Performance Analysis of Fisher-Snedecor $\mathcal{F}$ Composite Fading Channels

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***Abstract*—In this paper, we consider the Fisher-Snedecor $\mathcal{F}$ composite fading channel model and derive exact closed-form expressions for the symbol error rate (SER) of M-ary pulse amplitude modulation (M-PAM) and M-ary quadrature amplitude modulation (M-QAM). We also derive asymptotic expressions for the SER of M-PAM and M-QAM to study the behavior of SER at high values of signal-to-noise ratio. Moreover, we derive an exact closed-form expression for the average capacity. The derived expressions are evaluated for different values of the fading parameters to show the effects of shadowing and small-scale fading on the performance of SER and capacity. Simulation results are also provided to show the accuracy of the derived expressions.**

***Index Terms*—Composite fading; capacity; SER; D2D.**

I. Introduction

In Device-to-Device (D2D) communications, mobile terminals are allowed to exchange data between them directly without the need for a base station. Therefore, D2D communication has been identified as an enabling technology to off-load high traffic volumes in the fifth generation (5G) wireless networks by utilizing good channel conditions in proximity. D2D communications can be also used to provide end users with public safety information in cases where infrastructure network is not available.

The received signal in D2D communications suffers from both fading and shadowing simultaneously. Several composite fading models have been proposed in the literature to characterize the statistics of the simultaneous effects of fading and shadowing. Rayleigh-lognormal is a classical composite fading model that uses Rayleigh distribution to characterize the small-scale variations of the envelope of the received signal while it uses lognormal distribution to characterize the variations in the mean power of the received signal [1]. Other composite fading models use the Gamma distribution to characterize shadowing in order to render mathematically tractable expressions for performance metrics like average symbol error rate (SER) [2]. In [3], the authors approximate the generalized-K probability density function (PDF) by a Gamma PDF to simplify the analysis of performance metrics over such fading channels.

Recently, the Fisher distribution has been introduced in [4] to accurately model the composite effects of both small- and large-scale variations of the faded signal. In Fisher composite fading model, it is assumed that small-scale variations follow Nakagami-m distribution whereas shadowing follows inverse Nakagami-m distribution. It has been shown in [4] that the Fisher composite fading model fits experimental channel measurements for D2D communications at 5.8 GHz better than K_G fading model in both line-of-sight (LOS) and non-LOS (NLOS) scenarios.

Average SER and capacity are fundamental performance metrics in wireless communication systems. Closed-form expressions for the average bit error rate (BER) of differential phase shift keying (DPSK) and binary phase shift keying (BPSK) over Fisher composite fading channels have been provided in [4]. The performance of the composite α-μ/α-μ multipath-shadowing distribution is investigated in [5]. In [6], the authors derive closed-form expressions for the average channel capacity and the average BER over K_G fading channels.

In this paper, we derive exact closed-form expressions for the average SER of M–ary pulse amplitude modulation (M-PAM) and M-ary quadrature amplitude modulation (M-QAM) of a point-to-point communication system assuming Fisher composite fading model. We also provide closed-form expression for the average capacity. The derived expressions can be represented either in terms of the univariate and bivariate Fox H-function or univariate and bivariate Meijer G-function. Moreover, we provide closed-form expressions for the asymptotic behavior of the average SER for both M-PAM and M-QAM modulation techniques.

The rest of the paper is organized as follows. Section II introduces the channel model. SER of M-QAM is analyzed in Section III while SER of M-PAM is analyzed in Section IV and capacity analysis is provided in Section V. Numerical results are provided in Section VI while Section VII concludes the paper.

