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Optimal Tradeoff between Efficiency and Jain's Fairness Index in Resource Allocation

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***Abstract*—In this paper, we study tradeoff policies between efficiency and the Jain's fairness index of the benefits received by M users in general resource allocation scenarios. Analyzing the commonly-used α-fair tradeoff policy, it is shown that, except for the case of $M = 2$ users, this policy does not necessarily achieve the optimal Efficiency-Jain tradeoff. In particular, it is shown that, when the number of users $M > 2$, the gap between the efficiency achieved by the α-fair and the optimal Efficiency-Jain tradeoff policy can be unbounded, for the same Jain's index. Finding the optimal Efficiency-Jain tradeoff for arbitrary set of admissible benefits is generally difficult. To alleviate this difficulty, we derive sufficient conditions, which, when satisfied by the set of admissible benefits, lead to efficiently computable optimal tradeoff and benefit vectors. Numerical results for a typical communication network scenario are provided to confirm analytical findings.**

I. Introduction

In several systems, including wireless communication ones, certain services, which are referred to as benefits, are provided to multiple users based on the allocation of shared resources that are typically scarce. The allocation of these resources typically encounters conflicting goals. For instance, favouring a certain class of users may increase the system's efficiency, but would result in the dissatisfaction of other classes of users. In contrast, providing equal benefits to all users may result in higher fairness, but will potentially result in low efficiency. To control the emphasis placed on various goals, the provider uses a tradeoff policy, which, unless properly chosen, can result in wasteful allocation of resources. In particular, an unoptimal tradeoff policy can be less efficient and, at the same time, less fair to the users [1]–[3].

In the downlink of wireless communication systems, the benefits can be defined as the rates of the data delivered to the users. These rates are controlled by appropriate allocation of radio resources at the transmitters. For instance, the transmitter may allocate its resources in such a way that maximizes the sum of the rates delivered to the users. This allocation favours users that are geographically closer to the transmitter, but "starves" farther users, and although more efficient from the system's perspective, such an allocation is unfair to the users at less advantageous locations. A fairer allocation is one in which the minimum rate received by the users is maximized. However, this allocation can result in unacceptable system efficiency; i.e., low sum rate. Hence, it is desirable to find an optimal tradeoff policy whereby the system provider allocates its resources in such a way that no other allocations can provide a strictly higher efficiency and at the same time be fairer to the users. The focus of this paper is to provide a technique for obtaining the optimal efficiency-fairness tradeoff and to derive sufficient conditions, which, when satisfied by the set of admissible benefits, lead to efficiently computable optimal tradeoff and benefit vectors.

Although our analysis is performed with communication-related frameworks in mind, it applies to a wider scope of frameworks, including social and economics ones [2], [4], [5].

To study the tradeoff between efficiency and fairness, these quantities must be defined in a mathematically precise way. While efficiency is usually well-defined depending on the context, several definitions are used to quantify fairness, including, e.g., min-max ratio and entropy-based indices [2]. Among these definitions is the so-called Jain's index [6]. This index is a bounded continuous function in the benefits and a metric that conforms with standard fairness benchmarks. For instance, it was shown in [6] that the Jain's index corresponding to a situation in which $p\%$ of users receives equal benefits and the remaining $(100 - p)\%$ receives zero is equal to $p/100$. Motivated by these features, in this paper the Jain's index will be used as the basis for comparing fairness. A geometric interpretation that commends its use will be provided hereinafter.

A common approach to trading off efficiency with fairness in wireless networks, is to allocate the resources in a way that maximizes the network efficiency while ensuring that the minimum rates achieved by the users exceed some prescribed bounds, e.g., [3], [5]. Varying these bounds over the set of admissible rates provides a means for controlling fairness [3]. One extreme corresponds to maximum efficiency, whereas the other extreme corresponds to the so-called max-min fairness [3]. Another approach is to allocate the resources in a way that maximizes a parametric utility, whereby one, or multiple, parameters are used to control the emphasis on efficiency and fairness. A commonly used policy is the α-fair one (also known as the α-fair utility) [1], wherein various settings of a parameter α yield allocations that achieve established efficiency-fairness tradeoffs. For instance, setting $\alpha = 0$ yields maximum efficiency, setting $\alpha = 1$ yields proportionally fair allocations [7], and setting $\alpha = \infty$ yields allocations that are fair in the max-min sense [1]. Motivations for using the α-fair policy are provided in [2]. Generally speaking, increasing α results in allocations that are fairer [2] in a sense, that does not

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necessarily conform to the Jain's index one, as will be shown hereinafter. Other parametric utilities for trading off efficiency and fairness are considered in [8], [9], and a comparison between multiple tradeoff criteria is provided in [10].

As mentioned earlier and highlighted in [6], compared with other measures, Jain's index provides a fairness criterion that takes into consideration all the users of the system, not only those users that are assigned minimal resources. Hence, maximizing this index while avoiding wasteful allocation of resources requires the determination of the optimal tradeoff between efficiency and this index. A question that arises in this case is whether maximizing the well-studied α-fair policy yields such an optimal tradeoff. To address this question, we begin in this paper by showing that α-fair allocations are not guaranteed to achieve the optimal tradeoff between efficiency and Jain's index except for the case of $M = 2$ users. To overcome this drawback, we develop a generic technique for obtaining allocations that are optimal from a tradeoff perspective. Unfortunately, solving the set of optimization problems that underlie this technique is generally difficult. To alleviate this difficulty, we derive sufficient conditions that enable us to identify sets of admissible benefits for which these problems are relatively easy to solve. Interestingly, it is shown that these sufficient conditions are satisfied by a wide range of resource allocation problems in communications networks. Numerical results are provided for confirming our theoretical findings and for demonstrating the advantage of the optimal tradeoff provided by our technique over the α-fair one.

