Multi-Resolution Broadcasting Over the Grassmann and Stiefel Manifolds

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Motivation

- A receiver in a broadcast scenario in MIMO systems may or may not be able to acquire reliable CSI.
- In such cases, the transmitter may wish to send
  - Basic low-resolution (LR) information that can be detected by all receivers, including those without CSI.
  - Incremental high-resolution (HR) information to receivers with reliable CSI.

Motivation (cont'd)

- In this paper, we address the problem of designing space-time codes that allow the simultaneous transmission of information to two classes of receivers:
  - HR receivers, which have access to reliable CSI and can perform coherent detection.
  - LR receivers, which do not have access to reliable CSI and can only perform non-coherent detection.

Preliminaries

- For $T \geq M$, **the Stiefel manifold** $\mathbb{S}_{T,M}(\mathbb{C})$ is defined as the set of all unitary $T \times M$ matrices.

$$\mathbb{S}_{T,M}(\mathbb{C}) = \{\mathbf{Q} \in \mathbb{C}^{T \times M} : \mathbf{Q}^{\mathscr{H}}\mathbf{Q} = \mathbf{I}_M\}. \quad (1)$$

- The Stiefel manifold $\mathbb{S}_{T,M}(\mathbb{C})$ is a submanifold of $\mathbb{C}^{T \times M}$ of $TM - M^2/2$ complex dimensions.

Preliminaries (cont'd)

- **The Grassmann manifold** $\mathbb{G}_{T,M}(\mathbb{C})$ is defined as the quotient space of $\mathbb{S}_{T,M}(\mathbb{C})$ with respect to the equivalence relation that renders two elements $\mathbf{P}, \mathbf{Q} \in \mathbb{S}_{T,M}(\mathbb{C})$ equivalent if their T-dimensional column vectors span the same subspace, i.e.,

$$\mathbf{P} = \mathbf{Q}\mathbf{V} \tag{2}$$

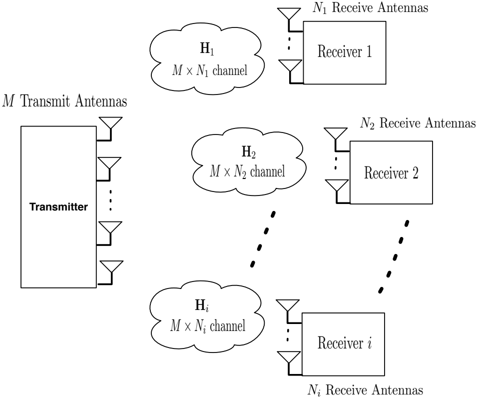
  for some matrix $\mathbf{V}$ in the unitary group $\mathbb{U}_M = \mathbb{S}_{M,M}(\mathbb{C})$.

- The number of complex dimensions of the Grassmann manifold can be expressed as:

$$\begin{aligned}\dim(\mathbb{G}_{T,M}(\mathbb{C})) &= \dim(\mathbb{S}_{T,M}(\mathbb{C})) - \dim(\mathbb{S}_{M,M}(\mathbb{C})) \\ &= M(T-M).\end{aligned} \tag{3}$$

System Model

- We consider a broadcast MIMO communication system with M transmit antennas with two classes of receivers operating over the block Rayleigh flat-fading channel.



System Model (cont'd)

The communication system can be modeled as

$$\begin{aligned}\mathbf{Y}_i &= \mathbf{X}\mathbf{H}_i + \mathbf{W}_i \\ &= \mathbf{U}\mathbf{A}\mathbf{H}_i + \mathbf{W}_i,\ i = 1, 2, \cdots,\end{aligned} \tag{4}$$

- The channel is assumed to be constant over T consecutive time slots.
- N_i denotes the number of receive antennas of the i-th receiver.
- $\mathbf{Y}_i$ is the $T \times N_i$ received matrix of the i-th receiver.
- The matrices $\mathbf{H}_i$ and $\mathbf{W}_i$ represent the channel and noise observed by receiver i (independent entries).

System Model (cont'd)

- $\mathbf{X} = \mathbf{U}\mathbf{A}$ is the $T \times M$ transmitted matrix,
    - The LR information is encoded in the matrix $\mathbf{U} \in \mathbb{G}_{T,M}(\mathbb{C})$. This layer has the advantage that it can be decoded coherently if the receiver has reliable CSI or non-coherently if CSI is not available.
    - The HR information is encoded in the matrix $\mathbf{A} \in \mathbb{U}_M$. This layer can only be decoded coherently by a receiver that has access to reliable CSI.

