Polar Codes for Noncoherent MIMO Signalling

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  - Multilevel Polar Code Design Methodology
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Introduction

- Under multiple-input multiple output (MIMO) fast fading scenarios, channel estimation may not be easily/efficiently obtained.
- Grassmannian constellations, specifically designed for such scenarios, approach the ergodic channel capacity at high signal-to-noise ratio (SNR).
- Polar codes are known to achieve capacity for a wide range of communication channels with low encoding and decoding complexity.
- A novel methodology for designing multilevel polar codes that work effectively with a multidimensional Grassmannian signalling and a novel set partitioning algorithm that works for arbitrary, not necessarily structured, multidimensional signalling schemes are proposed.
- Simulation results confirm that substantial gains in performance over existing techniques are realized.

Grassmannian Signalling

- For noncoherent communication over block fading MIMO channels.
- Transmitted symbols, $\mathbf{X}$, are $T \times N_t$ complex matrices, isotropically distributed on a compact Grassmann manifold. $\mathbf{X}^\dagger\mathbf{X} = \mathbf{I}_{N_t}$.

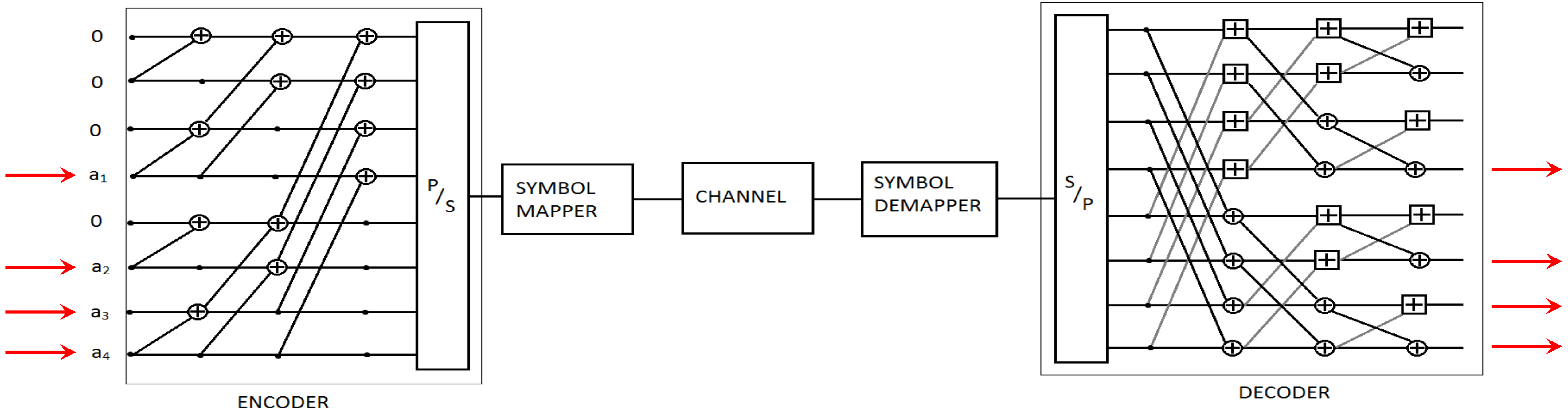
  T = number of time slots

  N_t = number of transmit antennas
- The number of symbols in the constellation is ideally large.
- The system model is $\mathbf{Y} = \mathbf{XH} + \mathbf{W}$
- No channel state information is required at the receiver or transmitter.
- In the uncoded case, the receiver maximizes the likelihood function

$$\Pr\{\mathbf{Y}|\mathbf{X}\} = \kappa \times exp\left\{\frac{\|\mathbf{X}^\dagger\mathbf{Y}\|^2}{\sigma_W^2(1+\sigma_W^2)}\right\}$$

Polar Codes

- Polar codes are the first provably capacity-achieving codes for binary-input symmetric memoryless channels.
- They require relatively low decoding complexity compared to other state-of-the-art coding techniques.
- Number and position of information bits in encoder define code rate and code design.

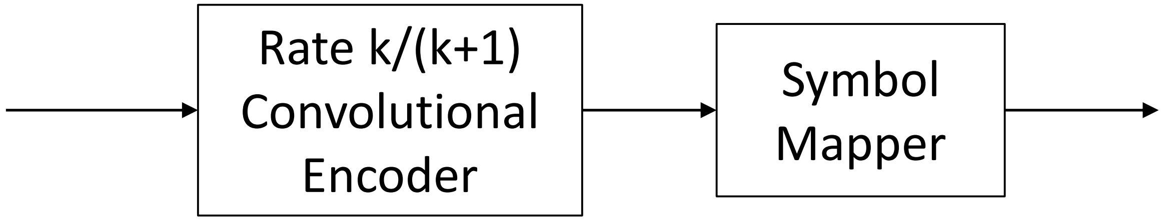


Polar Codes

- In a polar code with codeword length N and rate R, RN bit channels carry data while the rest are frozen (set to zero).
- The polar code performance is affected by which bit channels are chosen to send data over. Only the best RN bit channels should be used.
- Every change in the code length and channel characteristics affects the choice of bit channels.
- The encoder and decoder are defined by the choice of bit channels.

Spectrally Efficient Coded Modulation

- Involves combining error correcting codes with non-binary signalling.
- Techniques include trellis coded modulation (TCM), bit-interleaved coded modulation (BICM) and multi-level coding (MLC).
- TCM combines a high-rate convolutional code with non-binary constellations such as 8-PSK or 16-QAM:



Set partitioning is used to determine the bit-to-symbol mapping

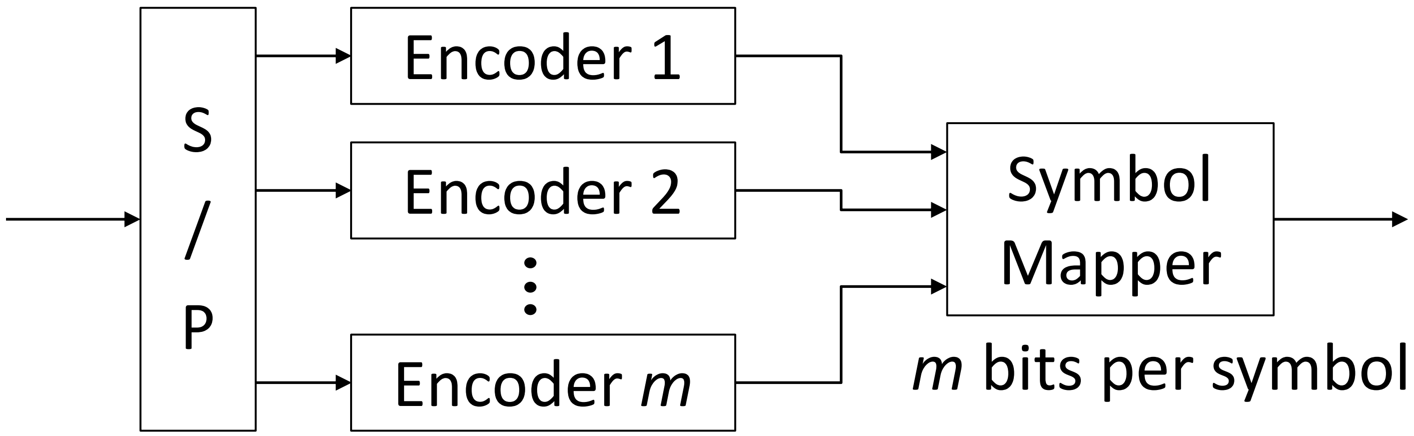
- BICM uses an interleaver between encoder and mapper:

Gray labelling is used for bit-to-symbol mapping

  - Can use any code, of any rate, with any constellation.
  - Interleaver must be carefully designed for compatibility with encoder and mapper.

Multilevel Coding

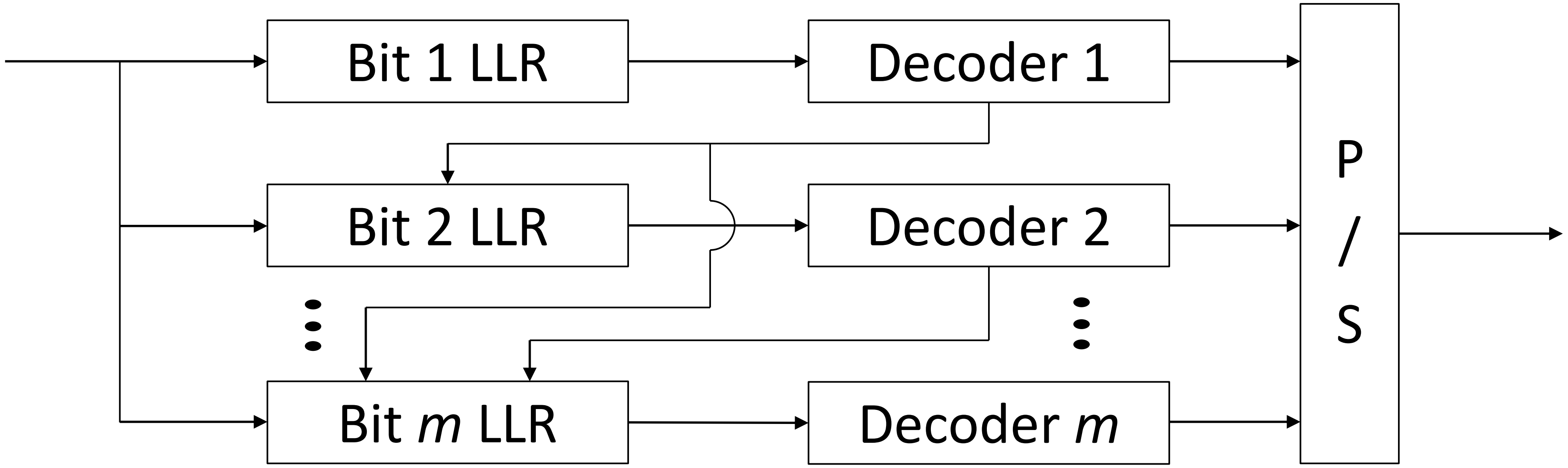
- Whereas convolutional codes work well with TCM and BICM, and LDPC and turbo codes work well with BICM, polar codes work better with multilevel coding.
- Uses a bank of encoders, each with a different rate.
- Number of encoders same as number of bits per channel symbol ($m = \log_2 M$)
- Each code bit from encoder 1 is transmitted in the first bit position of each symbol, each code bit from encoder 2 is transmitted in the second position, and so on.



