An Upper Bound on BER in a Coded Two-Transmission Scheme with Same-size Arbitrary 2D Constellations

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***Abstract*—Constellation design is a well studied topic. For instance, it is known that, when 2-dimensional (2D) signalling schemes are used, the performance gains that the optimal constellations yield in comparison to the commonly employed square constellations are rather small. However, most of the earlier literature in this area considers rather simple communication protocols (for instance, without retransmissions), even in the absence of channel coding. With the advent of advanced communication protocols as well as signal processing techniques, there has been a rejuvenated interest in constellation design in recent years.**

In this paper, we consider a generic two-transmission scheme which may correspond to a relay, HARQ (hybrid automatic repeat request), or CoMP (coordinated multipoint) based transmission scenario, with a maximum likelihood receiver. The system has the flexibility of using a different 2D constellation in each transmission (however, the constellation size, i.e., the number of bits per symbol, remains the same). A generic channel coding scheme for which an encoder transfer function can be written (such as, convolutional and turbo codes) is considered for the versatile Nakagami-*m* fading channel. The main contribution of this paper is the derivation of an upper bound on the bit error rate (BER) which is expressed as a function of the distances between the constellation points. Using proper optimization techniques, this bound (tight for the high SNR values) which captures the impact of the distances between the constellation points can enable the design of good constellations for a given coding scheme in the above explained two-transmission setting.

I. Introduction

Signal constellation design is one of the most fundamental problems in digital communications; as a matter of fact, the history of constellation design goes back even before Shannon's landmark work [1]. Having said that, constellation design has become a more structured problem only after the signal space analysis has become the standard design tool with the seminal textbook of Wozencraft and Jacobs in 1965 [2]. For a good overview of the area until 1980s, please refer to [3], [4] and the references therein.

Interestingly, in recent years there has been a rejuvenated interest in this most classic digital communications problem due to at least three reasons:

- In the last two decades, the more advanced optimization tools have become increasingly available to the physical layer researchers; in particular, another seminal textbook played an important role here [5]. In the absence of optimization as a tool, most of the earlier constellation design work has been confined to cases in which the possible signal point locations are on regular lattices [6] or which have other restrictive constraints. With the utilization of the optimization techniques, on the other hand, it is possible to construct constellations with minimal or no constraints on the signal point locations except for the inevitable ones to make the problem realistic, such as the average energy or peak power constraint, or the ones introduced to study a particular aspect, such as the constant-modulus or the peak-to-average-power ratio constraint.
- Conventionally, modulation (and thus, signal constellation design) has been considered as a standalone operation in the transmitter design. With the advent of coded modulation, it has become possible to consider channel coding and modulation as a joint process [7]–[9]. This view creates new opportunities in the joint design or in designing good constellations for a given channel encoder.
- The application and scenario possibilities in communications are ever-increasing especially in the context of the perceived 5G wireless networks. The early work in digital communications in general, and in constellation design in particular, has naturally been on rather simple point-to-point communications scenario. The scenarios are getting diversified as point-to-multipoint and multipoint-to-multipoint with more advanced architectures, including the relay, mesh, CoMP (coordinated multi-point), distributed antennas, multi-user, and network coding scenarios. The envisioned applications necessitate a broad range of devices especially with the advent of machine-type communications. The device costs may range from less than a dollar (low end sensors) to thousands of dollars (virtual reality and presence gadgets), the consequence of which is that the energy constraints and signal processing capabilities of these devices will be orders of magnitude different. Moreover, some of the novel design aspects in future networks may have implications on constellation design. For instance, the use of millimeter waves has been articulated in 5G networks. In those frequencies phase noise is often a concern; therefore, constellation design robust against phase noise may emerge as a relevant problem.

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The above factors, compounded with the progressed understanding in the foundational dynamics of constellation design as a result of decades of research (such as, the realization thatthere is no single optimum constellation for all signal-to-noise ratio (SNR) levels [10]), have resulted in a rejuvenated interest in this area.

As pointed out in the second bullet given above, the earlier studies have focused on mainly the uncoded systems with rather simple point-to-point scenarios [3], [6], [11]–[13]. These studies differ from each other with respect to the constellation structure, (such as, the non-structured constellations [6], [14] and the pre-defined ones [15]) and the constellation dimensionality (such as, 2D [6] and multidimensional ND constellations including lattice codes [13], [15]–[17]).

