of $\lambda > 0$. A victim receiver is assumed to be located within the field. There might be an exclusion region of radius r_o around this victim receiver in which there are no active nodes.

The victim receiver receives an interference power of I_i from the transmission of an active node i. Under the assumption of incoherent addition of the interfering signals, the aggregate interference power received by the victim receiver can be expressed as

$$I_A = \sum_{i\in\Lambda} I_i = \sum_{i\in\Lambda} X_i g(r_i), \tag{1}$$

where Λ is the set of active nodes, X_i is a random variable resulting from the multiplication of some deterministic parameters and random variables. Without loss of generality, we assume the deterministic parameters in X_i to be equal to one. Therefore, we consider X_i as random variables modeling small-scale fading, large-scale fading, or composite fading. X_i are assumed to be i.i.d. which is a common assumption in similar works such as [2], [7], [9], [10]. The function $g(r_i)$ models distance-dependent attenuation where r_i is the Euclidean distance between node i and the victim receiver. In the rest of the paper, we call $g(r_i)$ the path-loss.

Some path-loss models used in literature suffer from a singularity at $r_i = 0$ which may affect the performance measures of the wireless network [11]. Therefore, we consider in our analysis the following non-singular model [12]

$$g(r_i) = \begin{cases} \alpha r_i^{-n}, & r_i \geq r_c \\ \alpha r_c^{-n}, & r_i < r_c \end{cases}, \tag{2}$$

where $r_c > 0$ is a critical distance below which $g(r_i)$ becomes constant. The constant parameter α can be assumed to be one without loss of generality. The path-loss exponent is denoted by n and assumed to be greater than 2.

III. Berry-Esseen Bound for the Distribution of the Aggregate Interference Power

Our investigation of the Gaussianity of the distribution of the aggregate interference power is based on the Berry-Esseen bound. Formulations for the Berry-Esseen bound when the underlying process is a stationary PPP are obtained in [8]. According to our assumptions, the underlying process for the aggregate interference is a homogenous PPP which implies that it is a stationary PPP [4]. Therefore, we can apply the results in [8] to the distribution of I_A, which leads to

$$|F_Z(y) - F_N(y)| \leq 2.21 \frac{\kappa_3(I_A)}{[\kappa_2(I_A)]^{\frac{3}{2}}}, \tag{3}$$

where $F_Z(y)$ is the CDF of the normalized aggregate interference power Z, i.e., $Z = \frac{I_A - \tilde{\mu}_A}{\sigma_A}$, $\kappa_2(I_A)$ and $\kappa_3(I_A)$ are the second and third cumulants of I_A, respectively. The function $F_N(y)$ denotes the CDF of the standard normal distribution, i.e., $\mathcal{N}(0,1)$.

Results obtained in [13] can be used to find $\kappa_2(I_A)$ and $\kappa_3(I_A)$. From [13], the mth cumulant of I_A can be written as

$$\kappa_m(I_A) = \lambda\pi \left[r_{\text{eff}}^2(m) - r_o^2\right] g(r_o)^m \tilde{\mu}_m(X_i), \tag{4}$$

where

$$r_{\text{eff}}(m) = \max(r_c, r_o)\sqrt{\frac{mn}{mn-2}}. \tag{5}$$

The parameter $\tilde{\mu}_m(X_i)$ denotes the mth raw moment of X_i, i.e., E$[X_i^m]$.

There are three possible topologies with respect to the exclusion region: an exclusion region with $r_o \geq r_c$, no exclusion region ($r_o = 0$), and an exclusion region with $0 < r_o < r_c$. We do not discuss the third topology here due to space limitations and since results for the third topology are bounded by the results of the first two topologies. Results for the third topology can be easily obtained from (3), (4) and (5).

