The ergodic high SNR capacity of the spatially-correlated non-coherent MIMO channel within an SNR-independent gap

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Introduction

- Non-coherent MIMO communication system: No channel state information (CSI) is available at either the Tx or Rx.
- The analysis of non-coherent systems accounts for the communication resources expended to acquire accurate CSI.
- Training cost tolerable in static and slow fading scenarios, but not in fast and block fading ones.
- In fast fading, more beneficial to use signalling strategies that do not require Rx to know CSI. (Hochwald et al. ’00, Zheng et al. ’02)

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Previous Work

- For spatially-white channel, input matrices that achieve capacity have the following structures:
  - At any SNR and any coherence interval, the product of an isotropically distributed unitary component and a diagonal component with non-negative entries. (Hochwald et al. '00)
  - At high SNRs and coherence interval greater than a threshold, τ, isotropically distributed unitary on the Grassmann manifold. (Zheng et al. '02)
  - At high SNRs and coherence interval less than τ, the product of an isotropically distributed unitary component and a diagonal component with random entries distributed as the square root of the eigenvalues of a beta matrix. (Yang et al. '13)
  - At low SNRs, only one entry of the diagonal component is potentially non-zero. (Srinivasan et al. '09)
- What about spatially-correlated channels?

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Spatial correlation

- Spatial correlation arises due to proximity of physical antennas, especially in prospective massive MIMO systems.
- Correlation nonnegligible, even when spacing exceeds multiple wavelengths.
- Kronecker model: left and right multiplication of the spatially-white channel matrix with Tx and Rx correlation matrices.
- Correlation matrices, vary much more slowly than instantaneous channel parameters. Can be estimated accurately and assumed known. (Yu et al. ’04)
- Correlation: significant impact on signalling methodology and achievable rate.
- Kronecker correlation noncoherent model considered in (Jafar et al. ’05) at any SNR.

Non-coherent communication on spatially correlated channels: What is not known?

- No closed-form expressions for capacity, or bounds thereof.
- No constructive signalling strategy to approach capacity.

This Work

- Derive an expression for the ergodic high SNR non-coherent capacity for block Rayleigh fading channels with Kronecker correlation.
- Expression accurate within an SNR-independent gap and an error that decays as $1/\text{SNR}$.
- Derive an upper bound on the gap to the actual capacity. Gap decreases monotonically with logarithm of condition number of Tx correlation.
- Show that input signals that achieve capacity lower bound can be expressed as product of isotropically distributed random Grassmannian component and deterministic component comprising eigenvectors and inverse of eigenvalues of Tx correlation matrix.

System Model

- Frequency-flat block Rayleigh fading channel with equal number of Tx and Rx antennas, M.
- Correlated signals emitted from Tx and correlated signals impinging on Rx. Channel

$$\boldsymbol{H} = A^{1/2}\boldsymbol{H}_w B^{1/2},$$

where A and B are Tx and Rx pd correlation matrices, and $\boldsymbol{H}_w$ random with zero-mean unit-variance i. i. d. circularly-symmetric complex Gaussian entries.

- We assume A and B are full rank and $\operatorname{Tr} A = \operatorname{Tr} B = 1$.
- Block fading model with coherence time T.

System Model (cont'd)

- The received signal matrix can be expressed as

$$\boldsymbol{Y} = \boldsymbol{X}A^{1/2}\boldsymbol{H}_w B^{1/2} + \boldsymbol{V},$$

where $\boldsymbol{X} \in \mathbb{C}^{T\times M}$ is Tx signal, and $\boldsymbol{V} \in \mathbb{C}^{T\times M}$ additive noise; the entries of $\boldsymbol{V}$ are i. i. d. standard complex Gaussian random variables.

- Tx power constraint:

$$\mathrm{E}\{\mathrm{Tr}(\boldsymbol{X}\boldsymbol{X}^\dagger)\} \leq TP.$$

- The matrices A and B are known but $\boldsymbol{H}_w$ is not.