II. Preliminaries

Let $\mathbf{x} \in \mathcal{C} \subseteq \mathbb{R}_+^M$ denote a length-M vector of non-negative real entries $\{x_i\}_{i=1}^M$, where x_i denotes the benefit received by user i and $\mathcal{C}$ is the set of admissible benefit vectors. Generally, the benefits $\{x_i\}$ and the set $\mathcal{C}$ depend on the application and the resources allocated to each user [6, Sec. 5]. For example, in the downlink of wireless communication systems, x_i can be the rate of user i resulting from a particular allocation of the radio resources, and $\mathcal{C}$ is the set of all achievable rates.

In this paper, the efficiency, $\eta(\mathbf{x})$, of a resource allocation is defined to be the sum of the resulting benefits (i.e., $\eta(\mathbf{x}) = \sum_{i=1}^M x_i$), and its fairness is defined to be the Jain's index defined below.

Definition 1 (Jain's Index). *For a given vector $\mathbf{x} \in \mathbb{R}_+^M$, the Jain's fairness index $J : \mathbb{R}_+^M \to \mathbb{R}_+$ is given by [6]*

$$J(\mathbf{x}) = \frac{\left(\sum\limits_{i=1}^M x_i\right)^2}{M \sum\limits_{i=1}^M x_i^2}. \tag{1}$$

□

From this definition it can be readily verified that $J(\mathbf{x})$ is continuous and that its range is the interval $\left[\frac{1}{M}, 1\right]$. In this interval, $J = \frac{1}{M}$ corresponds to the least fair allocation in which only one user receives a non-zero benefit, and $J = 1$ corresponds to the fairest allocation in which all users receive the same benefit.

In many cases, depending on the set $\mathcal{C}$, there is an inherent tradeoff between $\eta(\mathbf{x})$ and $J(\mathbf{x})$. Hence, to ensure efficient utilization of resources, we seek the optimal tradeoff, which is defined next.

Definition 2 (Optimal Efficiency-Jain tradeoff). *A tradeoff is said to be optimal if it results in a benefit vector $\mathbf{x}^*$ such that there is no $\mathbf{x} \in \mathcal{C}$ for which: 1) $\eta(\mathbf{x}) > \eta(\mathbf{x}^*)$ and at the same time $J(\mathbf{x}) \geq J(\mathbf{x}^*)$; or 2) $\eta(\mathbf{x}) \geq \eta(\mathbf{x}^*)$ and at the same time $J(\mathbf{x}) > J(\mathbf{x}^*)$.* □

This definition will be used in the next section to determine whether the α-fair tradeoff policy achieves the optimal Efficiency-Jain tradeoff.

III. Does α-fair achieve the optimal Efficiency-Jain tradeoff?

Given an $\alpha \in [0, \infty)$, the benefit vector $\mathbf{x}_\alpha^\star$ generated by the α-fair tradeoff policy maximizes the α-fair utility [1], i.e.,

$$\mathbf{x}_\alpha^\star = \arg\max_{\mathbf{x}\in\mathcal{C}} U_\alpha(\mathbf{x}), \tag{2}$$

where

$$U_\alpha(\mathbf{x}) = \begin{cases} \sum\limits_{i=1}^M \log x_i, & \alpha = 1, \\ \frac{1}{1-\alpha} \sum\limits_{i=1}^M x_i^{1-\alpha}, & \alpha \geq 0, \alpha \neq 1. \end{cases} \tag{3}$$

The α-fair policy thus described was considered in [2]. It was shown therein that, for $\alpha \neq 1$, $\mathbf{x}_\alpha^\star$ generated by (2) is the same as that generated by

$$\mathbf{x}_\alpha^\star = \arg\max_{\mathbf{x}\in\mathcal{C}} \Big(\Big| \frac{\alpha}{1-\alpha} \Big| L\big(H_\alpha(\mathbf{x})\big) + L\big(\eta(\mathbf{x})\big) \Big), \tag{4}$$

where

$$H_\alpha(\mathbf{x}) = \operatorname{sgn}(1-\alpha) \sqrt[\alpha]{\sum_{i=1}^M \Big(\frac{x_i}{\eta(\mathbf{x})}\Big)^{1-\alpha}}, \tag{5}$$

and $L(\cdot) \triangleq \operatorname{sgn}(\cdot)\log(|\cdot|)$. This equivalent formulation of the α-fair policy provides insight into the role of α. In particular, it can be seen that $L(\cdot)$ is monotonically increasing and that, for any $\alpha \neq 1$, $H_\alpha(\mathbf{x})$ provides a homogeneous fairness measure [2]. Hence, it can be seen that increasing α places more emphasis on fairness at the expense of efficiency.

Using the above observations, it was argued in [2] that solving (4) yields a benefit vector that achieves the optimal tradeoff between $H_\alpha(\mathbf{x})$ and $\eta(\mathbf{x})$. Although this explanation offers a better understanding, it presents the fairness component of the α-fair policy as being parametrized by α. Hence, according to this explanation, varying α not only controls the emphasis placed on fairness, but also changes the fairness measure. A question that arises is whether the α-fair policy achieves optimal efficiency-fairness tradeoffs in practical resource allocation scenarios wherein the fairness measure does not depend on extrinsic parameters like α.