1) Exclusion Region ($r_o \geq r_c$): For this topology, it can be shown from (4), (5), and (2) that

$$\kappa_m(I_A) = \frac{2\pi\lambda}{mn-2} r_o^{2-mn} \tilde{\mu}_m(X_i). \tag{6}$$

Thus, the Berry-Esseen bound yields

$$| F_Z(y) - F_N(y) | < 2.21 \frac{2(n-1)^{\frac{3}{2}}}{3n-2} \frac{1}{\sqrt{\lambda\pi r_o^2}} \frac{\tilde{\mu}_3(X_i)}{[\tilde{\mu}_2(X_i)]^{\frac{3}{2}}}. \tag{7}$$

Remarks:

- It is observed from (7) that the bound is mainly controlled by four variables: the path-loss exponent, the active node density, the radius of the exclusion region, and the fading distribution.
- The active node density is an important parameter in the convergence of the distribution of I_A to a Gaussian distribution. As λ increases, the bound becomes tighter and the distribution of I_A becomes closer to the Gaussian distribution, as shown in Fig. 1. It is observed from (7) that the rate of Gaussian convergence with respect to the increase in λ is $\sqrt{\lambda}$, which agrees with the findings in [8].
- Similarly, as the exclusion region increases, the bound in (7) becomes tighter. Hence, the distribution of I_A becomes closer to a Gaussian distribution, as shown in Fig. 2. However, the convergence caused by the increase in r_o is faster than the one caused by increasing λ. The rate of convergence with respect to the size of exclusion region is $\sqrt{\pi r_o^2}$.
- An explanation for this convergence with respect to the increase in λ and r_o is as follows. What really matters for the convergence to Gaussianity is the number of the dominant interferers around the victim receiver not the total number of interferes in the field. The number of dominant interferers is controlled mainly by λ and an effective area around the victim receiver. As λ increases, the number of dominant interferers increases. Similarly, as r_o increases the effective area increases and, hence the number of the dominant interferers increases. By virtue

of the CLT, as the number of the dominant interferers increases, the distribution of I_A converges to a Gaussian distribution.

- The lower n is, the better the convergence becomes. However, this effect on convergence is minor since the range of n is practically limited. Assuming $n \in (2, 6]$,

$$0.5 < \frac{2(n-1)^{\frac{3}{2}}}{3n-2} \leq 1.4. \qquad (8)$$

The effect of the fading distribution on the Gaussianity of I_A is discussed in Section IV.

2) No Exclusion Region ($r_o = 0$): If there is no exclusion region around the victim receiver, i.e., $r_o = 0$, then

$$\kappa_m(I_A) = \frac{mn\pi\lambda}{mn-2} r_c^{2-mn} \tilde{\mu}_m(X_i). \qquad (9)$$

Therefore, the Berry-Esseen bound can be expressed as

$$| F_Z(y) - F_N(y) | < 2.21 \frac{3(n-1)^{\frac{3}{2}}}{\sqrt{n}(3n-2)} \frac{1}{\sqrt{\lambda\pi r_c^2}} \frac{\tilde{\mu}_3(X_i)}{[\tilde{\mu}_2(X_i)]^{\frac{3}{2}}}. \qquad (10)$$

Remarks:

- From (10), it might be concluded that increasing the value of r_c improves the Gaussian approximation. However, r_c is used in the path-loss model to avoid the singularity at $r_i = 0$. Therefore, the value of r_c should be kept relatively small.
- To justify the Gaussian approximation, $\sqrt{\lambda\pi r_c^2}$ should be large. However, since r_c is relatively small then λ should be very high.

To sum up, the Gaussian approximation could be possible for wireless networks with a sufficiently wide exclusion region. However, the validity of the approximation is questionable when there is no exclusion region or when the exclusion region is small, unless the active node density is very high, which might be practically infeasible.

IV. Effect of Fading Distributions on the Gaussian Convergence of I_A

In this section, we investigate the effect of the distribution of X_i, i.e., the fading distribution, on the Gaussian convergence of the distribution of I_A. We provide expressions for $\tilde{\mu}_3(X)/[\tilde{\mu}_2(X)]^{\frac{3}{2}}$ considering different fading cases. These expressions can be used with (7) or (10) to get the related Berry-Esseen bounds.