Set partitioning is used for bit-to-symbol mapping

Multilevel Coding

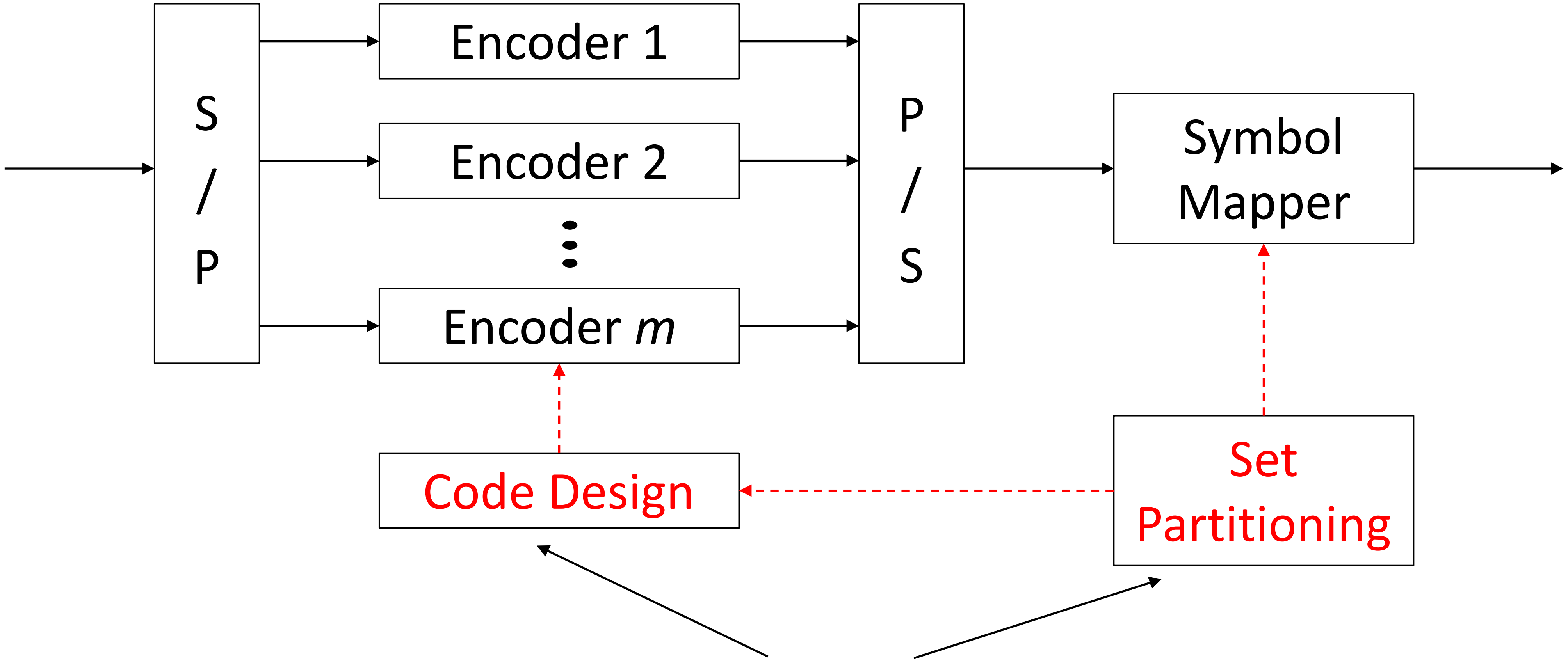
- Detect first bit in all the received symbols, and use them to decode first code. Use decoded code word to detect second bit in the symbols, and decode second code, and so on.
- Exploits differences in reliabilities between the different bits in the constellation.
  - Code rates selected to match reliabilities of the bit positions.
  - The overall code rate, R, of the encoder is determined by selecting the individual rates of the subcodes, R_i in such a way that $R = \frac{1}{m}\sum_{i=1}^{m} R_i$



Polar Codes for Irregular Multidimensional Constellations

- Multilevel polar codes have been proposed for regular 2-D constellations such as QAM or PSK.
- These regular constellations are easily set-partitioned in order to enable this method to work. However, this is not trivially extended to multidimensional constellations.
- We propose two novel techniques that enable the effective use of multilevel polar codes with multidimensional signal constellations.
- Irregular multidimensional constellations are used in:
  - Grassmannian signalling for noncoherent communication
  - Unitary space-time constellations for noncoherent communication
  - Golden codes for space-time block coding
  - Sparse code multiple access (SCMA)

Polar Codes for Irregular Multidimensional Constellations



Two new techniques for irregular multidimensional constellations:

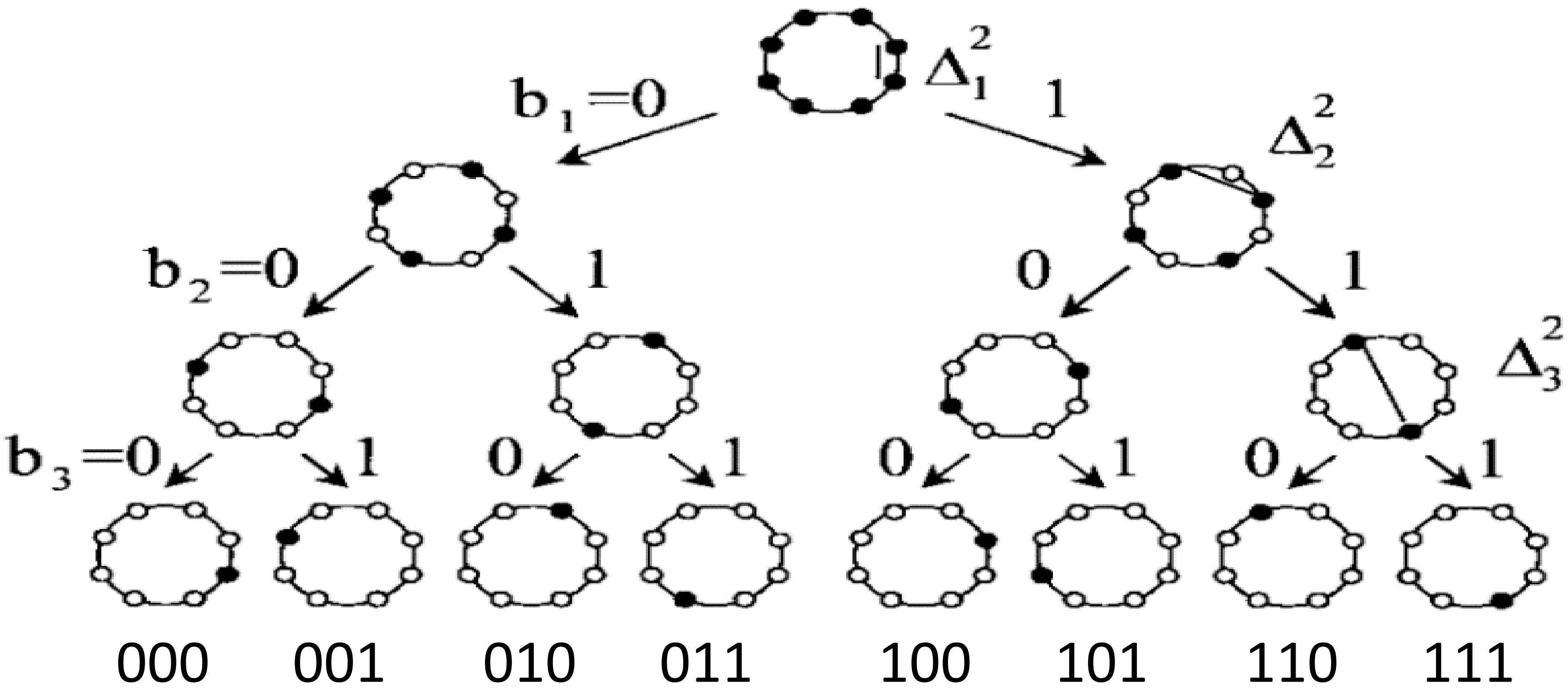
1. Generalized algebraic set partitioning algorithm, and
2. Multilevel polar code design methodology

Set Partitioning

- Ungerboeck proposed a simple set partitioning algorithm that works well for simple, two-dimensional signal constellations.
  - Ungerboeck's algorithm does not work with irregular multidimensional signal constellations.
  - Ungerboeck's algorithm only works with Euclidean distances as the distance metric.
- Forney proposed an algorithm that works with regular, lattice-based, multidimensional constellations.
- We propose the first generalized algebraic set partitioning algorithm
  - This algorithm works with any signal constellation, and with any distance metric.