In recent years, the constellation design problem has increasingly been studied through the use of optimization techniques. For instance, in [18], the maximization of the Euclidean distance between the signal points is casted as a non-convex optimization problem with quadratic constraints. It is shown in that study that a simple iterative method provides favourable performance results. Another example is [19] in which constellation design along with bit-to-symbol mapping (labelling) in the presence of phase noise is studied through simulated annealing.

The fact that utilizing properly designed constellations may result in considerable performance gains in uncoded systems as highlighted above, has resulted in rejuvenated interest in the good old constellation design problem, but this time for coded cases. The optimum 2D constellation design through trellis coded modulation (TCM) is investigated in [20]–[22]. The extended studies of constellation design on TCM encoders over multidimensional signal space are proposed in [23], [24]. The use of asymmetric 2D constellations for TCM encoders is first proposed in [20] by maximizing the minimum Euclidean distance over Gaussian channels; then in [21], an asymmetric 8-PSK design based on minimizing the pairwise error probability (PEP) in the high SNR regime is provided for fading channels. Another optimization framework based on a recently proposed BER bound expression is evaluated in [22]. In addition to the above mentioned system scenarios, the constellation design problem has also been investigated over more advanced channel coding techniques such as the low density parity check (LDCP) codes [25], [26], and the bit interleaved coded modulation (BICM) [27]–[29].

Constellation design problem has also been addressed in cooperative communication. It is shown in [30] that using different constellations over different transmission phases (called constellation rearrangement, CoRe) yields significant error performance enhancement for uncoded systems by utilizing a convex optimization framework through a union bound based symbol error rate (SER) expression.

As a matter of fact, developing constellation designs is not the goal of this paper. Rather, this paper is on the performance analysis with the aim of enabling the design of better constellations. Towards that end, the metric under consideration, BER (actually, an upper bound), is expressed in terms of the distances between the constellation points in a rather generic scenario in which two orthogonal transmission phases exist. These two orthogonal transmissions can correspond to a number of network realizations including relaying, CoMP, or HARQ systems. In each of the two transmissions, the same information is sent, but through different constellations whose points can be located anywhere on the 2D space; the only two restrictions are that the average symbol energy is kept fixed as well as the constellation size (i.e., the same fixed number of bits per symbol) in both transmissions. The underlying motivation is that the explicit dependence of distances in the BER upper bound can be exploited towards determining good constellations. Indeed, a Euclidean distance-based BER bound with inter-symbol distances has already been derived in [31] for a coded maximal ratio combining (MRC) system with one transmitter and M receivers in a Nakagami-m channel (since there is one transmission in [31], there is also only one constellation which is M-PSK). The BER bound in [31] is quite versatile as it is applicable to any coded system as long as a transfer function can be written for the coding scheme used (i.e., convolutional and turbo codes, and TCM).

Getting inspired from [31], we consider in this paper a two-transmission scheme (rather than multiple received signals) with a more general system set up as highlighted below, keeping the constellation design goal in mind. The two transmissions

- employ different and arbitrary 2D constellations;
- experience independent Nakagami-m channels, but not necessarily identically distributed (that is, the m values of the two channels can be different);
- experience different path-losses.

This more general model is necessary to cover the CoMP and relay scenarios.

The rest of the paper is organized as follows: In Section II, the two-transmission system model is described. The BER upper bound expressions for independent and identically distributed as well as for independent but non-identically distributed Nakagami-m channels are obtained in Section III. The simulation results validating the analytical results are presented in Section IV. The conclusions are given in Section V.

II. System Model

We consider a coded two transmission system model where two orthogonal transmissions occur. This model captures the CoMP, relaying, and HARQ scenarios as shown in Fig. 1. It is assumed that the same convolutional encoders are used during both the first and second transmissions and the channels during both transmissions are modelled as frequency non-selective Nakagami-m fading. The same output bits of the encoder can map into different output symbols in the case of two different constellations used for two orthogonal transmissions. The lth coded symbol in the codeword is denoted as s_l^i where i refers to corresponding signal points assigned from ith signal constellations. The received signals for lth symbol in the codeword at the destination after the first and second transmission phases completed can be formulated as

$$\begin{aligned} r_{l,1} &= h_{l,1}s_l^{(1)} + n_{l,1} \\ r_{l,2} &= h_{l,2}s_l^{(2)} + n_{l,2}, \end{aligned} \tag{1}$$