A. Case 1: Without Multipath Fading and without Shadow Fading

For the case without fading (neither multipath nor shadow fading), X_i becomes deterministic. Therefore,

$$\frac{\tilde{\mu}_3(X_i)}{[\tilde{\mu}_2(X_i)]^{\frac{3}{2}}} = 1. \qquad (11)$$

We considered this case as the baseline to judge the effect of fading on the Gaussian convergence.

B. Case 2: With Multipath Fading but without Shadow Fading

The effect of multipath fading on the received individual interference power I_i can be modeled by a Gamma random variable (which is a result of the assumption that the interference signal can be modeled by the versatile Nakagami distribution). In this case, X_i is a Gamma distributed random variable with the PDF [14]

$$f_X(x) = \left(\frac{\nu}{\Omega}\right)^{\nu} \frac{x^{\nu-1}}{\Gamma(\nu)} e^{-\frac{\nu}{\Omega}x}, \ x > 0, \nu \geq \tfrac{1}{2}, \qquad (12)$$

where Ω is the average received power, ν is the shape parameter, and $\Gamma(.)$ is the Gamma function. The mth moment of X_i can be expressed as

$$\tilde{\mu}_m(X_i) = \left(\frac{\Omega}{\nu}\right)^{m} \frac{\Gamma(\nu+m)}{\Gamma(\nu)}. \qquad (13)$$

Thus,

$$\frac{\tilde{\mu}_3(X_i)}{[\tilde{\mu}_2(X_i)]^{\frac{3}{2}}} = \frac{\Gamma(\nu+3)}{[\Gamma(\nu+2)]^{\frac{3}{2}}} \Gamma(\nu)^{\frac{1}{2}} = \frac{\nu+2}{\sqrt{\nu(\nu+1)}}. \qquad (14)$$

We may conclude that the multipath fading shifts the distribution of the I_A away from Gaussianity. However, the shift is limited since, for $\frac{1}{2} \leq \nu < \infty$,

$$1 \leq \frac{\nu+2}{\sqrt{\nu(\nu+1)}} \leq 2.89. \qquad (15)$$

C. Case 3: With Shadow Fading but without Multipath Fading

The shadow fading is commonly modeled by a lognormal random variable with logarithmic mean 0 dB and standard deviation σ_s dB. Therefore, the effect of shadow fading on the aggregate interference power can be reflected by assuming that X_i is a lognormal random variable, and hence it has the following expression for its mth moment:

$$\tilde{\mu}_m(X_i) = e^{\frac{1}{2}\left(m\frac{\ln 10}{10}\sigma_s\right)^2}, \qquad (16)$$

which leads to

$$\frac{\tilde{\mu}_3(X_i)}{[\tilde{\mu}_2(X_i)]^{\frac{3}{2}}} = e^{\frac{3}{2}\left(\frac{\ln 10}{10}\sigma_s\right)^2}. \qquad (17)$$

For typical values of σ_s, e.g., $\sigma_s \in [4, 10]$ dB, the effect of shadow fading on $\tilde{\mu}_3(X)/[\tilde{\mu}_2(X)]^{\frac{3}{2}}$ and consequently on the Berry-Esseen bound could be dominant compared to the effect of multipath fading.

D. General Remarks

- Having multipath or shadow fading shifts the distribution of I_A away from Gaussianity. To maintain Gaussianity, the density of active nodes should be increased or the exclusion region should be extended.
- Figure 3 reflects the divergence from the Gaussian distribution that the fading distribution may cause. It is clear from the figure that the divergence caused by the shadow fading with $\sigma_s = 6$ dB is more than the one caused by the Rayleigh fading.

V. Conclusions

The emergence of different types of wireless networks has promoted the interest in characterizing the distribution of the aggregate interference power in large wireless networks. In this paper, we studied the convergence of this distribution to a Gaussian distribution. Based on the Berry-Esseen bound, we casted in a single mathematical framework some observations scattered across literature about the Gaussianity of the distribution of the aggregate interference power. We showed that an increase in the size of the exclusion region brings the distribution of the aggregate interference power closer to the Gaussian distribution. Increasing the active node density has a similar effect. However, the convergence is faster with the increase in the size of the exclusion region compared to the increase in the active node density. In contrast, channel fading causes divergence from Gaussianity. Shadow fading typically causes more divergence as compared to multipath fading.