Set Partitioning

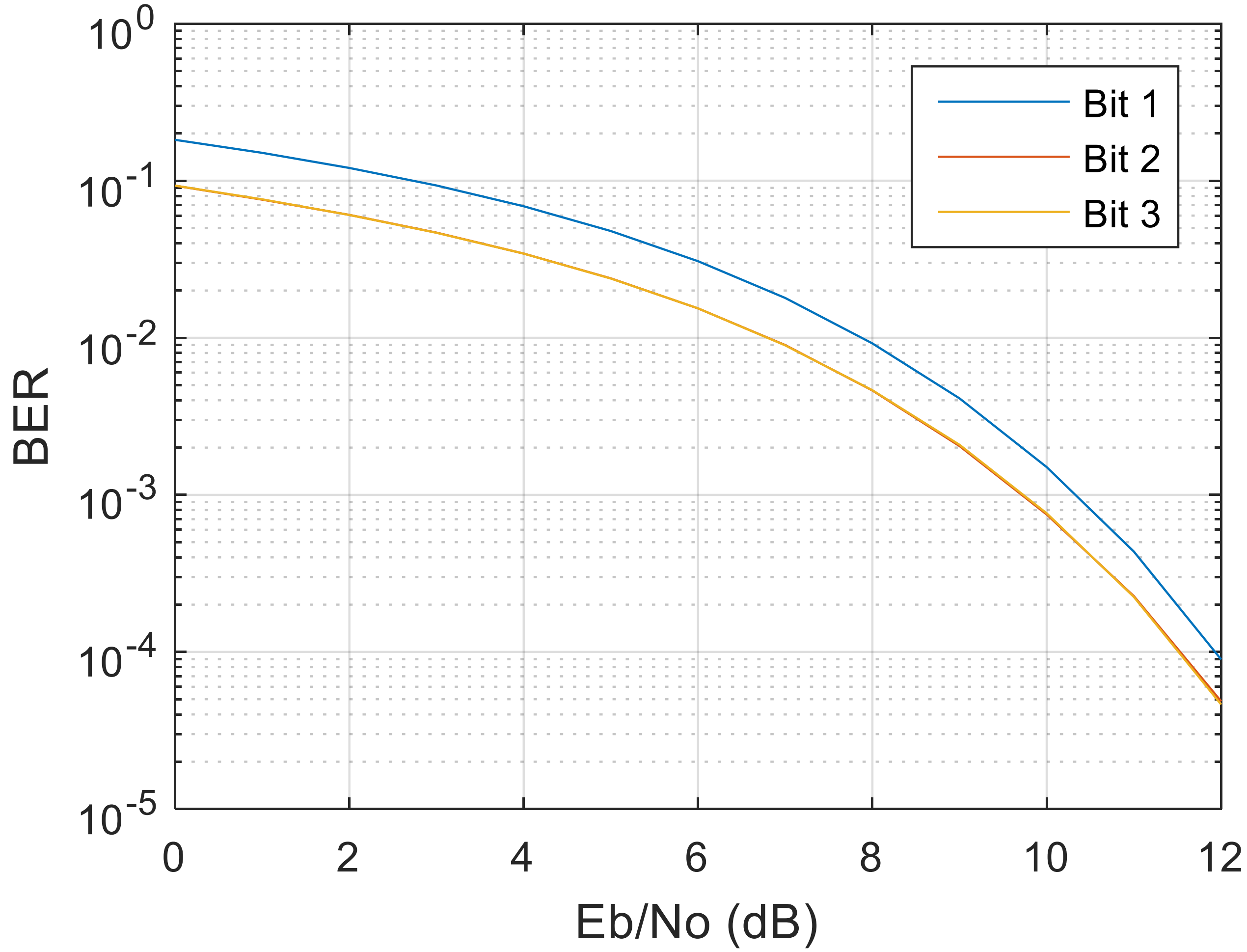
- Recursively divide constellation into subsets.
- Points in each divided subset have a larger minimum distance between points than the parent subset.
- Value of each bit determines which subset.



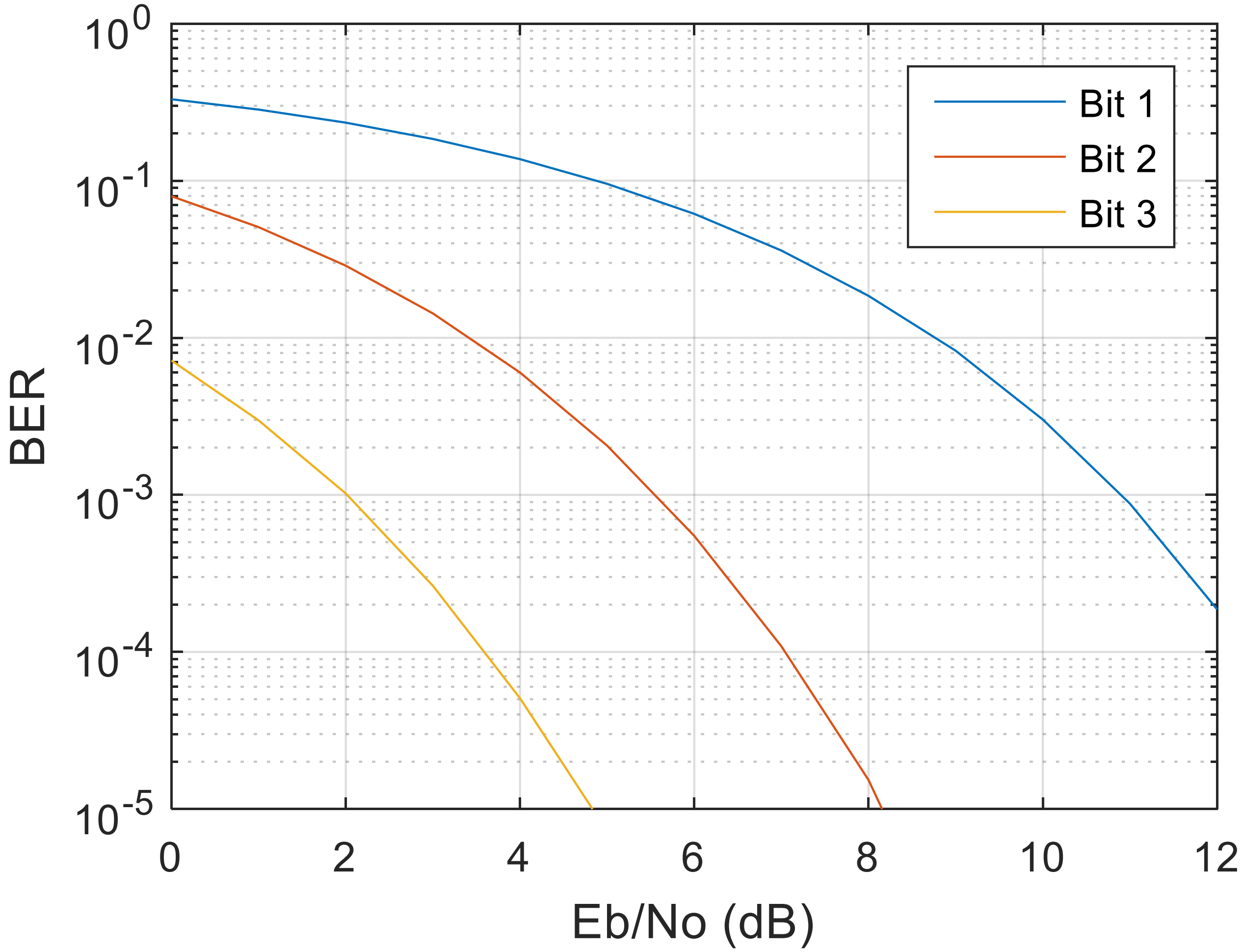
Example: Set partitioning of an 8-PSK constellation

Set Partitioning

- Each bit position has a different probability of error.
  - Use high-rate codes for reliable bit positions, low rate for unreliable ones.



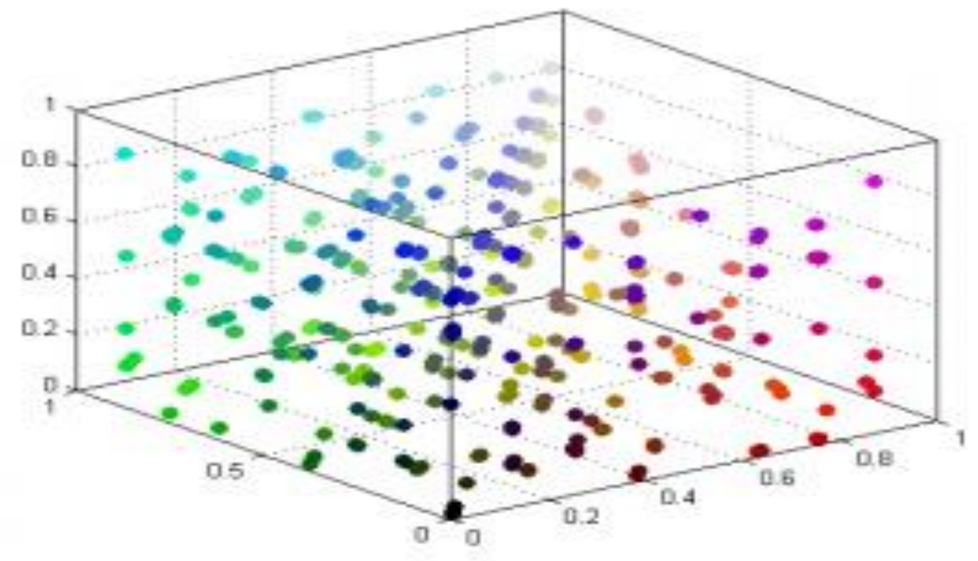
8-PSK with Gray labelling



8-PSK with set partitioning

Generalized Algebraic Set Partitioning

- Ungerboeck's set partitioning algorithm is not easily extended beyond 2-D constellations with the Euclidean distance metric.
- We propose a novel, efficient (polynomial time), generalized set partitioning algorithm that works with any regular or irregular constellation.
  - Supports multidimensional signal spaces.
  - Any distance metric can be used, such as the chordal Frobenius norm which is best for noncoherent Grassmannian signalling.