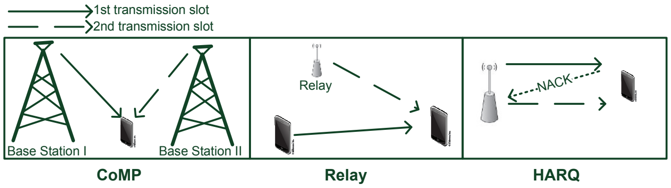

Fig. 1. Possible system realizations for the two orthogonal transmission phases scenario.

where $h_{l,i}$ ($i \in \{1,\ 2\}$) refers to small-scale fading coefficients during the first and second transmission phases which is modelled by Nakagami-*m* distribution with shaping parameters m_1 and m_2, respectively. $n_{l,i}$ are additive white Gaussian noise (AWGN) samples with zero-mean $N_0/2$ variance per dimension.

III. BER Bound Calculation

To find BER bound expression, pairwise error probability (PEP) expression is firstly obtained for the mentioned coding scenario. Generally, definition of the PEP is given as the probability of decoding erroneous codeword $\hat{S}$ corresponding to original transmitting codeword S and it can be formulated as

$$P\left(S \to \hat{S}\right) = K_c \times \prod_{l=1}^{L} W\left(s_l, \hat{s}_l\right), \tag{2}$$

where $W\left(s_l, \hat{s}_l\right)$ is error weight profile between s_l and $\hat{s}_l$, K_c is tightening constant independent from error sequence and $K_c = 1$ corresponds to Chernoff bound [31]. If maximum likelihood (ML) decision rule is selected to decode the two different received signals depending on different signal constellations from the first and second transmission slots, the conditional PEP expression can be explicitly written in the following form:

$$P\left(S \to \hat{S}\,|H\right) = P\left(\sum_{l=1}^{L}\sum_{i=1}^{2}\left(\left|r_{l,i} - h_{l,i}s_l^i\right|^2 - \left|r_{l,i} - h_{l,i}\hat{s}_l^i\right|^2 \geq 0\,|H\right)\right). \tag{3}$$

In the above, H denotes channel coefficient matrix that includes each channel coefficient corresponding to each symbol in the codeword where its length is L. After some mathematical manipulations, conditional PEP expression for ML decoder can be given as

$$P\left(S \to \hat{S}\,|H\right) = \frac{1}{2}\text{erfc}\left(\sqrt{\sum_{l=1}^{L}\sum_{i=1}^{2}\frac{\left|h_{l,i}\right|^2\left|s_l^i - \hat{s}_l^i\right|^2}{4N_0}}\right). \tag{4}$$

For simplification in (4), the new variables are introduced for (3) such that $d_{l,i} = \frac{\left|s_l^{(i)} - \hat{s}_l^{(i)}\right|^2}{4N_0}$, $\gamma_{l,i} = \left|h_{l,i}\right|^2$ where $i \in \{1,\ 2\}$ and then, the conditional PEP expression shown in (3) can be rewritten as

$$P\left(S \to \hat{S}\,|H\right) = \frac{1}{2}\text{erfc}\left(\sqrt{\sum_{l=1}^{L}\gamma_{l,1}d_{l,1} + \gamma_{l,2}d_{l,2}}\right). \tag{5}$$

By taking expectation over H channel coefficient matrix, each entry having Nakagami-*m* distributed variable, to eliminate conditional probability in (5), the unconditional PEP expression can be obtained as

$$\begin{aligned} P\left(S \to \hat{S}\right) = \tfrac{1}{2}\int_0^\infty \ldots \int_0^\infty \text{erfc}\left(\sqrt{\sum_{l=1}^{L}\gamma_{l,1}d_{l,1} + \gamma_{l,2}d_{l,2}}\right) \\ f_\gamma\left(\gamma_{1,1}\right) f_\gamma\left(\gamma_{2,1}\right)\ldots f_\gamma\left(\gamma_{L,1}\right) \\ f_\gamma\left(\gamma_{1,2}\right) f_\gamma\left(\gamma_{2,2}\right)\ldots f_\gamma\left(\gamma_{L,2}\right) \\ d_{\gamma_{1,1}} d_{\gamma_{2,1}}\ldots d_{\gamma_{L,1}} d_{\gamma_{1,2}}, \end{aligned} \tag{6}$$

where $f_\gamma\left(\gamma_{l,i}\right)$ denotes the probability density function of the channel coefficient corresponding to ith link (first or second) modelled by Nakagami-*m* and it is defined as [31]

$$f_{\gamma_{l,i}}\left(\gamma\right) = \frac{{m_i}^{m_i}}{\Gamma(m_i)}\gamma^{m_i - 1}e^{-m_i\gamma}. \tag{7}$$

