Non-coherent Multi-Layer Constellations for Unequal Error Protection

Speaker: Karim G. Seddik\textsuperscript{1}

in collaboration with Kareem M. Attiah\textsuperscript{2}, Ramy H. Gohary\textsuperscript{3}, and Halim Yanikomeroglu\textsuperscript{3}

\textsuperscript{1}American University in Cairo
\textsuperscript{2}Alexandria University
\textsuperscript{3}Carleton University

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5 Simplified Decoder
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- Transmitter chooses one of possible $N$ supersymbols from the set $S = \{s_1, \ldots, s_N\}$.  
- For simplicity, assume $L = 2$.
  Define the sets $S^1 = \{s^1_1, \ldots, s^1_{N_1}\}$ and $S^2 = \{s^2_1, \ldots, s^2_{N_2}\}$, where the set $S^1$ ($S^2$) contains the more (less) protected subsymbols.
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  Define the sets $S^1 = \{s^1_1, \ldots, s^1_{N_1}\}$ and $S^2 = \{s^2_1, \ldots, s^2_{N_2}\}$, where the set $S^1$ ($S^2$) contains the more (less) protected subsymbols.
- The chosen supersymbol $s_i$ is mapped onto the pair $(s^1_i, s^2_i)$ by some bijective function $f : S \rightarrow S^1 \times S^2$. 
Non-Coherent Signaling

Channel state information (CSI) is unknown at both transmitter and receiver. Transmission takes place over $T$ symbol durations. Channel coefficients are assumed constant during that interval. We only consider $T > M$. Transmitted symbols are represented by points on the Grassmannian manifold $\mathbb{G}_T^M(C)$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{non_coherent_signaling_diagram}
\caption{Non-Coherent Signaling}
\end{figure}
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The constellation is decomposed into disjoint subsets.
More important symbols encode subsets.
Less important symbols encode points within a chosen subset.
$d_1$ ($d_2$) controls the error probability of the more (less) protected symbols.
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For unitary signaling, the asymptotic error probability of mistaking $X_i$ for $X_j$ under GLRT detection is given by

$$ P_{ij} = (d_{cp}(X_i, X_j)\gamma T)^{-MN} M^{MN} \left(\frac{2MN - 1}{MN}\right), $$

$$ d_{cp}(X_i, X_j) = \left(\prod_{k=1}^{M} \sin^2 \theta_k\right)^{\frac{1}{M}}. $$

$\{\theta_i\}_{i=1}^{M}$ are the principle angles between the subspaces associated with $X_i$ and $X_j$. 

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2. I. Kammoun, A. M. Cipriano, and J. C. Belfiore, Non-coherent codes over the grassmannian, IEEE Transactions on Wireless Communications.
How to Measure the Distance between Constellation Points?

- For unitary signaling, the asymptotic error probability of mistaking $\mathbf{X}_i$ for $\mathbf{X}_j$ under GLRT detection is given by\(^1\)

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  \{\theta_i\}_{i=1}^{M} are the principle angles between the subspaces associated with $\mathbf{X}_i$ and $\mathbf{X}_j$.

- The **Chordal Product Distance**\(^2\) (not a true distance!) can be employed to define the coding gain of the more (less) protected symbols.

  \[
  d_1 = \min_{i \neq j} d_{cp}(\Omega_i, \Omega_j),
  \]

  \[
  d_2 = \min_{\mathbf{X}, \mathbf{Y} \in \mathcal{C}} d_{cp}(\mathbf{X}, \mathbf{Y}).
  \]

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\(^1\)M. Brehler and M. K. Varanasi, Asymptotic error probability analysis of quadratic receivers in rayleigh-fading channels with applications to a unified analysis of coherent and noncoherent space-time receivers, IEEE Transactions on Information Theory.

\(^2\)I. Kammoun, A. M. Cipriano, and J. C. Belfiore, Non-coherent codes over the grassmannian, IEEE Transactions on Wireless Communications.
A related quantity is the Chordal Distance

\[ d_c(X_i, X_j) = \sum_{k=1}^{M} \sin^2 \theta_k. \]
How to Measure the Distance between Constellation Points?

- A related quantity is the **Chordal Distance**

\[ d_c(\mathbf{x}_i, \mathbf{x}_j) = \sum_{k=1}^{M} \sin^2 \theta_k. \]

- The AM-GM inequality governs the relationship between the two metrics

\[ d_{cp}(\mathbf{x}_i, \mathbf{x}_j) \leq \frac{1}{M} d_c(\mathbf{x}_i, \mathbf{x}_j). \]
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Our design approach is entirely based on the more intuitive chordal distance. However, the connection between both metrics is essential since the coding gains are expressed in terms of the product chordal distance.
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5. Simplified Decoder
For an initial layer $\xi_1$, we define a sequence of $L - 1$ children layers $\xi_2, \xi_3, \ldots, \xi_L$. 