Example of an irregular 3D constellation

Generalized Algebraic Set Partitioning

- Instead of dividing the constellation into subsets, the proposed algorithm starts with subsets consisting of only one point, and merges subsets until only one (containing the whole subset) remains.
- The algorithm is initialized with the distances between each pair of symbols, $\mathbf{X}_i$ and $\mathbf{X}_j$, using whatever metric is most suitable for the communication system.
  - For coherent detection, the Euclidean distance is usually preferred:

$$\mathcal{D}_1(i,j) = \left\|\mathbf{X}_i - \mathbf{X}_j\right\|$$

  - For noncoherent detection of Grassmannian signals, the chordal Frobenius norm should be used:

$$\mathcal{D}_1(i,j) = \sqrt{2N_T - 2\mathrm{Tr}\left\{\Sigma_{\mathbf{X}_i^{\dagger}\mathbf{X}_j}\right\}}$$

where $\Sigma_{\mathbf{X}_i^{\dagger}\mathbf{X}_j}$ is a diagonal matrix containing the singular values of $\mathbf{X}_i^{\dagger}\mathbf{X}_j$.

Generalized Algebraic Set Partitioning

- For each symbol, the distance to the farthest other symbol is found, and then the minimum of these distances is found:

$$\Delta_1 = \arg\min_i \max_j D(i,j)$$

  - The algorithm the pairs every symbol with the closest other symbol that has a distance of at least Δ_1. That is, it pairs symbol i with symbol

$$j = \arg \min_{j, \mathcal{D}_1(i,j) \geq \Delta_1} \mathcal{D}_1(i,j)$$

  - Symbol i is labelled with a bit value of 0 in the first bit position, and symbol j is labelled with a bit value of 1.
- Once every symbol has been paired into subsets containing two points, the process is repeated, merging subsets together to create large subsets of size 4. The distance between table is updated as

$$\mathcal{D}_2(i,j) = \min\{\mathcal{D}_1(i_1,i_2), \mathcal{D}_1(i_1,j_2), \mathcal{D}_1(j_1,i_2), \mathcal{D}_1(j_1,j_2)\}$$

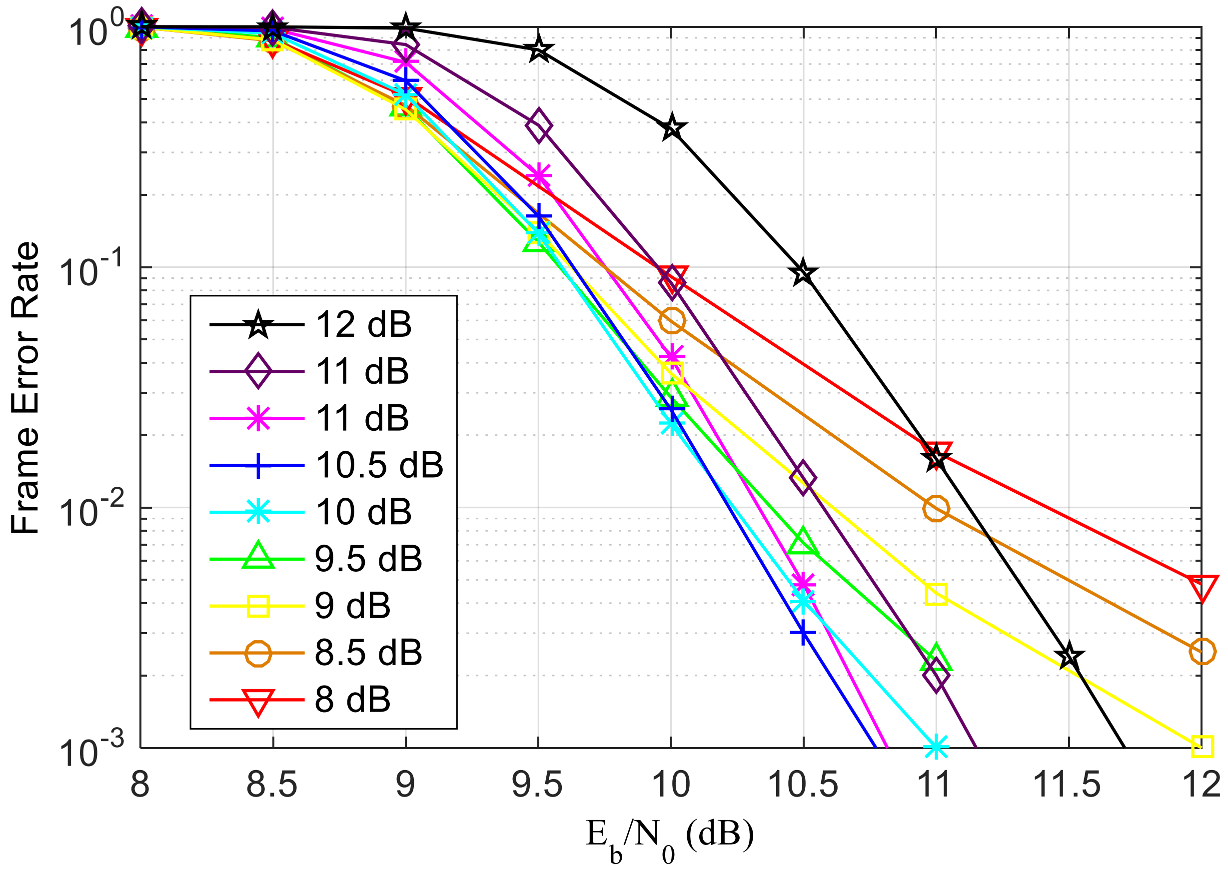
- This process is repeated until only one subset, of size M, remains.

Multilevel Polar Code Design Methodology

- Positions of frozen bit must be determined for each subcode based on the overall code rate. This choice is made for a given design SNR.
- The transmission of a large number of message frames is simulated at a specific design SNR and the first error probability for each bit channel is determined. In this stage, no bit channels are frozen and correct decision feedback is assumed within the decoders.
- The bit channels with the highest first error probabilities are frozen. The number of bit channels to freeze is $(1 - R)mN$, where R is the overall code rate, N is the subcode codeword length, and $m = \log_2 M$ is the number of subcodes.
- The rates of the individual subcodes is not determined in advanced, but is calculated from the number of non-frozen bit channels in each subcode.
- System performance depends on design SNR.

Multilevel Polar Code Design Methodology

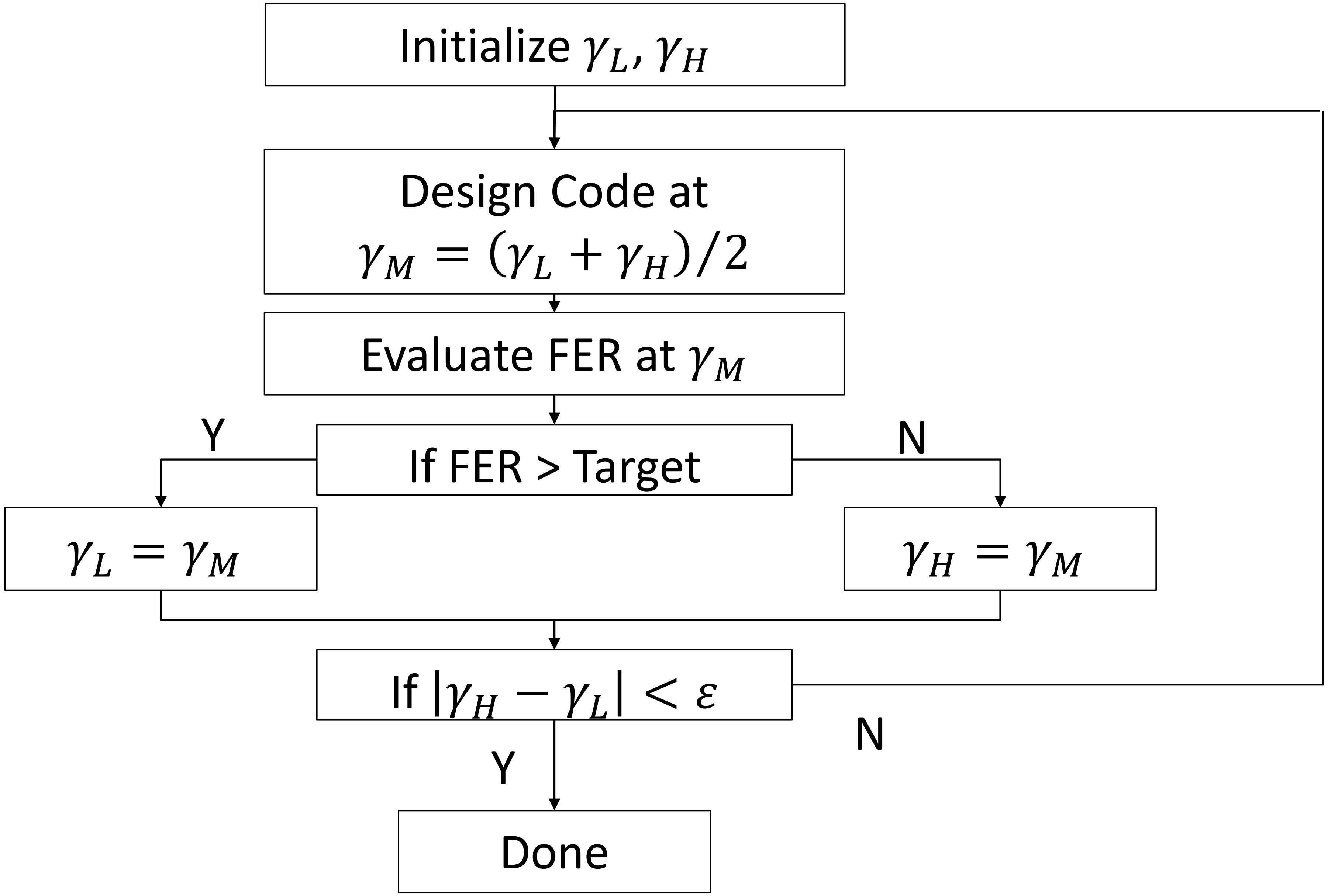
- Designing the code at an SNR that is too high or too low may yield a code that requires a needlessly high SNR to achieve a target FER.
- We proposed the use of the bisection algorithm to find the optimal design SNR for a target FER
  - If the code designed at a given SNR gives a FER less than the target FER at the design SNR, design a new code at a higher SNR. Otherwise, design a new code at a lower SNR.