Acknowledgment

The authors would like to thank Sebastian Szyszkowicz for his helpful comments.

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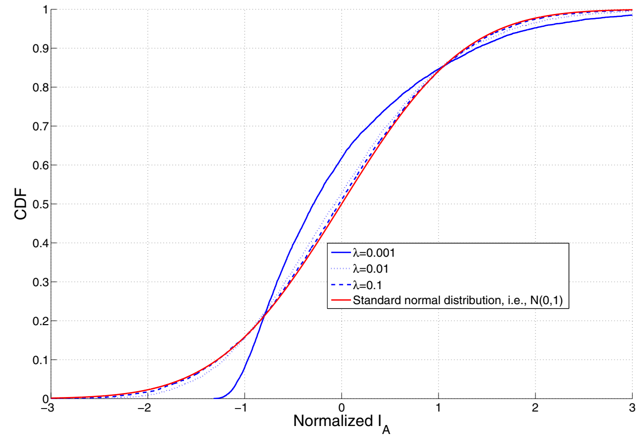

Fig. 1. Monte-Carlo simulation-based CDF for the normalized I_A, i.e., $\frac{I_A - \tilde{\mu}_A}{\sigma_A}$, for different values of λ ($r_o = 10$ m, $r_c = 1$ m, and $n = 3$).

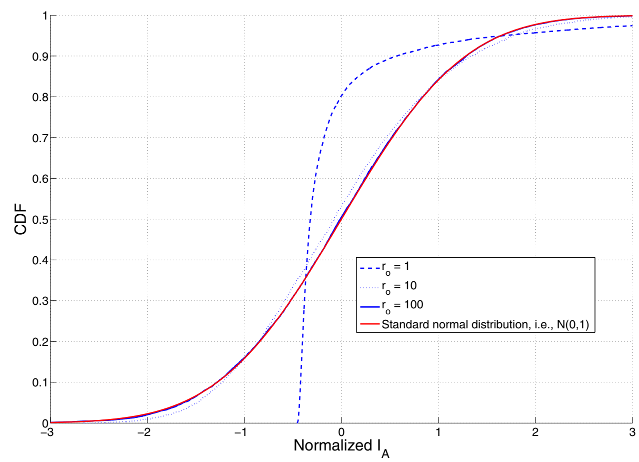

Fig. 2. Monte-Carlo simulation-based CDF for $\frac{I_A - \tilde{\mu}_A}{\sigma_A}$, for different values of r_o ($\lambda = 0.01$ nodes/m^2, $r_c = 1$ m, and $n = 3$).

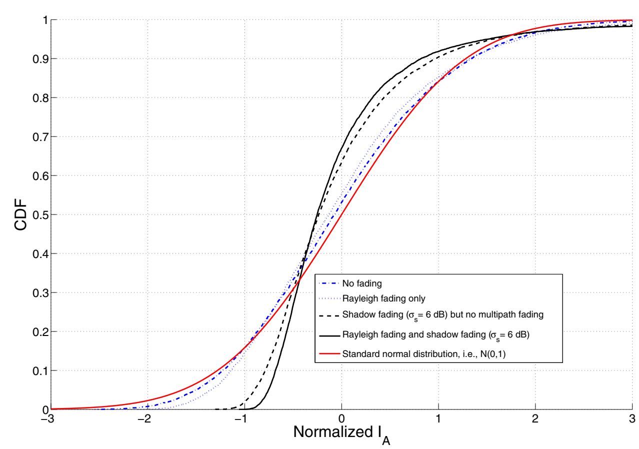

Fig. 3. Monte-Carlo simulation-based CDF for $\frac{I_A - \tilde{\mu}_A}{\sigma_A}$, for different fading scenarios ($r_o = 10$ m, $r_c = 1$ m, $n = 3$, and $\lambda = 0.01$ nodes/m^2).

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