II. Channel Model

In the Fisher composite fading model, the small-scale variations of the signal are assumed to follow the Nakagami-m distribution while the root mean square power of the received signal is assumed to follow the inverse Nakagami-m

distribution. The PDF of the instantaneous signal-to-noise ratio (SNR) denoted here by γ can be expressed as [4]

$$f_\gamma(\gamma) = \frac{m^m \left(m_s\overline{\gamma}\right)^{m_s} \gamma^{m-1}}{B\left(m, m_s\right)\left(m\gamma + m_s\overline{\gamma}\right)^{m+m_s}}, \qquad (1)$$

where $B\left(.,.\right)$ is the beta function [7], and $\overline{\gamma}$ is the average SNR. The PDF in (1) can be written with the help of [8, (8.4.2.5)] as

$$f_\gamma\left(\gamma\right) = \frac{m^m \gamma^{m-1}}{\left(m_s\overline{\gamma}\right)^m \Gamma\left(m_s\right)\Gamma\left(m\right)} H_{1,1}^{1,1}\left(\frac{m\gamma}{m_s\overline{\gamma}}\middle| \begin{matrix}(1-m-m_s,1)\\(0,1)\end{matrix}\right), \quad (2)$$

where $\Gamma\left(.\right)$ is the Gamma function [7, (8.310.1)], and $\mathrm{H}_{\mathrm{p,q}}^{\mathrm{m,n}}\left[.|.\right]$ is the Fox-H function defined in [9, (1.2)]. The parameter m represents the number of multipath clusters and m_s represents the amount of shadowing where $m_s \to 0$ for large amount of shadowing and $m_s \to \infty$ for no shadowing.

III. Symbol Error Rate Analysis For M-QAM

A. Exact SER Analysis for M-QAM

The SER of square M-QAM over additive white Gaussian noise (AWGN) can be expressed as [10]

$$\begin{aligned} P_s\left(e|\gamma\right) &= 4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3\gamma}{M-1}}\right) \\ &\quad - 4\left(1-\frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{\frac{3\gamma}{M-1}}\right). \end{aligned} \qquad (3)$$

In order to find the average SER assuming Fisher fading model, we average the conditional SER in (3) over the PDF in (2) as

$$P_s\left(e\right) = \int_0^\infty P\left(e|\gamma\right) f_\gamma(\gamma)d\gamma = I_1 - I_2, \qquad (4)$$

where

$$\begin{aligned} \mathcal{I}_1 = \int_0^\infty & \frac{4\left(1-\frac{1}{\sqrt{M}}\right)\left(\frac{m}{m_s\overline{\gamma}}\right)^m \gamma^{m-1}}{\Gamma\left(m\right)\Gamma\left(m_s\right)} Q\left(\sqrt{\frac{3\gamma}{M-1}}\right) \\ & \times \mathrm{H}_{1,1}^{1,1}\left(\frac{\mathrm{m}\gamma}{\mathrm{m_s}\overline{\gamma}}\middle| \begin{matrix}(1-m-m_s,1)\\(0,1)\end{matrix}\right) \mathrm{d}\gamma. \end{aligned} \qquad (5)$$

The integral $\mathcal{I}_1$ in (5) can be rewritten using [8, (8.4.14.2)] and the relationship between the complementary error function and the Q-function as

$$\begin{aligned} \mathcal{I}_1 = \int_0^\infty & \frac{2\left(1-\frac{1}{\sqrt{M}}\right)\left(\frac{m}{m_s\overline{\gamma}}\right)^m \gamma^{m-1}}{\sqrt{\pi}\Gamma\left(m\right)\Gamma\left(m_s\right)} \\ & \times \mathrm{H}_{1,1}^{1,1}\left(\frac{\mathrm{m}\gamma}{\mathrm{m_s}\overline{\gamma}}\middle| \begin{matrix}(1-m-m_s,1)\\(0,1)\end{matrix}\right) \\ & \times \mathrm{H}_{1,2}^{2,0}\left(\frac{3\gamma}{2\left(\mathrm{M}-1\right)}\middle| \begin{matrix}(1,1)\\(0,1),\left(\frac{1}{2},1\right)\end{matrix}\right) \mathrm{d}\gamma. \end{aligned} \qquad (6)$$