To address this question, in this section we will investigate the relationship between the α-fair policy and the optimal tradeoff between efficiency and Jain's index. We begin by studying the case of $M = 2$ users. The main result in this case is stated in the following theorem:

Theorem 1. *Let $\mathcal{C}$ be an arbitrary set, possibly discrete, and let $M = 2$. Then, for any $\alpha \in (0, \infty)$, the α-fair benefit vector $\mathbf{x}_\alpha^\star$ generated by (2) achieves the optimal Efficiency-Jain tradeoff.*

Before proving this result, we note that, in contrast with the explanation in [2], in Theorem 1, the α-fair policy is shown to yield optimal tradeoffs with respect to Jain's index, which is a fairness measure that does not depend on α.

Proof: We will proceed by contradiction. Let $\alpha \in (0, \infty)$ be given and suppose that $\mathbf{x}_\alpha^\star$ does not achieve the optimal Efficiency-Jain tradeoff, that is, there exists a non α-fair optimal vector $\mathbf{x}$ such that either 1) $\eta(\mathbf{x}) > \eta(\mathbf{x}_\alpha^\star)$ and $J(\mathbf{x}) \geq J(\mathbf{x}_\alpha^\star)$; or 2) $\eta(\mathbf{x}) \geq \eta(\mathbf{x}_\alpha^\star)$ and $J(\mathbf{x}) > J(\mathbf{x}_\alpha^\star)$. We will show that such a vector $\mathbf{x}$ results in $U_\alpha(\mathbf{x}) > U_\alpha(\mathbf{x}_\alpha^\star)$, which contradicts the definition of α-fair benefit vectors; cf. (2). We will focus on the first case. The proof for the second case follows similar lines and is omitted for brevity.

Since $M = 2$, we can define a parameter $\beta = \frac{\max \mathbf{x}}{\min \mathbf{x}}$. Using this β, we have $J(\mathbf{x}) = \frac{(1+\beta)^2}{2(1+\beta^2)}$. Now, $\frac{dJ}{d\beta} = -\frac{\beta^2-1}{(\beta^2+1)^2}$. Since, by definition, $\beta \geq 1$, it can be seen that J is monotonically decreasing in β. This with the fact that in the considered case $J(\mathbf{x}) \geq J(\mathbf{x}_\alpha^\star)$ implies that

$$\frac{\max \mathbf{x}}{\min \mathbf{x}} \leq \frac{\max \mathbf{x}_\alpha^\star}{\min \mathbf{x}_\alpha^\star}. \tag{6}$$

Since in this case we also have $\eta(\mathbf{x}) > \eta(\mathbf{x}_\alpha^\star)$, it follows that $\min \mathbf{x} + \max \mathbf{x} > \min \mathbf{x}_\alpha^\star + \max \mathbf{x}_\alpha^\star$, which is equivalent to $\left(1 + \frac{\max \mathbf{x}}{\min \mathbf{x}}\right) \min \mathbf{x} > \left(1 + \frac{\max \mathbf{x}_\alpha^\star}{\min \mathbf{x}_\alpha^\star}\right) \min \mathbf{x}_\alpha^\star$. This inequality implies that

$$\min \mathbf{x} > \frac{\left(1 + \frac{\max \mathbf{x}_\alpha^\star}{\min \mathbf{x}_\alpha^\star}\right)}{\left(1 + \frac{\max \mathbf{x}}{\min \mathbf{x}}\right)} \min \mathbf{x}_\alpha^\star. \tag{7}$$

Invoking (6) implies that the fraction on the right hand side is greater than 1, which further implies that we can write $\min \mathbf{x} = \min \mathbf{x}_\alpha^\star + \epsilon_1$, with $\epsilon_1 > 0$. Since $\mathbf{x}$ is not α-fair, we must have

$$U_\alpha(\mathbf{x}) < U_\alpha(\mathbf{x}_\alpha^\star). \tag{8}$$

We now observe that $U_\alpha(\mathbf{x})$ is strictly increasing in each x_i, $i = 1, 2$. This observation and (8) imply that $\max \mathbf{x} = \max \mathbf{x}_\alpha^\star - \epsilon_2$, with $\epsilon_2 > 0$. Combining this with the fact that $\min \mathbf{x} = \min \mathbf{x}_\alpha^\star + \epsilon_1$ and the fact that in the current case $\eta(\mathbf{x}) > \eta(\mathbf{x}_\alpha^\star)$ yields $\epsilon_1 > \epsilon_2$. Using this notation, it can be readily verified that

$$\nabla U_\alpha(\mathbf{x})^T(\mathbf{x}_\alpha^\star - \mathbf{x}) = -\epsilon_1 (\min \mathbf{x})^{-\alpha} \left(1 - \frac{\epsilon_2}{\epsilon_1}\left(\frac{\max \mathbf{x}}{\min \mathbf{x}}\right)^{-\alpha}\right) < 0,$$

where the last inequality follows because $\epsilon_1 > \epsilon_2$.

Now, direct computation of the Hessian of $U_\alpha(\mathbf{x})$ shows that U_α is concave for any $\alpha \in (0, \infty)$. Thus [11, p. 69], $U_\alpha(\mathbf{x}_\alpha^\star) \leq U_\alpha(\mathbf{x}) + \nabla U_\alpha(\mathbf{x})^T(\mathbf{x}_\alpha^\star - \mathbf{x})$, which yields

$$U_\alpha(\mathbf{x}_\alpha^\star) < U_\alpha(\mathbf{x}). \tag{9}$$

This with (8) establish the desired contradiction. ■

Theorem 1 shows that for an arbitrary set $\mathcal{C}$, the α-fair policy yields tradeoffs that are optimal from Jain's index perspective. However, this result does not necessarily carry over to cases with $M > 2$ users. To show this, we constructed counter examples for $M = 3$ and $M = 4$. The case of $M = 4$ yields deeper insight and will be explained in more detail.