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$^{3}$K. M. Attiah, K. Seddik, R. H. Gohary, and H. Yanikomeroglu, A systematic design approach for non-coherent grassmannian constellations, ISIT, 2016.
For an initial layer $\xi_1$, we define a sequence of $L - 1$ children layers $\xi_2, \xi_3, \ldots, \xi_L$.

Starting at $X = X(0)$, the motion of a point traveling along some direction $B$ is given by

$$X(t) = \begin{bmatrix} X & X_\bot \end{bmatrix} \begin{bmatrix} V \cos \Sigma t \\ U \sin \Sigma t \end{bmatrix}.$$ 

Where $B = UV^\dagger$ is the SVD of $B$, $X_\bot$ is the orthogonal complement of $X$ and $t$ is a step parameter.
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Starting at $X = X(0)$, the motion of a point traveling along some direction $B$ is given by

$$X(t) = [X \quad X_\perp] \begin{bmatrix} V \cos \Sigma t \\ U \sin \Sigma t \end{bmatrix}.$$ 

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$$\xi_i = \left\{ [X_p \quad X_p^\perp] \begin{bmatrix} V_k \cos \Sigma_k t_i \\ U_k \sin \Sigma_k t_i \end{bmatrix} \mid \forall X_p \in C_i, \forall k = 1, \ldots, K \right\}.$$
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$$\xi_i = \left\{ \begin{bmatrix} x_p & x_p \perp \end{bmatrix} \begin{bmatrix} V_k \cos \Sigma_k t_i \\ U_k \sin \Sigma_k t_i \end{bmatrix} \bigg| \forall x_p \in C_i, \forall k = 1, \ldots, K \right\}. $$

The $i^{th}$ parent layer is formed by augmenting all existing points at depth $i$

$$C_i = \bigcup_{j=1}^{i-1} \xi_j,$$
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The size of the constellation is given by $|C| = N'(K + 1)^{L-1}$, where $N'$ is the initial constellation size and $K$ is the number of geodesic directions.
Initial constellation.
Geodesic directions ensure maximal spacing.
Children points are found along the geodesic directions.
The process can be further repeated to add more points.
The step parameter $t$ controls how farther away are children points from their respective parents.
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With careful selection of this parameter, a specific coding gain can be ensured for the more protected layer over the Equal Error Protection scheme.
Selecting $t$ to Ensure a Desired Coding Gain

- The step parameter $t$ controls how farther away are children points from their respective parents.

- With careful selection of this parameter, a specific coding gain can be ensured for the more protected layer over the Equal Error Protection scheme.

- We propose two ways of doing this:
  - 1. Geometric Mean Approximation.
  - 2. Polynomial Regression.
A lower bound on minimum subset distance $d_1$ is established in terms of $\beta = \sin t$.

$$d_1 \geq \frac{1}{M} (d_{\text{pack}} - 2\sqrt{M\beta})^2 - \frac{M-1}{2M^2} (d_{\text{pack}} - 2\sqrt{M\beta})^4,$$

where $d_{\text{pack}} = \frac{M(T-M)}{T} \sqrt{\frac{N'}{N'-1}}$. 
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This bound is rather crude when the constellation size is large.
Polynomial Regression

Assume

\[ d_1 = \sum_{i=1}^{n} a_i \beta^i. \]
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With enough observations \((l \geq n)\), we can estimate the unknown coefficients \(a_i\).
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- With enough observations \((l \geq n)\), we can estimate the unknown coefficients \(a_i\).

- Performance analysis suggests that this is the more reliable approach.
Performance Analysis

- $R = 1.5$ bpcu, 33% of data is important

![Graph showing performance analysis with various error rates and symbol error rates across different SNR values. The graph includes lines for EEP, UEP - PR $\xi_1$, UEP - GMA $\xi_1$, UEP - PR $\xi_2$, and UEP - GMA $\xi_2$. The x-axis represents SNR in dB, and the y-axis represents symbol error rate.]
Performance Analysis

- $R = 2$ bpcu, 50% of data is important
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- The decoder initially examines all constellation points $X \in \xi_1$. 

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- The decoder initially examines all constellation points $\mathbf{X} \in \xi_1$.

- The decoder then investigates the children of the $q$ largest ML metric. The process is repeated for $L > 2$. 
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A decision is made in favor of some $X$, if $X$ maximizes the ML metric over all examined points.
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A reduction factor of \( \left( \frac{q}{K} \right)^{L-1} \) is realized in the number of operations needed.
Analysis of Simplified Decoder

- $K = 15$. 

![Graph showing Symbol Error Rate vs. SNR for different codes and noise levels]