The integral in (6) can be readily solved with the help of [11, (2.8.4) and (2.1.5)] as

$$\begin{aligned} \mathcal{I}_1 = & \frac{2\left(1-\frac{1}{\sqrt{M}}\right)}{\sqrt{\pi}\Gamma\left(m_s\right)\Gamma\left(m\right)} \\ & \times \mathrm{H}_{3,2}^{1,3}\left(\frac{2\mathrm{m}\left(\mathrm{M}-1\right)}{3\mathrm{m_s}\overline{\gamma}}\middle| \begin{matrix}(1-m_s,1),(1,1),\left(\frac{1}{2},1\right)\\(m,1),(0,1).\end{matrix}\right). \end{aligned} \qquad (7)$$

The integration $\mathcal{I}_2$ in (4) can be expressed as

$$\begin{aligned} \mathcal{I}_2 = \int_0^\infty & \frac{\left(1-\frac{1}{\sqrt{M}}\right)^2\left(\frac{m}{m_s\overline{\gamma}}\right)^m \gamma^{m-1}}{\pi\Gamma\left(m_s\right)\Gamma\left(m\right)} \\ & \times \mathrm{H}_{1,1}^{1,1}\left(\frac{\mathrm{m}\gamma}{\mathrm{m_s}\overline{\gamma}}\middle| \begin{matrix}(1-m-m_s,1)\\(0,1)\end{matrix}\right) \\ & \times \mathrm{H}_{1,2}^{2,0}\left(\frac{3\gamma}{2\left(\mathrm{M}-1\right)}\middle| \begin{matrix}(1,1)\\(0,1),\left(\frac{1}{2},1\right)\end{matrix}\right)^2 \mathrm{d}\gamma, \end{aligned} \qquad (8)$$

which can be solved using [12, (2.3)] as in (9), on the top of the next page, where $H\left[.|.|.\right]$ is the bivariate Fox H-function defined in [12]. Substituting (7) and (9) in (4) results in an exact form expression for the average SER of M-QAM which is not shown here due to space limitations. Note that the SER expressions in (7) and (9) are new and have not been reported in the literature before.

B. Asymptotic SER Analysis for M-QAM

The asymptotic behavior of the average SER at high values of average SNR $\overline{\gamma}$ is important for practical purposes. This can be achieved by studying the asymptotic behavior of the Fox-H functions in (7) and (9). By invoking [11, Th. 1.11], $\mathcal{I}_1$ in (7) can be approximated as

$$I_1 \approx \frac{2\left(1-\frac{1}{\sqrt{M}}\right)\Gamma\left(m+m_s\right)\Gamma\left(\frac{1}{2}+m\right)\left(\frac{2m(M-1)}{3m_s\overline{\gamma}}\right)^m}{\sqrt{\pi}\Gamma\left(m+1\right)\Gamma\left(m_s\right)}. \qquad (10)$$

Similarly, $\mathcal{I}_2$ in (9) can be approximated using the complex residue theorem. After some mathematical manipulations $\mathcal{I}_2$ can be expressed as

$$\begin{aligned} \mathcal{I}_2 \approx & \frac{\left(1-\frac{1}{\sqrt{M}}\right)^2\left(\frac{2m(M-1)}{3m_s\overline{\gamma}}\right)^m}{\pi B\left(m,m_s\right)} \\ & \times \mathrm{H}_{3,3}^{2,2}\left(1\middle| \begin{matrix}(1-m,1),\left(\frac{1}{2}-m,1\right),(1,1)\\(0,1),\left(\frac{1}{2},1\right),(-m,1)\end{matrix}\right). \end{aligned} \qquad (11)$$

IV. Symbol Error Rate Analysis For M-PAM

A. Exact SER Analysis for M-PAM

The SER of M-PAM in AWGN channels can be expressed as [10]

$$P_s\left(e|\gamma\right) = 2\left(\frac{M-1}{M}\right)Q\left(\sqrt{\frac{6\gamma}{M^2-1}}\right). \qquad (12)$$