Example 1. *Let $\mathcal{C}$ contain two benefit vectors, i.e., $\mathcal{C} = \{\mathbf{x}, \mathbf{y}\}$, where $\mathbf{x} = [8,\ 8,\ 90,\ 90]$ and $\mathbf{y} = [7,\ 14,\ 27,\ 86]$.*

For $\alpha = 2$, maximizing the α-fair utility yields $\mathbf{y}$ because $U_2(\mathbf{y}) > U_2(\mathbf{x})$. However, $\eta(\mathbf{x}) = 196$, $\eta(\mathbf{y}) = 134$, $J(\mathbf{x}) = 0.59$ and $J(\mathbf{y}) = 0.54$, that is, $\eta(\mathbf{x}) > \eta(\mathbf{y})$ and $J(\mathbf{x}) > J(\mathbf{y})$, which implies that $\mathbf{x}$ is the optimal Efficiency-Jain tradeoff benefit vector. This agrees with intuition since, by inspection, $\mathbf{x}$ offers 75% of the users higher benefits than $\mathbf{y}$. □

Drawing more insight from the above example, we will show that the gap between the benefit vectors generated by the optimal Efficiency-Jain tradeoff and those generated by the α-fair one can be unbounded. To show that, let $\hat{\mathbf{x}} = c\mathbf{x}$ and $\hat{\mathbf{y}} = c\mathbf{y}$, where $c > 0$ is some constant, be two other benefit vectors in $\mathcal{C}$. In this case, it can be easily verified that $\hat{\mathbf{y}}$ is the α-fair benefit vector. Furthermore, because Jain's index is invariant under scaling, $J(\hat{\mathbf{x}}) = J(\mathbf{x}) > J(\hat{\mathbf{y}}) = J(\mathbf{y})$. However, direct computation reveals that $\eta(\hat{\mathbf{x}}) - \eta(\hat{\mathbf{y}}) = c\big(\eta(\mathbf{x}) - \eta(\mathbf{y})\big)$. Hence, an unbounded c, results in an unbounded difference in efficiency. The existence of such c depends, of course, on $\mathcal{C}$. In fact, it will be shown later that the structure of $\mathcal{C}$ is intimately related to the optimal Efficiency-Jain tradeoff.

Another insight that can be drawn from the above example is that the α-fair benefit vector corresponding to $\alpha = 0$ is $\mathbf{x}$, which, from the Jain's perspective, is fairer than the α-fair benefit vector corresponding to $\alpha = 2$. This shows that, although increasing α results in benefit vectors that are fairer in the senses considered in [1] and [2], it does not necessarily improve fairness in the Jain's index sense.

Many applications, including wireless communications ones, involve the tradeoff between the benefit vectors received by more than two users. Since in these cases, maximizing α-fair utilities does not necessarily yield benefit vectors that achieve the optimal Efficiency-Jain tradeoff (cf. Theorem 1 and Example 1), in the next section we will develop another technique for achieving this tradeoff.

IV. Optimal Efficiency-Jain tradeoff policy

In this section, we develop a generic technique for obtaining the optimal Efficiency-Jain tradeoff for an arbitrary set $\mathcal{C}$. To enable practical implementation of this technique, we identify conditions, which, when satisfied by the set $\mathcal{C}$, renders the underlying optimization problems easy to solve. We will then provide instances in which these conditions are satisfied in practice, and finally, we will conclude this section by providing a geometric interpretation that commends the use of Jain's index as a fairness measure.

A. A Technique for Obtaining The Optimal Efficiency-Jain Tradeoff for an Arbitrary $\mathcal{C}$

Let σ be a threshold on the minimum efficiency, and let $\mathcal{X}_\sigma$ be the set of all benefit vectors that yield an efficiency greater than σ and, at the same time, maximize Jain's index, that is

$$\mathcal{X}_\sigma \triangleq \left\{ \mathbf{x} \middle| \mathbf{x} = \arg \max_{\eta(\mathbf{x}) \geq \sigma, \ \mathbf{x} \in \mathcal{C}} J(\mathbf{x}) \right\}. \quad (10)$$

We note that the cardinality of $\mathcal{X}_\sigma$ depends on $\mathcal{C}$. Furthermore, some elements in $\mathcal{X}_\sigma$ may satisfy the condition $\eta(\mathbf{x}) \geq \sigma$ in (10) with a strict inequality. Since we are seeking the benefit vectors that achieve the optimal Efficiency-Jain tradeoff, we pick those vectors in $\mathcal{X}_\sigma$ that yield the maximum efficiency. In particular, let $\mathbf{x}^\star_\sigma$ be a benefit vector that achieves the optimal Efficiency-Jain tradeoff corresponding to σ. Then,

$$\mathbf{x}^\star_\sigma = \arg \max_{\mathbf{x} \in \mathcal{X}_\sigma} \eta(\mathbf{x}). \quad (11)$$

From (10) and (11), it can be seen that $\mathbf{x}^\star_\sigma$ achieves the optimal Efficiency-Jain tradeoff in Definition 2. The set of all benefit vectors that achieve the optimal Efficiency-Jain tradeoff can be obtained by varying σ from $\sigma_{\max} = \max_{\mathbf{x} \in \mathcal{C}} \eta(\mathbf{x})$ to $\sigma_{\min} = \min_{\mathbf{x} \in \mathcal{C}} \eta(\mathbf{x})$, and solving the optimization problems in (10) and (11). This policy is presented formally in Procedure 1 below.