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The right singular vectors of $\boldsymbol{X}$

- Conditioned on $\boldsymbol{X}$, $\boldsymbol{Y}$ is Gaussian and

$$p(\boldsymbol{Y}|\boldsymbol{X}) = \frac{\exp\Big(-\operatorname{vec}^\dagger(\boldsymbol{Y})(B \otimes \boldsymbol{X}A\boldsymbol{X}^\dagger + I_{MT})^{-1}\operatorname{vec}(\boldsymbol{Y})\Big)}{\pi^{TM}\det(B \otimes \boldsymbol{X}A\boldsymbol{X}^\dagger + I_{MT})}.$$

- For deterministic Φ, $p(\Phi\boldsymbol{Y}|\Phi\boldsymbol{X}) = p(\boldsymbol{Y}|\boldsymbol{X})$, yielding optimal

$$\boldsymbol{X} = \boldsymbol{Q}_X\boldsymbol{D}U_A^\dagger, \qquad \text{where}$$

  - $\boldsymbol{Q}_X$ isotropically distributed unitary matrix;
  - $\boldsymbol{D}$ random diagonal with non-negative entries; and
  - U_A is the the matrix containing the eigenvectors of A.

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Conditional Entropy $h(\boldsymbol{Y}|\boldsymbol{X})$

$$C(P) = \max_{p(\boldsymbol{X}),\ \mathrm{E}\{\mathrm{Tr}(\boldsymbol{X}\boldsymbol{X}^\dagger)\} \leq TP} \frac{1}{T}\big(h(\boldsymbol{Y}) - h(\boldsymbol{Y}|\boldsymbol{X})\big).$$

Our goal is to evaluate $C(P)$ as $P \to \infty$.

- Evaluating $h(\boldsymbol{Y}|\boldsymbol{X})$ straightforward

$$h(\boldsymbol{Y}|\boldsymbol{X}) = MT \log \pi e + M \log\det AB + \mathrm{E}\{\log\det \boldsymbol{D}^2\} + O(1/P).$$

- Approximation valid when $\boldsymbol{D}$ full rank and its entries scale with P.

Nonconditional Entropy $h(\boldsymbol{Y})$

- Computing entropy of signal component plus noise component formidable task.
- At high SNR, write

$$\begin{aligned} h(\boldsymbol{Y}) &= h(\boldsymbol{X}A^{1/2}\boldsymbol{H}_w B^{1/2}) + O(1/P) \\ &= h(\boldsymbol{Q}_X \boldsymbol{D}\Lambda_A^{1/2}\boldsymbol{H}_w) + T \log\det B + O(1/P). \end{aligned}$$

  - Λ_A diagonal matrix of eigenvalues.
- The matrix $\boldsymbol{Q}_X \in \mathbb{C}^{T\times M}$, $T \geq M$.
- An expression for $h(\boldsymbol{Q}_X \boldsymbol{D}\Lambda_A^{1/2}\boldsymbol{H}_w)$ can be obtained by transforming from Cartesian to QR coordinates (Zheng et al. ’02).

- Coordinate change yields

$$h(\boldsymbol{Q}_X \boldsymbol{D} \Lambda_A^{1/2} \boldsymbol{H}_w) = h(\boldsymbol{\Psi} \boldsymbol{D} \Lambda_A^{1/2} \boldsymbol{H}_w) + \log |\mathbb{G}_M(\mathbb{C}^T)| + (T - M)\,\mathrm{E}\{\log\det \boldsymbol{H}_w^\dagger \boldsymbol{D}^2 \Lambda_A \boldsymbol{H}_w\}.$$

  - $\mathbb{G}_M(\mathbb{C}^T)$ is the Grassmann manifold; and
  - $\boldsymbol{\Psi} \in \mathbb{C}^{M \times M}$ isotropically distributed.
- Computing $h(\boldsymbol{\Psi} \boldsymbol{D} \Lambda_A^{1/2} \boldsymbol{H}_w)$ is the difficult part.
- Without spatial correlation $\Lambda_A = I_M$ and optimal $\boldsymbol{D} = I_M$.
- For case with spatial correlation, we develop bounds.

Upper Bound on Capacity

- Gaussian distribution maximizes entropy yields

$$h(\boldsymbol{\Psi D}\Lambda_A^{1/2}\boldsymbol{H}_w) \leq M^2 \log \frac{\pi e T}{M} \lambda_{A_1} P. \qquad (1)$$

  - Bound not achievable unless $A = \frac{1}{M} I_M$ and $\mathrm{E}\{\boldsymbol{D}^2\} = \frac{PT}{M} I_M$.