In order to find the average SER assuming Fisher fading model, we average the conditional SER in (12) over the PDF in (2) as

$$\begin{aligned} P_s\left(e\right) = \int_0^\infty & \frac{2\left(M-1\right)\left(\frac{m}{m_s\overline{\gamma}}\right)^m \gamma^{m-1}}{M\Gamma\left(m_s\right)\Gamma\left(m\right)} \\ & \times Q\left(\sqrt{\frac{3\gamma}{M-1}}\right) \mathrm{H}_{1,1}^{1,1}\left(\frac{\mathrm{m}\gamma}{\mathrm{m_s}\overline{\gamma}}\middle| \begin{matrix}(1-m-m_s,1)\\(0,1)\end{matrix}\right) \mathrm{d}\gamma. \end{aligned} \qquad (13)$$

$$\mathcal{I}_2 = \frac{\left(1-\frac{1}{\sqrt{M}}\right)^2\left(\frac{2m(M-1)}{3m_s\overline{\gamma}}\right)^m}{\pi\Gamma\left(m_s\right)\Gamma\left(m\right)} H\left[\begin{array}{c|c|c} \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} & (1-m;1,1)\,,\left(\frac{1}{2}-m;1,1\right) & \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & (-m;1,1) & \\ & (1-(m+m_s)\,,1) & \frac{2m\left(M-1\right)}{3m_s\overline{\gamma}},1 \\ & (0,1) & \\ \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} & (1,1) & \\ & (0,1)\,,\left(\frac{1}{2},1\right) & \end{array}\right]. \tag{9}$$

The integral in (13) can be solved similar to (5) as

$$P_s\left(e\right) = \frac{\left(M-1\right)}{\sqrt{\pi}M\Gamma\left(m_s\right)\Gamma\left(m\right)} \times \mathrm{H}_{3,2}^{1,3}\left(\frac{\mathrm{m}\left(\mathrm{M}^2-1\right)}{3\mathrm{m_s}\overline{\gamma}}\middle| \begin{array}{c}\left(1-m_s,1\right),\left(1,1\right),\left(\frac{1}{2},1\right) \\ \left(m,1\right),\left(0,1\right)\end{array}\right). \tag{14}$$

The expression in (14) is new and has not been reported in the literature before.

B. Asymptotic SER Analysis for M-PAM

Similar to (10), the asymptotic behavior of the average SER of M-PAM in (14) can be expressed as

$$P_s\left(e\right) \approx \frac{\left(M-1\right)\Gamma\left(m+m_s\right)\Gamma\left(\frac{1}{2}+m\right)\left(\frac{m\left(M^2-1\right)}{3m_s\overline{\gamma}}\right)^m}{M\sqrt{\pi}\Gamma\left(m+1\right)\Gamma\left(m_s\right)}. \tag{15}$$

V. Capacity

Channel capacity is defined as the maximum data rate in bits/sec/Hz that the channel can support error free and can be expressed as

$$C = \log_2\left(1+\gamma\right). \tag{16}$$

The average capacity for Fisher $\mathcal{F}$ fading channels is evaluated by averaging (16) over the PDF in (2). Which can be written with help of [8, (8.4.6.5)] as

$$C_{avg} = \int_0^\infty \frac{m^m\gamma^{m-1}}{\ln\left(2\right)\left(m_s\overline{\gamma}\right)^m\Gamma\left(m_s\right)\Gamma\left(m\right)} \times \mathrm{H}_{1,1}^{1,1}\left(\frac{\mathrm{m}\gamma}{\mathrm{m_s}\overline{\gamma}}\middle| \begin{array}{c}\left(1-m-m_s,1\right) \\ \left(0,1\right)\end{array}\right) \times \mathrm{H}_{2,2}^{1,2}\left(\gamma\middle| \begin{array}{c}\left(1,1\right),\left(1,1\right) \\ \left(1,1\right),\left(0,1\right)\end{array}\right)\mathrm{d}\gamma. \tag{17}$$