Here, $\Gamma\left(.\right)$ is the Gamma function [32] and $\gamma \geq 0, m_i \geq 0.5, \quad \{i \in \{1,\ 2\},\ l \in L\}$. By changing the variables like in [33] such that,

$$\delta_{l,i} = \frac{d_{l,i}}{1 + \frac{d_{l,i}}{m_i}}, \quad \omega_{l,i} = \gamma_{l,i}\left(1 + \frac{d_{l,i}}{m_i}\right) \tag{8}$$

and after rearranging the terms in (8), the unconditional PEP expression turns into

$$\begin{aligned} P\left(S \to \hat{S}\right) = \prod_{l=1}^{L_\eta}\left(1 + \tfrac{d_{l,1}}{m_1}\right)^{-m_1}\prod_{l=1}^{L_\eta}\left(1 + \tfrac{d_{l,2}}{m_2}\right)^{-m_2} \\ \times\int_0^\infty\ldots\int_0^\infty \text{erfc}\left(\sqrt{\sum_{l=1}^{L_\eta}\delta_{l,1}\omega_{l,1} + \delta_{l,2}\omega_{l,2}}\right) \\ \times\exp\left(\sum_{l=1}^{L_\eta}\delta_{l,1}\omega_{l,1} + \delta_{l,2}\omega_{l,2}\right) f_\omega\left(\omega_{1,1}\right)\ldots f_\omega\left(\omega_{L_\eta,1}\right) \\ f_\omega\left(\omega_{1,2}\right)\ldots f_\omega\left(\omega_{L_\eta,2}\right) d_{\omega_{1,1}}\ldots d_{\omega_{L_\eta,1}} d_{\omega_{1,2}}\ldots d_{\omega_{L_\eta,2}}. \end{aligned} \tag{9}$$

In the above, L_η denotes the minimum time diversity of the code, a path having the minimum Hamming distance among the all possible code words with zero sequences [34]. It is also important to note that the distribution of the variable ω_l have the same form of the variable γ_l that is given in (7). To obtain a closed form expression for (9), the following approximation is made as

$$\sum_{l=1}^{L_\eta}\delta_{l,1}\omega_{l,1} + \delta_{l,2}\omega_{l,2} \geq \left(\sum_{l=1}^{L_\eta}\omega_{l,1} + \omega_{l,2}\right)\delta_m, \tag{10}$$

where $\delta_m = \min\left\{\left(\delta_{l,1}, \delta_{l,1}\right),\ l \in L_\eta\right\}$ and then, it is seen in [31] that $\text{erfc}(\sqrt{x})e^x$ is monotonically decreasing function over x so the upper bound PEP expression can be calculated as

$$\begin{aligned} P\left(S \to \hat{S}\right) \leq \tfrac{1}{2}\prod_{l=1}^{L_\eta}\left(1 + \tfrac{d_{l,1}}{m_1}\right)^{-m_1}\prod_{l=1}^{L_\eta}\left(1 + \tfrac{d_{l,2}}{m_2}\right)^{-m_2} \\ \times\int_0^\infty \text{erfc}\left(\sqrt{\Omega\delta_m}\right) f_\Omega(\Omega)d\Omega, \end{aligned} \tag{11}$$

where $\Omega = \sum_{l=1}^{L_\eta}\omega_{l,1} + \omega_{l,2}$ and $f_\Omega(\Omega)$ can be given by

$$\begin{aligned} f_\Omega(\Omega) = \frac{e^{-m_2\Omega}m_1^{L_\eta}m_2^{L_\eta}\Omega^{L_\eta(m_1+m_2-1)}}{\Gamma\left(L_\eta(m_1+m_2)\right)} \\ \times{}_1F_1\left(L_\eta m_1;\ L_\eta(m_1+m_2);\ (m_2 - m_1)\,\Omega\right). \end{aligned} \tag{12}$$