Procedure 1 Optimal Efficiency-Jain tradeoff policy for arbitrary $\mathcal{C}$

Input: Arbitrary set $\mathcal{C}$, step size $\delta > 0$, $\sigma_{\min} = \min_{\mathbf{x} \in \mathcal{C}} \eta(\mathbf{x})$ and $\sigma_{\max} = \max_{\mathbf{x} \in \mathcal{C}} \eta(\mathbf{x})$

Output: $\mathbf{x}^\star_\sigma$

Initialize $\sigma \leftarrow \sigma_{\max}$.
while $\sigma \geq \sigma_{\min}$ **do**
  Find $\mathcal{X}_\sigma$ in (10).
  $\mathbf{x}^\star_\sigma \leftarrow \arg \max_{\mathbf{x} \in \mathcal{X}_\sigma} \eta(\mathbf{x})$.
  $\sigma \leftarrow \sigma - \delta$.
end while

Inspection of the above procedure reveals that the main difficulty in obtaining $\mathbf{x}^\star_\sigma$ lies in finding a solution of the optimization problem in (10), let alone finding the entire set $\mathcal{X}_\sigma$. This difficulty arises because $J(\mathbf{x})$ is a nonconcave function, even when $\mathcal{C}$ is a convex set. This motivates us to seek conditions, which, when satisfied by $\mathcal{C}$, this problem becomes tractable.

B. A Property for Ensuring Tractability

In order to render the optimization problems underlying (10) easy to solve, we begin by identifying a class of sets $\mathcal{C}$ which satisfy what we refer to as the "monotonic tradeoff property". To do so, let $J^\star_\sigma$ denote the maximum Jain's index corresponding to an efficiency $\eta(\mathbf{x}) = \sigma$, that is,

$$J^\star_\sigma = \max_{\eta(\mathbf{x}) = \sigma, \ \mathbf{x} \in \mathcal{C}} J(\mathbf{x}). \quad (12)$$

Using (12), we are now ready to provide our definition of the monotonic tradeoff property.

Definition 3 (Monotonic Tradeoff Property)**.** *A set $\mathcal{C}$ satisfies the monotonic tradeoff property if $J^\star_\sigma$ is strictly decreasing in σ, for σ greater than or equal to some efficiency $\sigma^\star$, and constant otherwise.* □

This definition implies that

$$J^\star_{\sigma^\star} = \max_{\eta(\mathbf{x}) = \sigma^\star, \ \mathbf{x} \in \mathcal{C}} J(\mathbf{x}) = \max_{\mathbf{x} \in \mathcal{C}} J(\mathbf{x}). \quad (13)$$

This definition states that a set that satisfies the monotonic tradeoff property is one in which any decrease in efficiency results in a strict increase in the Jain's index, until $\sigma^\star$ is reached. Decreasing efficiency beyond $\sigma^\star$ maintains Jain's index at its maximum. An instance in which $\mathcal{C}$ satisfies the monotonic tradeoff property is shown in Fig. 1(a) and the corresponding Efficiency-Jain tradeoff is shown in Fig. 1(b). These figures will be discussed in detail in the next section. Instances in which $\mathcal{C}$ does not satisfy this property have been omitted for space considerations.

We will now show how the monotonic tradeoff property facilitates finding the benefit vectors that achieve the optimal Efficiency-Jain tradeoff. When this property is satisfied, the inequality $\eta(\mathbf{x}) \geq \sigma$ in (10) is satisfied with equality when $\sigma > \sigma^\star$ because $J^\star_\sigma$ is strictly decreasing in σ. In this case, the optimization in (10) is equivalent to that in (12). Moreover, it can be readily verified that, with $\eta(\mathbf{x}) = \sigma$, the optimization in (12) can be cast in the following form

$$\min_{\eta(\mathbf{x}) = \sigma, \ x \in \mathcal{C}} \|\mathbf{x}\|^2, \quad (14)$$

where $\|\cdot\|$ is the Euclidean norm.

In contrast with (10), the objective in (14) is convex. This renders the optimization problem in (14) easy to solve, provided that the set $\mathcal{C}$ is convex. In addition, if $\mathcal{C}$ is not convex and (14) has multiple solutions, then all these solutions will achieve the same Efficiency-Jain tradeoff as they all have the same efficiency, σ, and the same Jain's index. This eliminates the requirement for finding all solutions in (10) since any solution of (14) achieves the optimal Efficiency-Jain tradeoff. To summarize, if the monotonic tradeoff property is satisfied, $\mathbf{x}^\star_\sigma$ can be found by solving (14), which is easier than solving the optimization problems in (10) and (11) for an arbitrary $\mathcal{C}$.

When $\mathcal{C}$ satisfies the monotonic tradeoff property, the benefit vectors that achieve the optimal Efficiency-Jain tradeoff can be obtained by varying σ from $\sigma_{\max}$ to $\sigma_{\min}$. For each σ, we find $\mathbf{x}^\star_\sigma$ by solving (14). This policy is presented formally in Procedure 2 below.