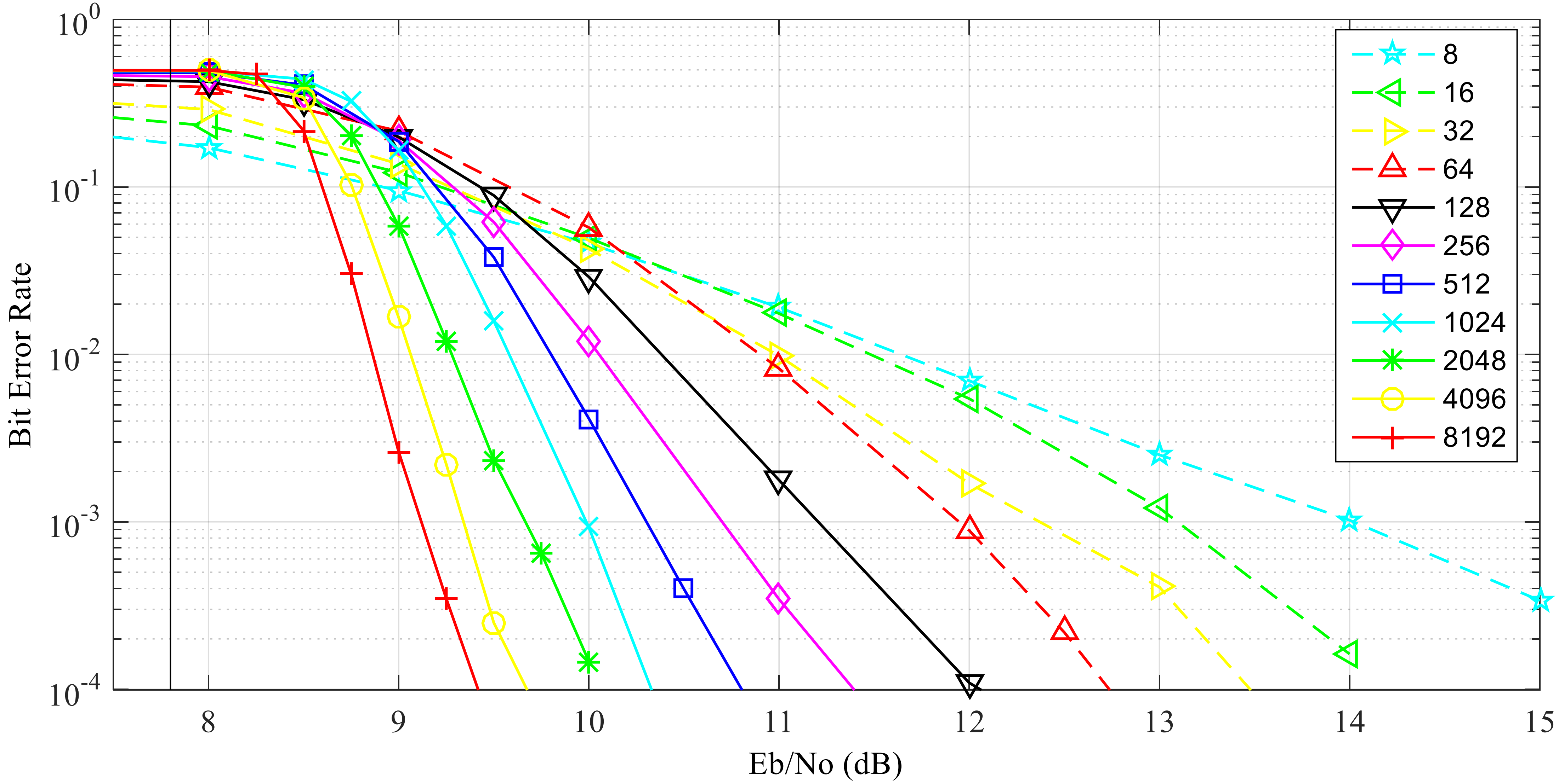
Example: Effect of design SNR on the Frame Error Rate performance of our system at various design SNRs. 4096-point Grassmannian signalling.

Multilevel Polar Code Design Methodology

γ_L = low design SNR
γ_H = high design SNR
ε = SNR tolerance

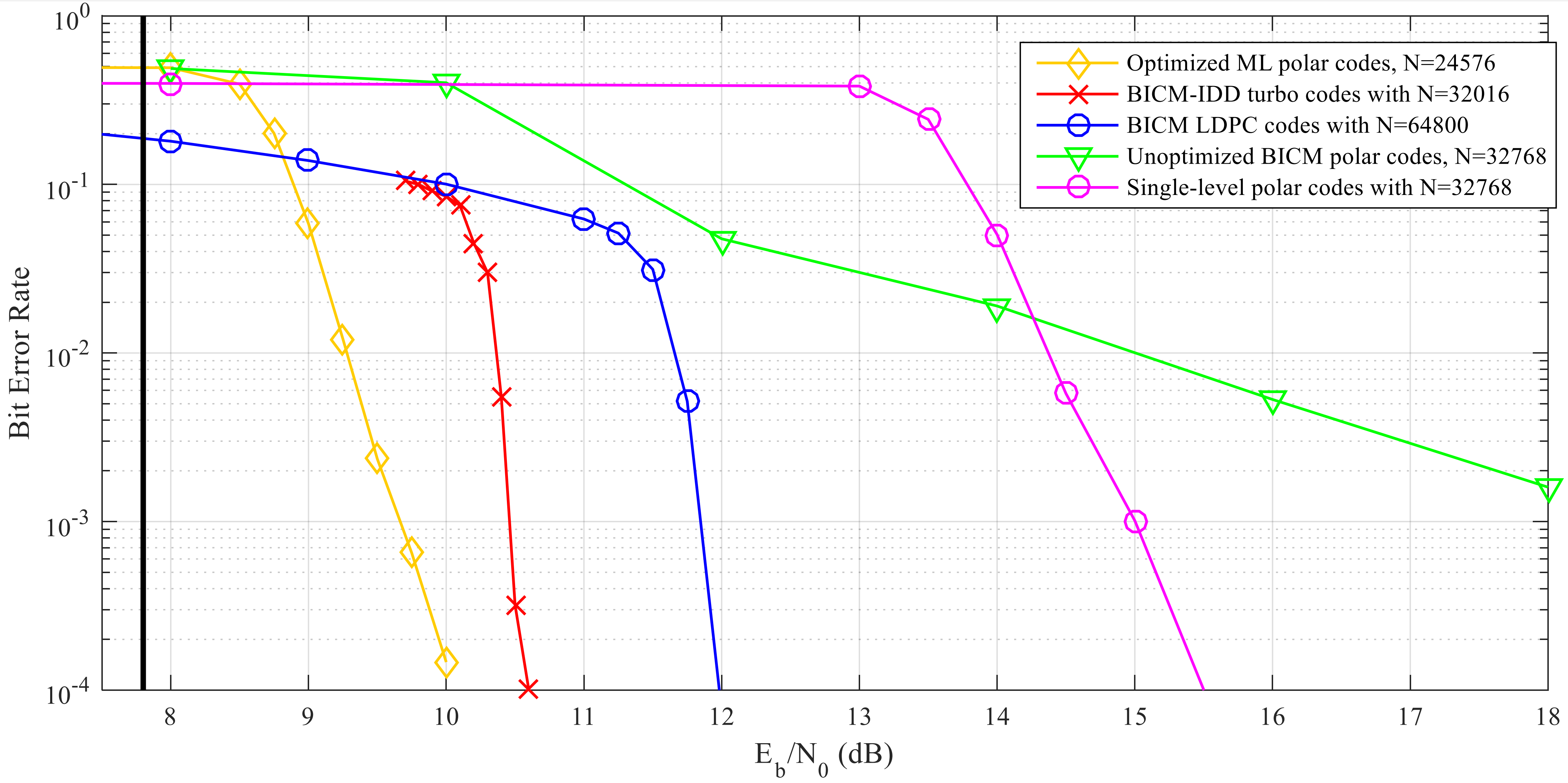


Performance Results



4096 point Grassmannian constellation with polar codes of different sub-code lengths with code rate 4/5. SNR threshold = 7.8 dB.