In the above, ${}_1F_1\left(a;\ b;\ z\right)$ denotes Kummer confluent hypergeometric function [32]. After substituting (12) into (11), unconditional upper bound PEP expression is given as

$$
\begin{aligned}
P\left(S\rightarrow\hat{S}\right) \leq & \prod_{l=1}^{L_\eta}\left(1+\frac{d_{l,1}}{m_1}\right)^{-m_1}\left(1+\frac{d_{l,2}}{m_2}\right)^{-m_2} \\
& \left\{\frac{1}{2}-\frac{1}{\sqrt{\pi}}\frac{m_1^{L_\eta m_1}m_2^{L_\eta m_2}}{\Gamma\left(L_\eta(m_1+m_2)\right)}\sum_{p=0}^{\infty}\frac{(-1)^p\delta_m^{2p+1}}{p!(2p+1)}\right. \\
& \times\Gamma\left(0.5+p+L_\eta\left(m_1+m_2\right)\right)\left(m_2-\delta_m\right)^{-0.5-p-L_\eta(m_1+m_2)} \\
& \left.\times{}_2F_1\left(L_\eta m_1,\ p+L_\eta\left(m_1+m_2\right)+0.5;\ L_\eta\left(m_1+m_2\right);\frac{m_1-m_2}{\delta_m-m_2}\right)\right\}
\end{aligned}
\tag{13}
$$

by utilizing $\operatorname{erfc}(x) = 1-\frac{2}{\sqrt{\pi}}\sum_{p=0}^{\infty}\frac{(-1)^p x^{2p+1}}{p!(2p+1)}$ and ${}_2F_1(a,b;c;z)$ denotes Gauss hypergeometric function [32]. After having PEP expression as in (13), the bound expression of BER for the two transmission coding case can written as

$$
\begin{aligned}
P_b \leq & \frac{1}{k}\left\{\frac{1}{2}-\frac{1}{\sqrt{\pi}}\frac{m_1^{L_\eta m_1}m_2^{L_\eta m_2}}{\Gamma\left(L_\eta(m_1+m_2)\right)}\sum_{p=0}^{\infty}\frac{(-1)^p\delta_m^{2p+1}}{p!(2p+1)}\right. \\
& \times\Gamma\left(0.5+p+L_\eta\left(m_1+m_2\right)\right)\left(m_2-\delta_m\right)^{-0.5-p-L_\eta(m_1+m_2)} \\
& \left.\times{}_2F_1\left(L_\eta m_1,\ p+L_\eta\left(m_1+m_2\right)+0.5;\ L_\eta\left(m_1+m_2\right);\frac{m_1-m_2}{\delta_m-m_2}\right)\right\} \\
& \times\left.\frac{\partial T(D,I)}{\partial I}\right|_{D=\left(1+\frac{d_{l,1}}{m_1}\right)^{-m_1}\left(1+\frac{d_{l,2}}{m_2}\right)^{-m_2},\,I=1.}
\end{aligned}
\tag{14}
$$

In (14), $T(D,I)$ corresponds to transfer function of convolutional encoder [35] where $D=D_1,D_2,...$ denotes the distance between a codeword and zero-sequence and the exponent of the I is the Hamming weight of corresponding input sequence [36]. For the case of $m_1=m_2=m$, (14) can be simplified

$$
\begin{aligned}
P_b \leq & \frac{1}{k}\left(\frac{m^{2L_\eta m}(m-\delta_m)^{-2L_\eta m}\Gamma\left(2L_\eta m\right)}{\Gamma\left(2L_\eta m\right)}-\right. \\
& \frac{2(m-\delta_m)^{-0.5-2L_\eta m}\sqrt{\delta_m}\,\Gamma\left(0.5(1+4L_\eta m)\right)}{\sqrt{\pi}} \\
& \left.\times{}_2F_1\left(0.5,0.5+2L_\eta m;1.5;-\frac{\delta_m}{m-\delta_m}\right)\right) \\
& \times\left.\frac{\partial T(D,I)}{\partial I}\right|_{D=\left(1+\frac{d_{l,1}}{m}\right)^{-m}\left(1+\frac{d_{l,2}}{m}\right)^{-m},\,I=1.}
\end{aligned}
\tag{15}
$$

It is important to note that both bound expression (14) and (15) can be also utilized under the impact of propagation path loss. In this case, only following change of variables is needed $d_{l,i}=\frac{\left|s_l^{(i)}-\hat{s}_l^{(i)}\right|^2}{4N_0}\rightarrow d_{l,i}=\frac{\left|s_l^{(i)}-\hat{s}_l^{(i)}\right|^2}{4N_0L_l}$ where L_l shows path-loss factor in the lth transmission phase.