Procedure 2 Optimal Efficiency-Jain tradeoff policy for $\mathcal{C}$ satisfying the monotonic tradeoff property

Input: A set $\mathcal{C}$ satisfying the monotonic tradeoff property, step size $\delta > 0$, $\sigma_{\min} = \min_{\mathbf{x}\in\mathcal{C}} \eta(\mathbf{x})$ and $\sigma_{\max} = \max_{\mathbf{x}\in\mathcal{C}} \eta(\mathbf{x})$

Output: $\mathbf{x}^\star_\sigma$

```
Initialize σ ← σ_max, and σ* ← σ_min.
while σ ≥ σ_min do
    x*_σ = arg min_{η(x)=σ, x∈C} ‖x‖²
    if J(x*_σ) = J(x*_{σ+δ}) then
        quit
    end if
    σ ← σ − δ.
end while
```

C. Sufficient Conditions for Satisfying the Monotonic Tradeoff Property

In the previous section, we have shown that finding the set of benefit vectors that achieve the optimal Efficiency-Jain tradeoff vectors is significantly simplified when the set $\mathcal{C}$ satisfies the monotonic tradeoff property. Hence, it is desirable to identify conditions that ensure that a given set satisfies this property. Such conditions are provided in Theorem 2 below.

Theorem 2. *The set $\mathcal{C}$ satisfies the monotonic tradeoff property if the following conditions are satisfied:*

i. $\mathcal{C}$ is convex; and

ii. $\mathbf{0}_M \in \mathcal{C}$,

where $\mathbf{0}_M$ is the length-M all zero vector.

Proof: For space considerations, we will only provide a sketch of the proof. The details thereof are provided in [12].

Let $\mathbf{x}^\star_{\sigma_1}$ and $\mathbf{x}^\star_{\sigma_2}$ be the benefit vectors obtained using (14) with σ_1 and σ_2, respectively, where $\sigma_{\min} \le \sigma_1 < \sigma_2 \le \sigma_{\max}$. To prove Theorem 2, it suffices to show that if conditions i and ii are satisfied, then the following statements hold:

1) If $J^\star_{\sigma_2} = J^\star_{\sigma^\star}$, then $J^\star_{\sigma_1} = J^\star_{\sigma_2}$; and
2) If $J^\star_{\sigma_2} < J^\star_{\sigma^\star}$, then $J^\star_{\sigma_1} > J^\star_{\sigma_2}$,

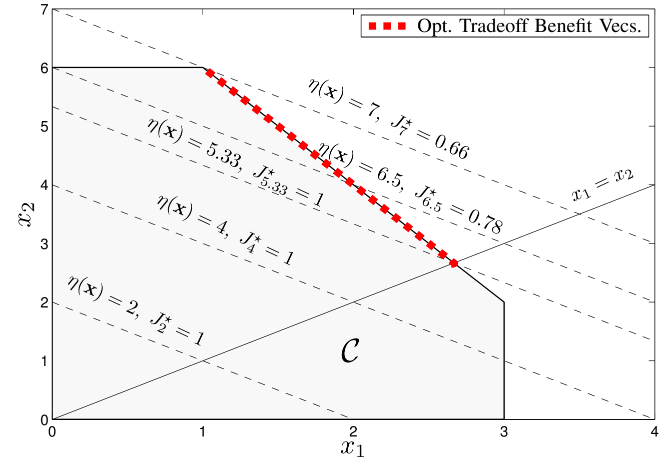
where $J^\star_\sigma$ and $J^\star_{\sigma^\star}$ are given by (12) and (13), respectively.

The proof of the first statement is based on using the scaling invariance of Jain's index, the conditions of the theorem, and the maximal property of $J^\star_{\sigma^\star}$ in (13) to show that $J^\star_\sigma$ is decreasing in σ, but not strictly.

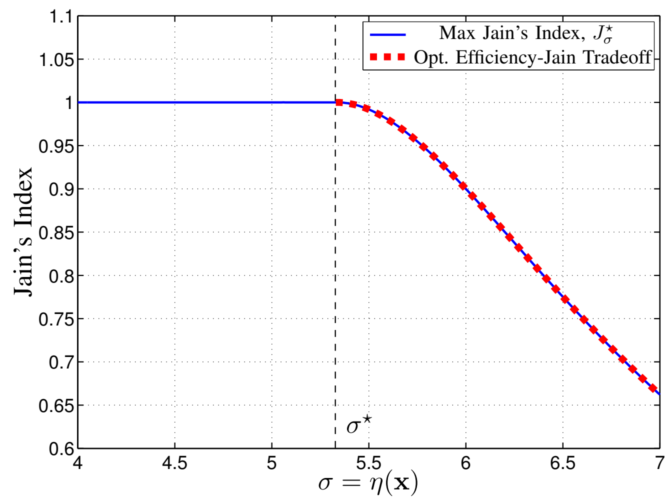
The proof of the second statement relies on showing that Jain's index increases along the line segment connecting $\mathbf{x}^\star_{\sigma_1}$ to $\mathbf{x}^\star_{\sigma^\star}$. Invoking again the scaling invariance of Jain's index and the conditions of the theorem yields the second statement. ■

We now provide a graphical illustration of Theorem 2 for a case with $M = 2$ users. Fig. 1(a) shows a set $\mathcal{C}$ that satisfies the theorem's conditions, and Fig. 1(b) shows the corresponding Efficiency-Jain tradeoff.

In Fig. 1(a), the maximum Jain's fairness line $x_1 = x_2$ passes through $\mathcal{C}$ and yields $J(\mathbf{x}) = 1$. The dashed lines in this figure represent the constant efficiency levels, $\eta(\mathbf{x}) = \sigma$, at different values of σ. For $\sigma \le 5.33$, the points at which the dashed lines intersect the $x_1 = x_2$ line lie inside $\mathcal{C}$. In this case, the maximal Jain's index, $J^\star_\sigma = 1$. For $\sigma > 5.33$, the dashed lines representing the $\eta(\mathbf{x}) = \sigma$ levels intersect the $x_1 = x_2$ line at points outside $\mathcal{C}$. For these efficiency levels, the maximal Jain's indices are strictly less than 1 and correspond to the points at which the dashed lines intersect with the boundary of $\mathcal{C}$. The optimal tradeoff benefit vectors are shown by the thick dashed line on the boundary of $\mathcal{C}$.