System Model (cont'd)

- At high SNR, the capacity of the LR channel observed by, say, receiver i can be achieved if $\mathbf{X}$ were isotropically distributed on $\mathbb{G}_{T,M}(\mathbb{C})$, provided that:
  - $N_i \geq M$.
  - $T \geq M + N_i$.
  - $M \leq \lfloor T/2 \rfloor$

The Optimum Non-coherent Detector

- Starting from GLRT detector

$$\hat{\mathbf{U}} = \arg\max_{\mathbf{U}} \sup_{\mathbf{H}} p(\mathbf{Y}|\mathbf{U},\mathbf{H}). \quad (5)$$

and using the facts that the matrix $\mathbf{U}$ is unitary and that the fading coefficients are i.i.d Gaussian-distributed random variables, the detector can be shown to be the following maximum likelihood (ML) detector

$$\hat{\mathbf{U}} = \arg\max_{\mathbf{U}} \text{Trace}(\mathbf{Y}^{\mathscr{H}}\mathbf{U}\mathbf{U}^{\mathscr{H}}\mathbf{Y}). \quad (6)$$

The Optimum Non-coherent Detector (cont'd)

- Encoding the HR information in the unitary matrix $\mathbf{A} \in \mathbb{U}_M$ does not compromise the performance of the non-coherent GLRT detector, since $\mathbf{H} \stackrel{d}{=} \mathbf{A}\mathbf{H}$.
- A Grassmannian-structured codebook then will exhibit a particular diversity order regardless of whether incremental HR information is transmitted.

The Pairwise Error Probability (PEP)

- The pairwise error probability (PEP) can be upper bounded by

$$\mathrm{PEP}(\mathbf{U}_1 \to \mathbf{U}_2) \leq \frac{1}{2}\prod_{m=1}^{M}\left[1 + \frac{\left(\frac{\mathrm{SNR}}{M}\right)^2 (1 - s_m^2)}{4(1 + \frac{\mathrm{SNR}}{M})}\right]^{-N}, \qquad (7)$$

where $\mathrm{SNR} \triangleq \mathbf{E}(\mathrm{Trace}\,\mathbf{X}\mathbf{X}^{\mathscr{H}})/(N_0 M T)$ and $1 \geq s_1 \geq \cdots \geq s_M \geq 0$ are the singular values of the $M \times M$ matrix $\mathbf{U}_2^{\mathscr{H}}\mathbf{U}_1$.

- The asymptotic SNR exponent equals MN, thereby ensuring that the Grassmannian codebook achieves **full diversity order**, as if the HR information were not transmitted.

The Optimum One-Step Coherent Detector

- The optimum coherent detector that "jointly" decodes the LR and HR information layers can be expressed as the detector that yields

$$\hat{\mathbf{X}} = \arg\min_{\mathbf{X}} \|\mathbf{Y} - \mathbf{X}\mathbf{H}\|^2. \tag{8}$$

- This detector requires an exhaustive search over $S_L S_H$ codewords.

The Two-Step Coherent Detector

- In the first step of this detector, the GLRT approach in (6) is used to detect the LR Grassmannian codeword.
- In the second step of the sequential detector, the GLRT output, $\hat{\mathbf{U}}$, is assumed to be the correct Grassmannian codeword . The output of this ML detector is given by

$$\hat{\mathbf{A}} = \arg\min_{\mathbf{A}} \|\mathbf{Y} - \hat{\mathbf{U}}\mathbf{A}\mathbf{H}\|^2. \quad (9)$$

- Less complex than the one-step detector in (8), as it requires searching over $S_L + S_H$ codewords.
- Both detectors yield the same diversity order.

Theorem

Theorem

Let the LR and HR codebooks, $\{\mathbf{U}\}$ and $\{\mathbf{A}\}$, satisfy the full diversity singular values criterion for non-coherent codes and the full diversity determinant criterion for coherent codes, respectively. Then, the sequential two-step coherent detector achieves a diversity of order MN, i.e., full diversity.

Remark:

Since full diversity is achieved by the suboptimal sequential two-step detector, this diversity order must be also achieved by the optimal one-step coherent detector.

Degrees of Freedom

Corollary

The achievable degrees of freedom for the conjoined LR and HR layers is $TM - M^2/2$, whereas the achievable degrees of freedom for the LR layer is $M(T-M)$.

- The proposed construction does not achieve the maximum number of degrees of freedom for the HR receivers TM.
- Due to restricting $\mathbf{X}$ to be in $\mathbb{S}_{T,M}(\mathbb{C})$ which is equivalent to restricting $\mathbf{A}$ to be in $\mathbb{U}_M$, this number is reduced by $M^2/2$.
- This reduction can be regarded as the price paid to ensure that the basic LR information rate, which can be decoded by all receivers, is maximized.

HR Layer (Coherent) Code Construction

- A candidate of such coherent codes is the standard 2×2 Alamouti scheme as follows:

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix}, \tag{10}$$

where s_1 and s_2 are two complex symbols drawn from any constant modulus constellation, e.g., PSK.

- For larger M, square orthogonal coherent code designs that exhaust all the $M^2/2$ degrees of freedom can be constructed directly on $\mathbb{U}_M$.