The integral in (17) can be solved in closed-form using [11, (2.8.4)] as

$$C_{avg} = \frac{1}{\ln\left(2\right)\Gamma\left(m\right)\Gamma\left(m_s\right)} \times \mathrm{H}_{2,2}^{1,2}\left(\frac{\mathrm{m_s}\overline{\gamma}}{\mathrm{m}}\middle| \begin{array}{c}\left(1,1\right),\left(1,1\right),\left(1-m,1\right) \\ \left(1,1\right),\left(m_s,1\right),\left(0,1\right)\end{array}\right). \tag{18}$$

Note that (7), (14), and (18) can be rewritten in terms of the Meijer G-function using [8, (8.3.2.21)], which is a built-in function in well-known software packages such as MATLAB and MATHEMATICA. On the other hand, the bivariate Fox H-function in (9), whose implementation is outlined in [13], can be expressed in terms of the bivariate Meijer G-function using [14, (2.3.1)], whose implementation is outlined in [15].

VI. Numerical Results

In this section, we evaluate the derived expressions for the average SER and capacity for different values of the parameters m and m_s to show the effects of different amounts of fading and shadowing. All results show excellent agreement between analytical and simulation results which verifies our derivations. Moreover, asymptotic results for the average SER are shown to follow simulation results at high values of SNR.

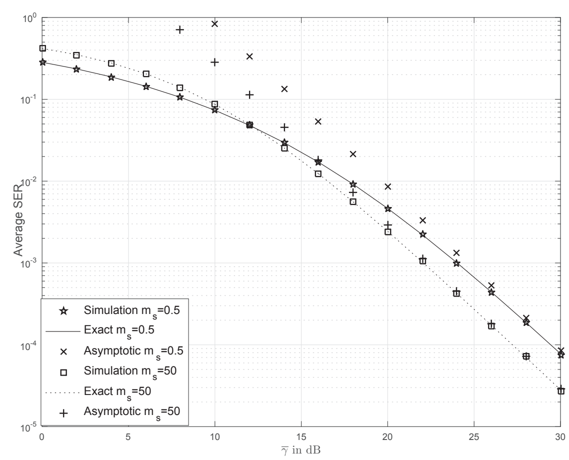

Fig. 1: SER of 4-PAM in the presence of mild fading ($m = 10$) with different amounts of shadowing.

Figs. 1 and 2 show the exact and asymptotic average SER of 4-PAM compared to simulations for different amounts of fading and shadowing. Fig. 1 shows the average SER of 4-PAM in the presence of mild fading ($m = 10$) with different amounts of shadowing. It is evident from the figure that for SNR values below 12 dB, heavy shadowing ($m_s = 0.5$) has lower SER than light shadowing ($m_s = 50$). However, for SNR values larger than 12 dB, light shadowing has lower SER than heavy shadowing. Fig. 2 shows the average SER of 4-PAM in the presence of light shadowing ($m_s = 50$) with different amounts of fading. It is clear from the figure that the SER decreases as m increases (less fading).

Figs. 3 and 4 show the exact and asymptotic average SER of 4-QAM and 16-QAM compared to simulations for different amounts of fading and shadowing. Fig. 3 shows that the SER of 4-QAM with $m = 1.5$ and different amounts of shadowing. It is clear from the figure that the SER performs better in

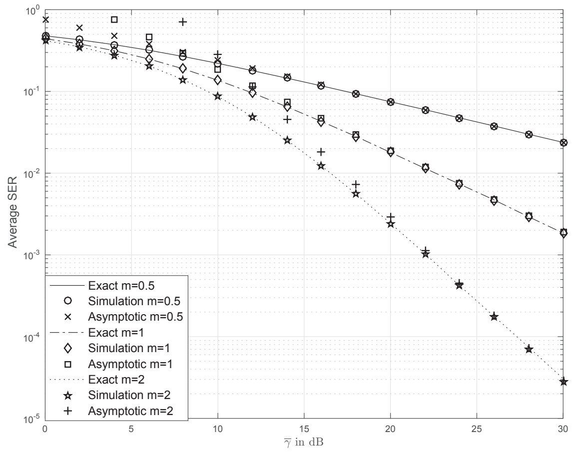

Fig. 2: SER of 4-PAM in the presence of light shadowing ($m_s = 50$) with different amounts of fading.