Performance Results



Different codes running with 4096 point Grassmannian constellation with rate R=4/5. All BICM figures use quasi-Gray labelling for the constellation. Multilevel code uses set partitioned labelling. Un-optimized BICM codes are optimized for a BPSK AWGN channel only.

Summary

- The generalized set partitioning algorithm is the first that can work with any signal constellation and any distance metric.
- The multilevel polar code design methodology allows for design of powerful polar codes. Previous polar code design methodologies minimize the FER at one design SNR.
- Multilevel polar codes work very well with irregular multidimensional signal constellations such as Grassmannian signalling.
- Polar codes designed using the proposed methodology with constellations that are labelled with the proposed set partitioning algorithm given better performance than BICM schemes with LDPC and turbo codes.
- The designed system provides better performance than other schemes and does so at a much lower receiver complexity.

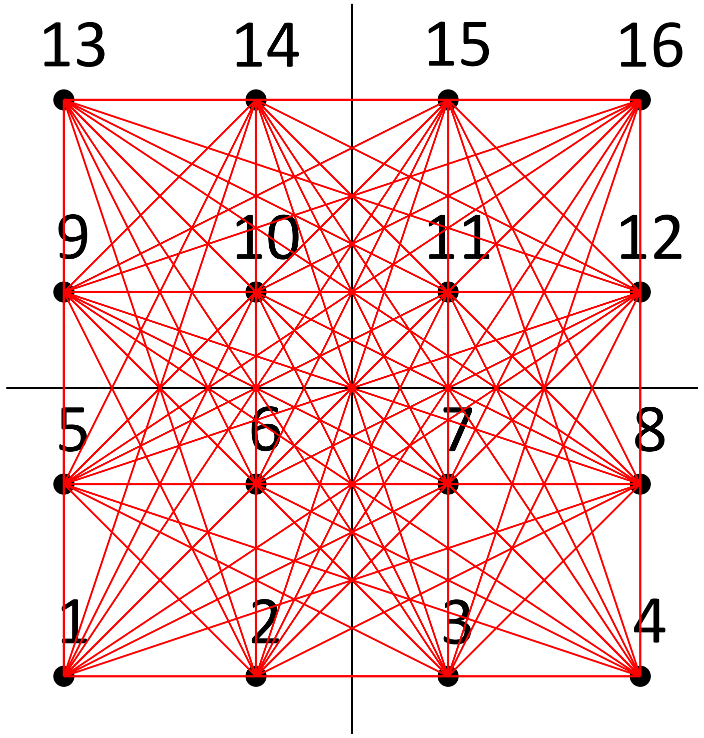
Thank you!

Generalized Algebraic Set Partitioning

$\mathcal{D}_1^2$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	4	16	36	4	8	20	40	16	20	32	52	36	40	52	72
2	4	0	4	16	8	4	8	20	20	16	20	32	40	36	40	52
3	16	4	0	4	20	8	4	8	32	20	16	20	52	40	36	40
4	36	16	4	0	40	20	8	4	52	32	20	16	72	52	40	36
5	4	8	20	40	0	4	16	36	4	8	20	40	16	20	32	52
6	8	4	8	20	4	0	4	16	8	4	8	20	20	16	20	32
7	20	8	4	8	16	4	0	4	20	8	4	8	32	20	16	20
8	40	20	8	4	36	16	4	0	40	20	8	4	52	32	20	16
9	16	20	32	52	4	8	20	40	0	4	16	36	4	8	20	40
10	20	16	20	32	8	4	8	20	4	0	4	16	8	4	8	20
11	32	20	16	20	20	8	4	8	16	4	0	4	20	8	4	8
12	52	32	20	16	40	20	8	4	36	16	4	0	40	20	8	4
13	36	40	52	72	16	20	32	52	4	8	20	40	0	4	16	36
14	40	36	40	52	20	16	20	32	8	4	8	20	4	0	4	16
15	52	40	36	40	32	20	16	20	20	8	4	8	16	4	0	4
16	72	52	40	36	52	32	20	16	40	20	8	4	36	16	4	0

1) Generate distance table:

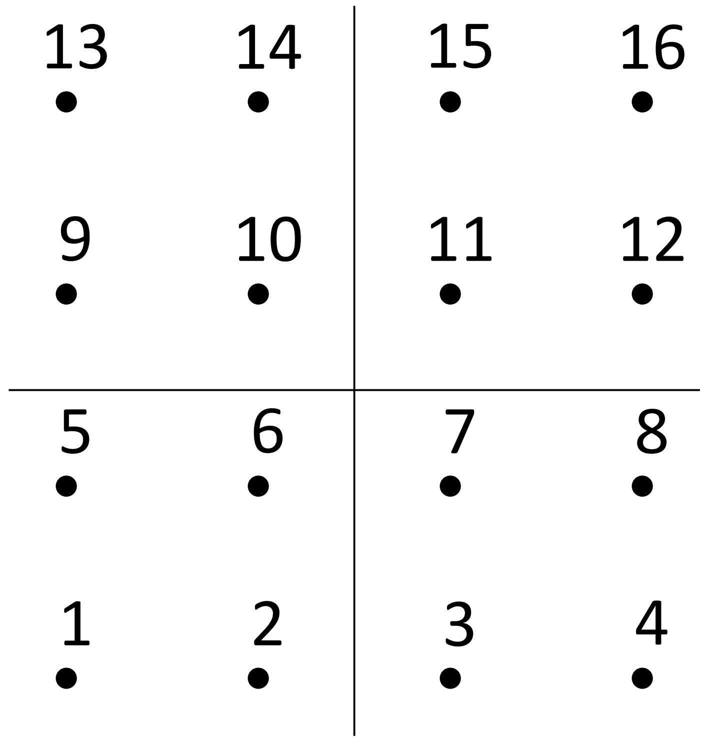
$$\mathcal{D}_1^2\,(i,j) = \begin{cases} \left\|\mathbf{X}_i - \mathbf{X}_j\right\|^2 & \text{(coherent)} \\ 2N_t - 2\,\mathrm{Tr}\left\{\mathbf{\Sigma}_{\mathbf{X}_i^\dagger \mathbf{X}_j}\right\} & \text{(noncoherent)} \end{cases}$$



Generalized Algebraic Set Partitioning

$\mathcal{D}_1^2$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	4	16	36	4	8	20	40	16	20	32	52	36	40	52	72
2	4	0	4	16	8	4	8	20	20	16	20	32	40	36	40	52
3	16	4	0	4	20	8	4	8	32	20	16	20	52	40	36	40
4	36	16	4	0	40	20	8	4	52	32	20	16	72	52	40	36
5	4	8	20	40	0	4	16	36	4	8	20	40	16	20	32	52
6	8	4	8	20	4	0	4	16	8	4	8	20	20	16	20	32
7	20	8	4	8	16	4	0	4	20	8	4	8	32	20	16	20
8	40	20	8	4	36	16	4	0	40	20	8	4	52	32	20	16
9	16	20	32	52	4	8	20	40	0	4	16	36	4	8	20	40
10	20	16	20	32	8	4	8	20	4	0	4	16	8	4	8	20
11	32	20	16	20	20	8	4	8	16	4	0	4	20	8	4	8
12	52	32	20	16	40	20	8	4	36	16	4	0	40	20	8	4
13	36	40	52	72	16	20	32	52	4	8	20	40	0	4	16	36
14	40	36	40	52	20	16	20	32	8	4	8	20	4	0	4	16
15	52	40	36	40	32	20	16	20	20	8	4	8	16	4	0	4
16	72	52	40	36	52	32	20	16	40	20	8	4	36	16	4	0