LR Layer (Non-coherent) Code Construction

- For the LR (non-coherent) code construction, we consider two approaches:
  - The exponential parameterization.
  - The direct design.

The Exponential Parameterization

- In this approach non-coherent codes are obtained from coherent block codes using the exponential map.
- The non-coherent code matrices $\{\mathbf{U}\}$ in (4) are constructed using

$$\mathbf{U} = \left[\exp\begin{pmatrix} \mathbf{0}_M & \alpha\mathbf{V} \\ -\alpha\mathbf{V}^{\mathscr{H}} & \mathbf{0}_M \end{pmatrix}\right]\mathbf{I}_{T,M}, \tag{11}$$

where $\mathbf{V} \in \mathbb{C}^{M\times(T-M)}$ is the matrix representing the coherent code and α is a homothetic factor which ensures that the singular values of $\mathbf{V}$ are less than $\pi/2$.

- Although this approach facilitates the design of non-coherent codes, it does not provide performance guarantees.

The Direct Design

- The minimum chordal Frobenius norm between the spaces spanned by any two matrices $\mathbf{U}_i, \mathbf{U}_j \in \mathbb{G}_{T,M}(\mathbb{C})$ is maximized.
- This norm is given by $\sqrt{2M - 2\,\mathrm{Trace}\,(\boldsymbol{\Sigma}_{ij})}$, where $\boldsymbol{\Sigma}_{ij}$ is the matrix containing the singular values of $\mathbf{U}_i^{\mathscr{H}}\mathbf{U}_j$.
- The S_L Grassmannian constellation points required for the non-coherent code of the LR layer can be cast as the following optimization problem:

$$\begin{aligned} &\min_{\{\mathbf{U}_r\}_{r=1}^{S_L}} \quad \max_{1\leq i,j\leq S_L} \quad \mathrm{Trace}(\boldsymbol{\Sigma}_{ij}) \\ &\text{subject to} \quad \mathbf{U}_k \in \mathbb{G}_{T,M}(\mathbb{C}), \quad k = 1,\ldots,S_L. \end{aligned} \tag{12}$$

Simulation Results

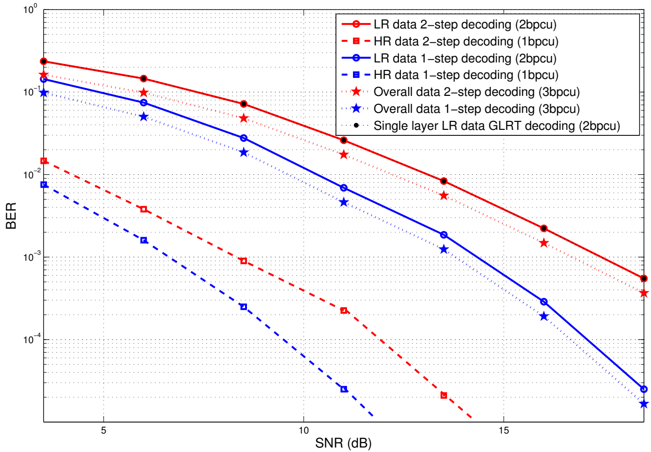


Figure: The LR layer is constructed on $\mathbb{G}_{4,2}(\mathbb{C})$ using the exponential parameterization method and the HR layer is constructed using the 2×2 Alamouti code..

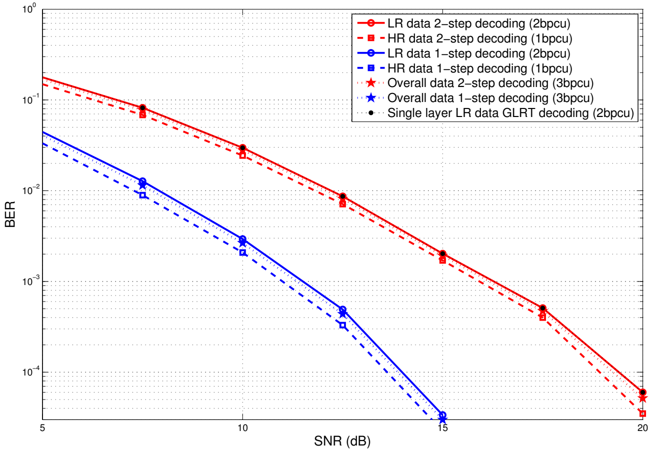


Figure: The LR layer is constructed on $\mathbb{G}_{4,2}(\mathbb{C})$ using the direct design technique and the HR layer is constructed using the 2×2 Alamouti code.

Conclusions

- We proposed a new layered multi-resolution broadcast space-time coding scheme.
- The proposed scheme ensures that the communication of the HR layer is transparent to the underlying LR layer.
- We showed that both the non-coherent and coherent receivers achieve full diversity.
- The proposed scheme achieves the maximum number of communication degrees of freedom for non-coherent LR channels and coherent HR channels with unitarily-constrained input signals.

Thank you for your time and attention. **Questions?**