(a)



(b)

Fig. 1. (a) A convex set that satisfies the conditions in Theorem 2 and (b) its Efficiency-Jain curve.

The variation of $J^\star_\sigma$ with σ is depicted in Fig. 1(b). It can be seen from this figure, that in agreement with Theorem 2, the set $\mathcal{C}$ satisfies the monotonic tradeoff property in Definition 3 with $\sigma^\star = 5.33$. In this figure, the optimal tradeoff corresponding to the thick dashed line on the boundary of $\mathcal{C}$ in Fig. 1(a) is represented by the thick dashed line to the right of $\sigma^\star$.

To show that the conditions of Theorem 2 are relatively sharp, we make the following remark:

Remark 1. *If the lower left corner of $\mathcal{C}$ in Fig. 1(a) is shifted* *to* $(6,0)$*, the condition that* $\mathbf{0}_M \in \mathcal{C}$ *in Theorem 2 is violated and the monotonic tradeoff property is not satisfied.* □

An illustration of this remark has been omitted for space considerations. See [12] for details.

D. Practical Applications of Theorem 2

The sufficient conditions given in Theorem 2 are naturally satisfied in various resources allocation problems in communication networks. For instance, in congestion control in elastic traffic communication networks [1], [7] the users share finite-capacity links and the goal is to assign the benefit vector $\mathbf{x}$, which represents the rates delivered to the users, in an efficient and fair manner.

The set of feasible rates in this case is given by $\mathcal{C} = \{\mathbf{x}|\mathbf{A}\mathbf{x} \preceq \mathbf{c}, \mathbf{0}_M \preceq \mathbf{x}\}$, where the j-th entry of $\mathbf{c} \in \mathbb{R}_+^M$ is the capacity of link j, $j = 1, \ldots, M$, and $\mathbf{A}$ is a matrix with binary entries that represents the assignment of users to links, and $\preceq$ is the element-wise inequality.

In this case, the set $\mathcal{C}$ is a convex polyhedron [11, p. 31] containing $\mathbf{0}_M$, and thereby satisfying the conditions of Theorem 2. Hence, $\mathcal{C}$ satisfies the monotonic tradeoff property and Procedure 2 can be used to find all the optimal Efficiency-Jain tradeoff rate vectors.

Another example is the allocation of radio resources in the downlink of cellular networks, which will be discussed in Section V in more detail.

E. Geometric Interpretation of the Optimal Efficiency-Jain Tradeoff

When $\mathcal{C}$ satisfies the sufficient conditions given in Theorem 2, optimal Efficiency-Jain benefit vectors $\{\mathbf{x}_\sigma^\star\}$ have an interesting geometric interpretation. To see that, we use (14) to write

$$\mathbf{x}_\sigma^\star = \arg \min_{\eta(\mathbf{x})=\sigma,\ \mathbf{x}\in\mathcal{C}} \sum_{i=1}^{M} x_i^2 \tag{15}$$

$$= \arg \min_{\eta(\mathbf{x})=\sigma,\ x\in\mathcal{C}} \sum_{i=1}^{M} \left(x_i^2 - 2\frac{\sigma}{M}\eta(\mathbf{x}) + \frac{\sigma^2}{M^2} \right) \tag{16}$$

$$= \arg \min_{\eta(\mathbf{x})=\sigma,\ x\in\mathcal{C}} \left\| \mathbf{x} - \frac{\sigma}{M}\mathbf{1}_M \right\|^2. \tag{17}$$

where $\mathbf{1}_M$ is the all one length-M vector. The last equality states that $\mathbf{x}_\sigma^\star$ is the unique Euclidean projection [11, p. 397] of the equal allocation vector $\frac{\sigma}{M}\mathbf{1}_M$ onto the set $\{\mathbf{x}|\eta(\mathbf{x}) = \sigma,\ \mathbf{x} \in \mathcal{C}\}$. In other words, a benefit vector $\mathbf{x}_\sigma^\star$ achieves the optimal Efficiency-Jain tradeoff if there is no other benefit vector $\mathbf{y}$ such that $\eta(\mathbf{y}) = \sigma$ is closer to the fairest solution $\frac{\sigma}{M}\mathbf{1}_M$. This interpretation commends the use of Jain's index as a fairness measure and is illustrated in Fig. 2.

V. Numerical Results

In this section we consider the problem of allocating the radio resources in the downlink of wireless communication networks. The available transmission bandwidth is divided into N-subchannels, which are allocated by the transmitter to M users. The date rate of user m on subchannel n is denoted by r_{mn}, $m = 1, \ldots, M$, $n = 1, \ldots, N$, where the values of $\{r_{mn}\}$ depend on the channel conditions experienced by the users. Let $\rho_{mn} \in [0, 1]$ be a time-sharing variable that assigns subchannel n to user m for a fraction ρ_{mn} of the signalling interval [13]. At each time instant, each sub-channel is used by at most one user, and thus $\sum_{m=1}^{M} \rho_{mn} \leq 1$. The total data rate (benefit) of user m is given by $x_m = \sum_{n=1}^{N} \rho_{mn} r_{mn}$ and the efficiency of the network is given by the total sum rate, which is given by $\eta(\mathbf{x}) = \sum_{i=1}^{M} x_m$. The set of achievable rates (benefits) for the users is given by