2) Find maximum in each row

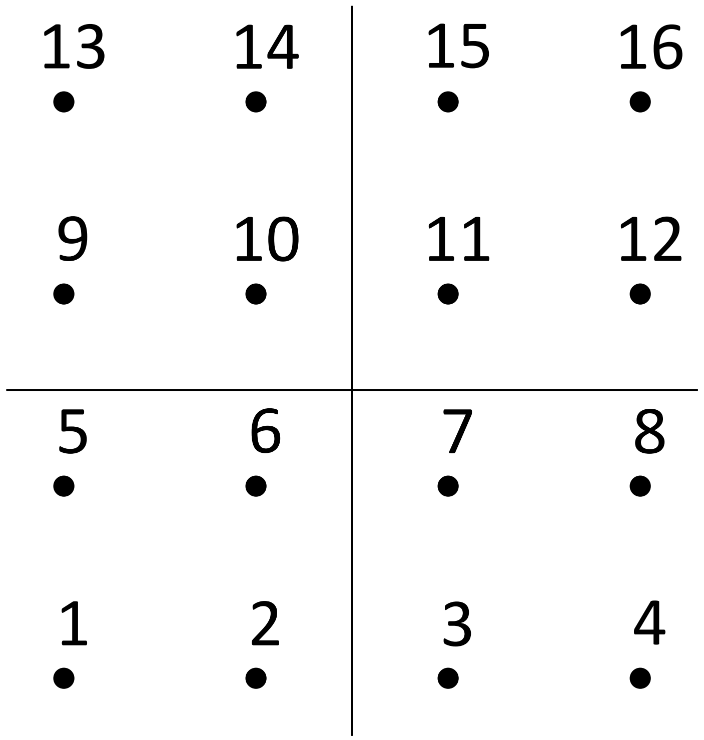


Generalized Algebraic Set Partitioning

$\mathcal{D}_1^2$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	4	16	36	4	8	20	40	16	20	32	52	36	40	52	72
2	4	0	4	16	8	4	8	20	20	16	20	32	40	36	40	52
3	16	4	0	4	20	8	4	8	32	20	16	20	52	40	36	40
4	36	16	4	0	40	20	8	4	52	32	20	16	72	52	40	36
5	4	8	20	40	0	4	16	36	4	8	20	40	16	20	32	52
6	8	4	8	20	4	0	4	16	8	4	8	20	20	16	20	32
7	20	8	4	8	16	4	0	4	20	8	4	8	32	20	16	20
8	40	20	8	4	36	16	4	0	40	20	8	4	52	32	20	16
9	16	20	32	52	4	8	20	40	0	4	16	36	4	8	20	40
10	20	16	20	32	8	4	8	20	4	0	4	16	8	4	8	20
11	32	20	16	20	20	8	4	8	16	4	0	4	20	8	4	8
12	52	32	20	16	40	20	8	4	36	16	4	0	40	20	8	4
13	36	40	52	72	16	20	32	52	4	8	20	40	0	4	16	36
14	40	36	40	52	20	16	20	32	8	4	8	20	4	0	4	16
15	52	40	36	40	32	20	16	20	20	8	4	8	16	4	0	4
16	72	52	40	36	52	32	20	16	40	20	8	4	36	16	4	0

3) Find minimum of the maxima

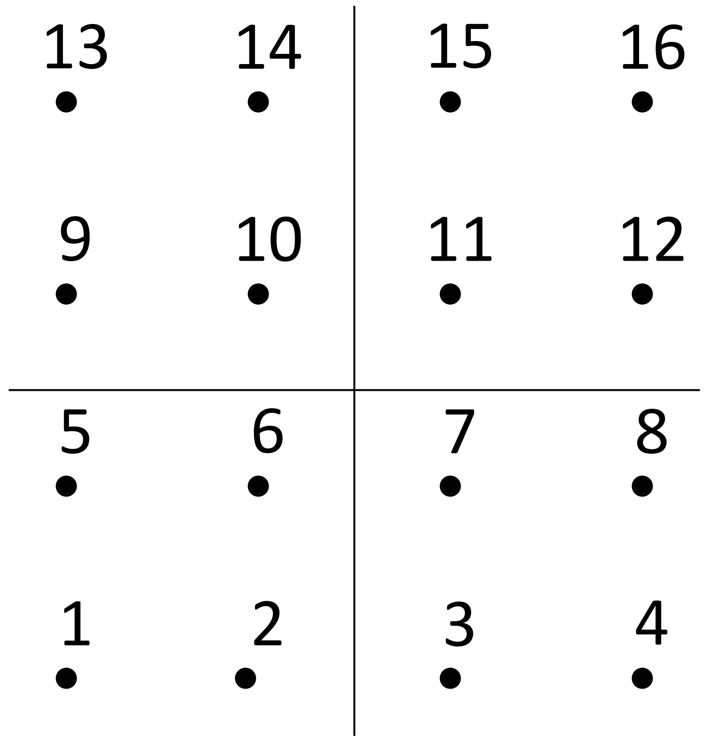
$$\Delta_1 = \min_i \max_j \mathcal{D}_1 \quad (i,j)$$



Generalized Algebraic Set Partitioning

$\mathcal{D}_1^2$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	4	16	36	4	8	20	40	16	20	32	52	36	40	52	72
2	4	0	4	16	8	4	8	20	20	16	20	32	40	36	40	52
3	16	4	0	4	20	8	4	8	32	20	16	20	52	40	36	40
4	36	16	4	0	40	20	8	4	52	32	20	16	72	52	40	36
5	4	8	20	40	0	4	16	36	4	8	20	40	16	20	32	52
6	8	4	8	20	4	0	4	16	8	4	8	20	20	16	20	32
7	20	8	4	8	16	4	0	4	20	8	4	8	32	20	16	20
8	40	20	8	4	36	16	4	0	40	20	8	4	52	32	20	16
9	16	20	32	52	4	8	20	40	0	4	16	36	4	8	20	40
10	20	16	20	32	8	4	8	20	4	0	4	16	8	4	8	20
11	32	20	16	20	20	8	4	8	16	4	0	4	20	8	4	8
12	52	32	20	16	40	20	8	4	36	16	4	0	40	20	8	4
13	36	40	52	72	16	20	32	52	4	8	20	40	0	4	16	36
14	40	36	40	52	20	16	20	32	8	4	8	20	4	0	4	16
15	52	40	36	40	32	20	16	20	20	8	4	8	16	4	0	4
16	72	52	40	36	52	32	20	16	40	20	8	4	36	16	4	0

4) Pair each symbol with its closest neighbour with a distance of at least Δ_1



Generalized Algebraic Set Partitioning

$\mathcal{D}_1^2$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	4	16	36	4	8	20	40	16	20	32	52	36	40	52	72
2	4	0	4	16	8	4	8	20	20	16	20	32	40	36	40	52
3	16	4	0	4	20	8	4	8	32	20	16	20	52	40	36	40
4	36	16	4	0	40	20	8	4	52	32	20	16	72	52	40	36
5	4	8	20	40	0	4	16	36	4	8	20	40	16	20	32	52
6	8	4	8	20	4	0	4	16	8	4	8	20	20	16	20	32
7	20	8	4	8	16	4	0	4	20	8	4	8	32	20	16	20
8	40	20	8	4	36	16	4	0	40	20	8	4	52	32	20	16
9	16	20	32	52	4	8	20	40	0	4	16	36	4	8	20	40
10	20	16	20	32	8	4	8	20	4	0	4	16	8	4	8	20
11	32	20	16	20	20	8	4	8	16	4	0	4	20	8	4	8
12	52	32	20	16	40	20	8	4	36	16	4	0	40	20	8	4
13	36	40	52	72	16	20	32	52	4	8	20	40	0	4	16	36
14	40	36	40	52	20	16	20	32	8	4	8	20	4	0	4	16
15	52	40	36	40	32	20	16	20	20	8	4	8	16	4	0	4
16	72	52	40	36	52	32	20	16	40	20	8	4	36	16	4	0

5) Calculate new distance table

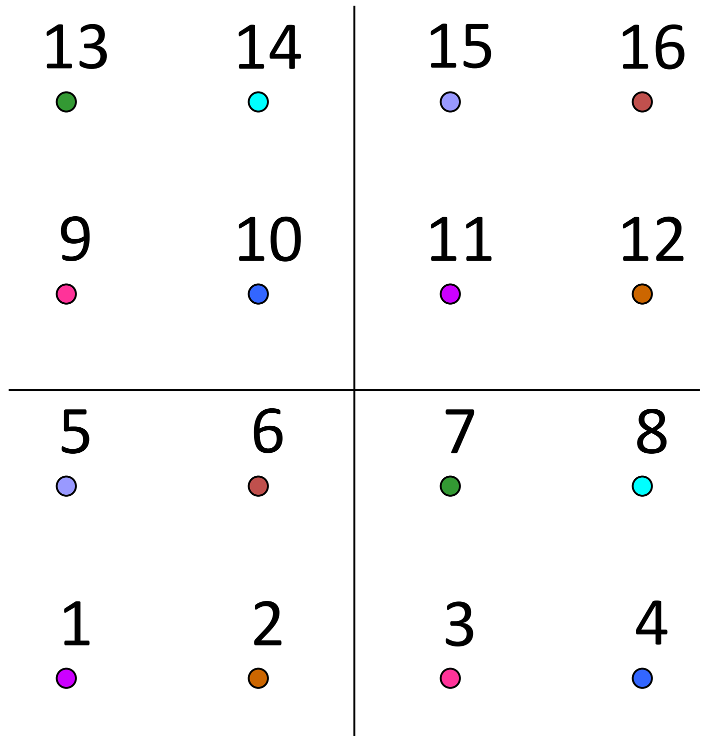
Each element of $\mathcal{D}_2$ is the minimum of 4 elements of $\mathcal{D}_1$

$\mathcal{D}_2^2$	1	2	3	4	5	6	7	8
1	0	4	16	4	4	8	4	8
2	4	0	4	16	8	4	8	4
3	16	4	0	4	4	8	4	8
4	4	16	4	0	8	4	8	4
5	4	8	4	8	0	4	16	4
6	8	4	8	4	4	0	4	16
7	4	8	4	8	16	4	0	4
8	8	4	8	4	4	16	4	0

Generalized Algebraic Set Partitioning

6) Find maximum in each row

$\mathcal{D}_2^2$	1	2	3	4	5	6	7	8
1	0	4	16	4	4	8	4	8
2	4	0	4	16	8	4	8	4
3	16	4	0	4	4	8	4	8
4	4	16	4	0	8	4	8	4
5	4	8	4	8	0	4	16	4
6	8	4	8	4	4	0	4	16
7	4	8	4	8	16	4	0	4
8	8	4	8	4	4	16	4	0