IV. Numerical Results

In this section, upper bound BER expression expressed as (15) is validated by numerical results over computer simulations. For the simplicity, it is considered first and second transmission links have same Nakagami-*m* shaping parameter so upper bound BER curves are obtained from (15). In numerical evaluations, both transmission slots use the same convolutional encoder defined as $[17,13,5,7]_8$ in the octal form and the encoded bits are modulated by 16-QAM by natural bit to symbol mapping. It is also assumed that channel coefficients are constant during one symbol transmission and soft-decision Viterbi decoding is used in the receiver side.

The design of specific non-uniformly distributed constellations differing from conventional M-QAM or M-PSK cases is beyond the scope of this paper. For just an example in simulations, the technique given in [20] is exploited. Specifically, after the conventional 16-QAM is separated into two subset like in the case the set partitioning process, one of subset is rotated by specific angle θ. Then, the new asymmetric constellation sets can be obtained by combining the original and rotated subsets for $\theta=\pi/4$ (arbitrarily chosen) shown in Fig. 2.

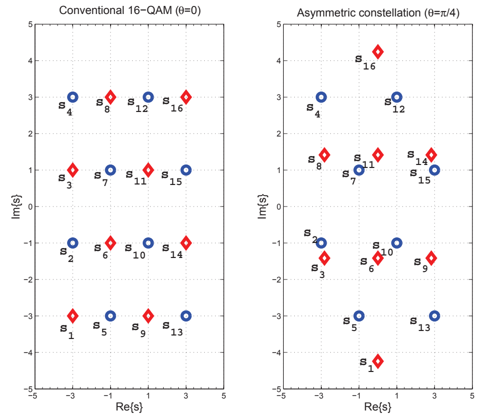

Fig. 2. Conventional 16-QAM and asymmetric constellation cases.

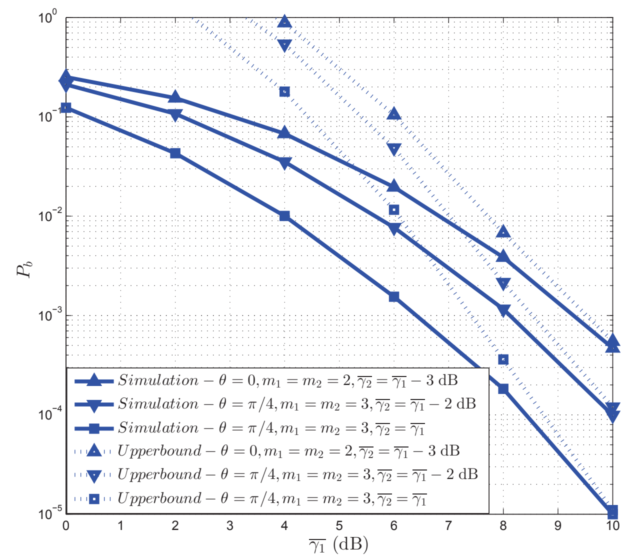

Fig. 3. Bit error probability of the two-transmission scheme over the Nakagami-m cases.

In Fig. 3, BER results obtained by Monte Carlo simulations and analytical expression given in (15) respect to the first transmission SNR values ($\overline{\gamma_1}$) are plotted together in the case of using conventional 16-QAM in the first transmission and either conventional 16-QAM or the constellation shown in Fig. 2 with $\theta=\pi/4$ in the second one. It can be seen that derived upper BER bound expression becomes tighter with increasing values of SNR. Also, the BER bound and simulation curves are also given for asymmetric scenario where second transmission phase ($\overline{\gamma_2}$) has path loss component and bound expression also shows the same manner in this asymmetric scenario. As seen from figure, bound expression has considerable potential to find optimized constellations for the future research.

V. Conclusions

This paper presents an upper bound expression on BER for a generic two-transmission scheme which can be considered as relay, HARQ or CoMP scenarios. The bound expression is based on the distances between constellation points so it enables working with any form of 2D constellation over two orthogonal transmission phases. To validate the tightness of the upper bound expression, Monte Carlo simulation curves are shown in the case of conventional M-QAM and also arbitrarily chosen asymmetric 2D constellations.

Since the BER bound expression herein only needs a generic transfer function form for any given encoder i.e., convolutional and turbo codes, it can be exploited on the purpose of finding optimum constellations over various coded two-transmission scenarios to obtain better error performance.

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