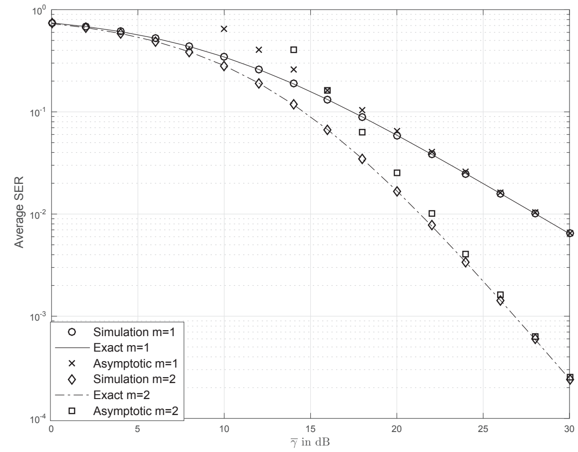

Fig. 4: SER of 16-QAM in the presence of heavy shadowing ($m_s = 5$) with different amounts of fading.

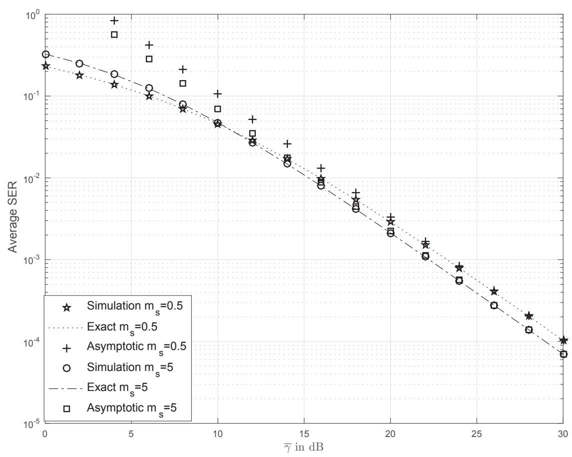

Fig. 3: SER of 4-QAM with $m = 1.5$ and different amounts of shadowing.

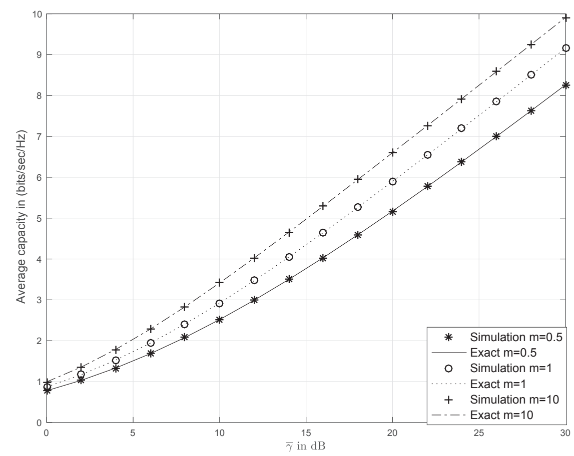

Fig. 5: Average capacity in the presence of light shadowing ($m_s = 50$) with different amounts of fading.

the presence of heavy shadowing ($m_s = 0.5$) for SNR values less than 12 dB. However, at SNR values larger than 12 dB, the SER performs better at less shadowing ($m_s = 5$). Fig. 4 shows the SER performance of 16-QAM with $m_s = 5$ and for different values of the fading parameter m. It is clear from the figure that the SER decreases with less fading (i.e. larger values of m).

Fig. 5 shows the average capacity in the presence of light shadowing environments ($m_s = 50$) with different values of the fading parameter m. It is evident from the figure that capacity increases with m (i.e. less fading).

VII. Conclusion

In this paper, we derived exact and asymptotic closed-form expressions for the average symbol error rate of M-PAM and M-QAM over the Fisher-Snedecor composite fading channels. We have also provided closed-form expression for the corresponding average capacity. The results showed excellent match between exact and simulation results.

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