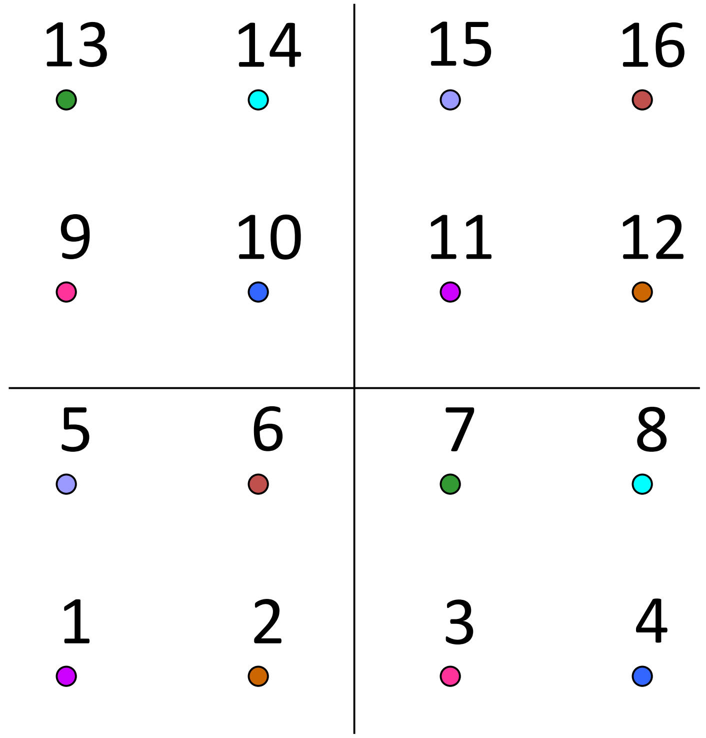


Generalized Algebraic Set Partitioning

7) Find minimum of the maxima

$\mathcal{D}_2^2$	1	2	3	4	5	6	7	8
1	0	4	16	4	4	8	4	8
2	4	0	4	16	8	4	8	4
3	16	4	0	4	4	8	4	8
4	4	16	4	0	8	4	8	4
5	4	8	4	8	0	4	16	4
6	8	4	8	4	4	0	4	16
7	4	8	4	8	16	4	0	4
8	8	4	8	4	4	16	4	0

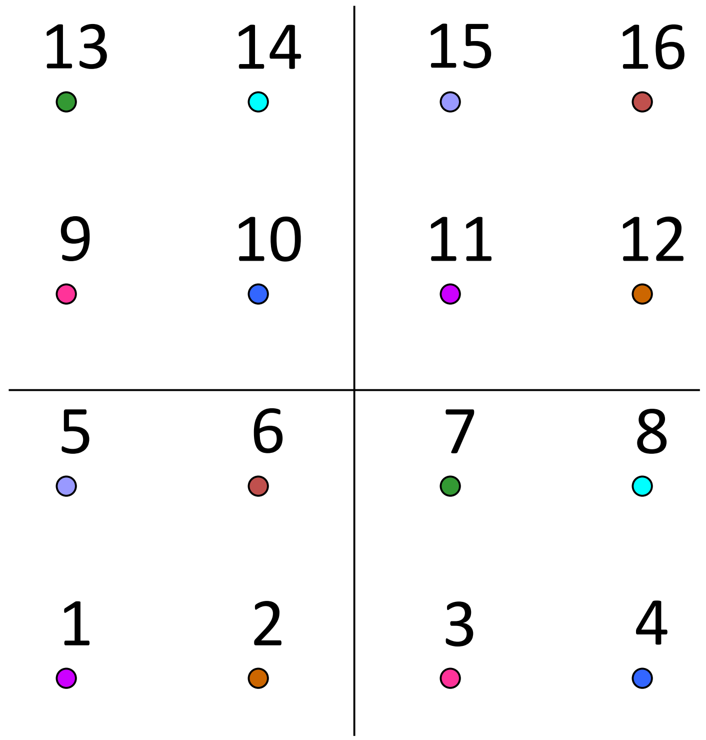
$$\Delta_2 = \min_i \max_j \mathcal{D}_2 \ (i,j)$$



Generalized Algebraic Set Partitioning

8) Pair each symbol with its closest neighbour with a distance of at least Δ_2

$\mathcal{D}_2^2$	1	2	3	4	5	6	7	8
1	0	4	16	4	4	8	4	8
2	4	0	4	16	8	4	8	4
3	16	4	0	4	4	8	4	8
4	4	16	4	0	8	4	8	4
5	4	8	4	8	0	4	16	4
6	8	4	8	4	4	0	4	16
7	4	8	4	8	16	4	0	4
8	8	4	8	4	4	16	4	0



Generalized Algebraic Set Partitioning

9) Calculate new distance table

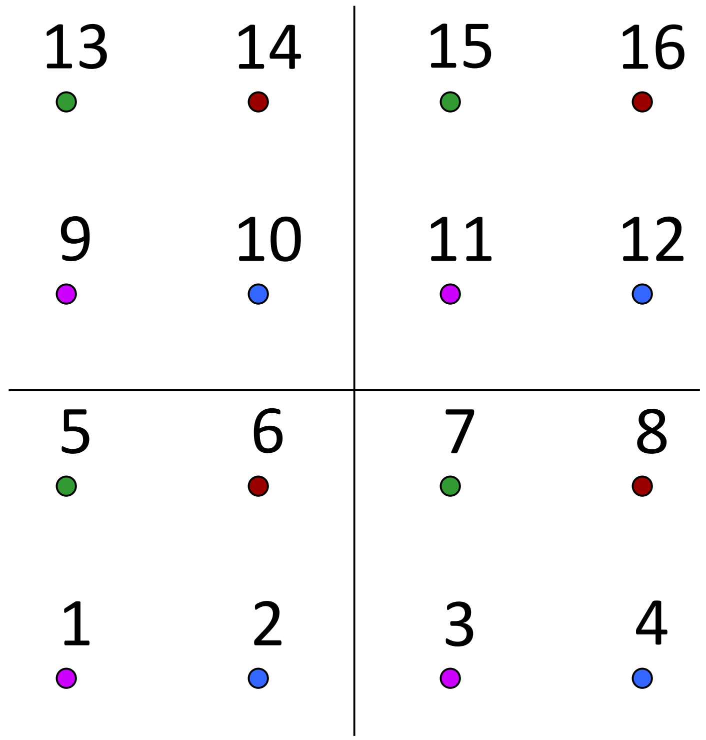
$\mathcal{D}_2^2$	1	2	3	4	5	6	7	8
1	0	4	16	4	4	8	4	8
2	4	0	4	16	8	4	8	4
3	16	4	0	4	4	8	4	8
4	4	16	4	0	8	4	8	4
5	4	8	4	8	0	4	16	4
6	8	4	8	4	4	0	4	16
7	4	8	4	8	16	4	0	4
8	8	4	8	4	4	16	4	0

$\mathcal{D}_3^2$	1	2	3	4
1	0	4	4	8
2	4	0	8	4
3	4	8	0	4
4	8	4	4	0

Generalized Algebraic Set Partitioning

10) Find maximum in each row

$\mathcal{D}_3^2$	1	2	3	4
1	0	4	4	8
2	4	0	8	4
3	4	8	0	4
4	8	4	4	0

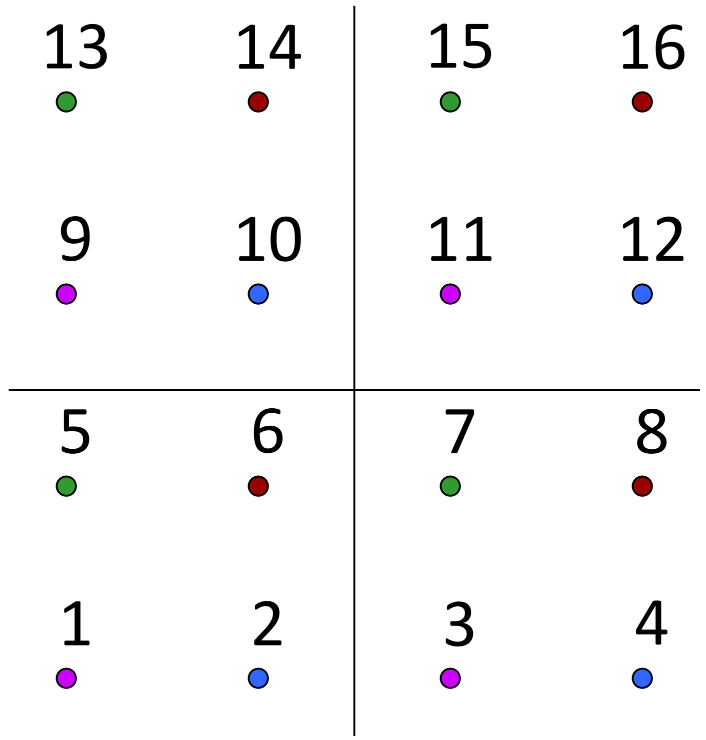


Generalized Algebraic Set Partitioning

11) Find minimum of the maxima

$$\Delta_3 = \min_i \max_j \mathcal{D}_3 \ (i,j)$$

$\mathcal{D}_3^2$	1	2	3	4
1	0	4	4	8
2	4	0	8	4
3	4	8	0	4
4	8	4	4	0



Generalized Algebraic Set Partitioning

12) Pair each symbol with its closest neighbour with a distance of at least Δ_3

$\mathcal{D}_3^2$	1	2	3	4
1	0	4	4	8
2	4	0	8	4
3	4	8	0	4
4	8	4	4	0