- Upper bound on capacity:

$$C(P) \leq M\left(1 - \frac{M}{T}\right) \log \frac{TP}{\pi e M} + \left(1 - \frac{2M}{T}\right) \log \det A + \frac{M^2}{T} \log \lambda_{A_1} + \frac{1}{T} \log |\mathbb{G}_M(\mathbb{C}^T)| + \left(1 - \frac{M}{T}\right) \mathrm{E}\{\log \det B \boldsymbol{H}_w \boldsymbol{H}_w^\dagger\} + O(1/P).$$

Lower Bound on Capacity

- Restricting $\boldsymbol{D}$ to particular distribution yields lower bound on capacity.
- Set $\boldsymbol{D}$ to deterministic

$$D = \sqrt{\frac{TP}{\operatorname{Tr}\Lambda_A^{-1}}}\Lambda_A^{-1/2}$$

- Choice ensures $\boldsymbol{\Psi D}\Lambda_A^{1/2}\boldsymbol{H}_w$ Gaussian, i.i.d. entries

$$h(\boldsymbol{\Psi D}\Lambda_A^{1/2}\boldsymbol{H}_w) = M^2 \log \frac{\pi e TP}{\operatorname{Tr}\Lambda_A^{-1}}.$$

- Lower bound on capacity:

$$C(P) \geq M\left(1 - \frac{M}{T}\right) \log \frac{TP}{\pi e \operatorname{Tr}\Lambda_A^{-1}} + \frac{1}{T} \log |\mathbb{G}_M(\mathbb{C}^T)| + \left(1 - \frac{M}{T}\right) \mathrm{E}\{\log\det B\boldsymbol{H}_w\boldsymbol{H}_w^\dagger\} + O(1/P).$$

How tight are the bounds?

- Let gap between bounds be Δ

$$\Delta \leq M\left(1 - \frac{M}{T}\right) \log \kappa_A.$$

  - κ_A condition number of A.

Main Result

$$\log \frac{TP}{\pi e \operatorname{Tr} \Lambda_A^{-1}} \leq \frac{C(P) - c}{M(1 - M/T)} \leq \log \frac{TP\kappa_A}{\pi e \operatorname{Tr} \Lambda_A^{-1}}$$

- $c = \frac{1}{T} \log |\mathbb{G}_M(\mathbb{C}^T)| + \left(1 - \frac{M}{T}\right) \mathrm{E}\{\log \det \boldsymbol{B} \boldsymbol{H}_w \boldsymbol{H}_w^\dagger\} + O(1/P)$.
- Lower bound achieved by input signals $\boldsymbol{X} = \boldsymbol{Q}_X \boldsymbol{D} \boldsymbol{U}_A^\dagger$, and $\boldsymbol{D} = \sqrt{\frac{TP}{\operatorname{Tr} \Lambda_A^{-1}}} \Lambda_A^{-1/2}$.

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Comments on Results

- Rate achieved by setting $D = \sqrt{\frac{TP}{\operatorname{Tr}\Lambda_A^{-1}}\Lambda_A}^{-1/2}$ is within $M\left(1 - \frac{M}{T}\right)\log\kappa_A$ bits from capacity.
- Signalling strategy optimal when channel coefficients possibly correlated at Rx but independent at Tx; more likely in downlink scenarios.
- For channels with κ_A slightly greater than 1, rate loss relatively small.
- Rate loss is unbounded as $\kappa_A \to \infty$. Message: use $D = \sqrt{\frac{TP}{\operatorname{Tr}\Lambda_A^{-1}}\Lambda_A}^{-1/2}$ only to excite non-negligible eigenmodes of the channels.

Conclusions

- Closed form expressions for upper and lower bounds on ergodic non-coherent capacity of spatially correlated MIMO systems with M transmit and receive antennas.
- Lower bound achievable using deterministic precoding.
- Gap between bounds does not depend on SNR and increases monotonically with transmit condition number.
- Results are tight within $O(1/P)$.