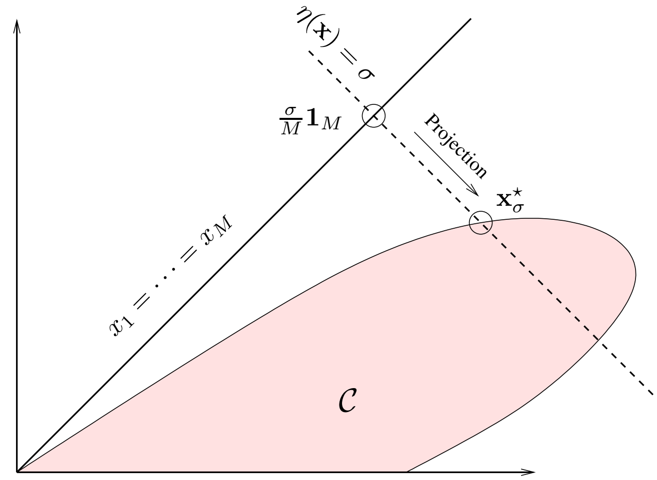


Fig. 2. The optimal Efficiency-Jain tradeoff benefit vector, $\mathbf{x}_\sigma^\star$, is the unique projection of the fairest vector $\frac{\sigma}{M}\mathbf{1}_M$ onto the set $\{\mathbf{x}|\eta(\mathbf{x}) = \sigma,\ \mathbf{x} \in \mathcal{C}\}$.

$$\mathcal{C} = \left\{ \mathbf{x} | x_m = \sum_{n=1}^{N} \rho_{mn} r_{mn}, \sum_{m=1}^{M} \rho_{mn} \leq 1, \rho_{mn} \in [0, 1] \right\}. \tag{18}$$

The goal is to determine ρ_{mn} that results in an efficient and fair rate vector $\mathbf{x}$. Since the α-fair policy is commonly used to obtain different efficiency-fairness tradeoffs [14], we will use it as a benchmark for comparing our results.

We consider the case of $M = 4$ users and $N = 5$ subchannels. For simplicity, we consider one realization of a quasi static network in which the rate matrix $\mathbf{R} = [r_{mn}]$ is the one given below

$$\mathbf{R} = \begin{bmatrix} 544 & 648 & 807 & 544 & 722 \\ 388 & 92 & 223 & 388 & 56 \\ 35 & 544 & 35 & 722 & 56 \\ 35 & 56 & 35 & 92 & 35 \end{bmatrix}. \tag{19}$$

The rates in this matrix are given in Kbit/sec and were obtained from simulating a practical Long-Term Evolution (LTE) system in which users 1 and 2 are closer to the transmitter than users 3 and 4.

First, we note that the set $\mathcal{C}$ is convex and contains the $\mathbf{0}_M$ vector. Hence, the conditions of Theorem 2 are satisfied and thus $\mathcal{C}$ satisfies the monotonic tradeoff property. Using Procedure 2 we obtained the optimal Efficiency-Jain tradeoff shown in Fig. 3. In this figure, we also plot the Efficiency-Jain tradeoff achieved by the α-fair policy. From this figure, it can be seen that while these tradeoffs are close to each other for small and large values of α, for intermediate values, the tradeoff generated by the optimal tradeoff policy is significantly better than that generated by the α-fair policy. For example, for a Jain's index of 0.7, the optimal tradeoff policy provides 33% gain in efficiency as compared to the α-fair policy.

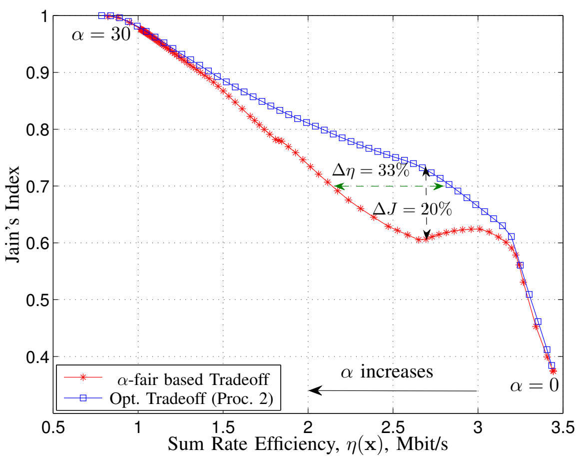


Fig. 3. Optimal and α-fair Efficiency-Jain tradeoffs.

When the subchannels are not time-shared by the users, i.e., $\rho_{mn} \in \{0, 1\}$, the corresponding set $\mathcal{C}$ is not convex and does not satisfy the monotonic tradeoff property. In this case, Procedure 1 can be used to obtain the optimal Efficiency-Jain tradeoff. An example that considers this case is provided in [12] and was omitted for space considerations.

VI. Conclusions

In this paper, we considered multiuser resource allocations that achieve the optimal tradeoff between efficiency and fairness from the Jain's index perspective. We have shown that, in general, the commonly-used α-fair policy does not yield optimal Efficiency-Jain tradeoffs except for the two-user case. To achieve the optimal Efficiency-Jain tradeoffs in the general case, we developed two procedures. In the first procedure, the set of admissible allocations is arbitrary, but finding the allocations that achieve the optimal Efficiency-Jain tradeoffs involves solving potentially difficult optimization problems. In contrast, in the second procedure, the set of admissible allocations is assumed to have a monotonic property that arises in many practical scenarios. This property is exploited to facilitate the search for allocations achieving optimal Efficiency-Jain tradeoffs. Our analysis is supported by illustrations, geometric interpretations and numerical examples.

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