Irregular Multidimensional Constellations for Orthogonal STBCs

Hossein Khoshnevis, Ian Marsland, and Halim Yanikomeroglu
Department of Sytems and Computer Engineering
Carleton University, Canada
Email: {khoshnevis, ianm, halim}@sce.carleton.ca.

***Abstract*—Utilizing multiple antennas at the transmitter and receiver provides higher data rates and better reliability by exploiting spatial diversity. Space-time block codes (STBCs) is a simple approach for using multiple transmit and receive antennas that has been widely employed in standards. The STBCs introduced in the literature use independent two-dimensional constellations, while the performance of orthogonal STBCs may be improved with multidimensional constellations. These constellations are transmitted by combining multiple space-time resources to form a multidimensional signal space. In this paper, we propose a method for finding optimized multidimensional constellations for orthogonal STBCs. Optimization is performed by minimizing a novel bound on the block or symbol error rate. We show that a substantial improvement in the error probability can be achieved with these novel constellations.**

I. Introduction

In future wireless communication systems, higher quality, capacity and reliability are among the essential demands. Multiple-input multiple-output (MIMO) techniques, based on multiantenna multiplexing and diversity, can be used to provide high data rates and better reliability. In 1998, Tarokh et al. introduced space-time codes [1] and subsequently space-time block codes (STBC) [2] as methods for the efficient use of the available space-time resources in multiantenna communication systems. Orthogonal space-time block codes (OSTBCs), as one of the main types of space-time codes, are used to provide full diversity with a linear complexity decoder [2], [3]. Because of their low complexity in encoding and decoding, OSTBCs have been used widely in standards [4]. In the last decade, there have been extensive studies on designing improved space-time codes, and some STBCs with higher performance and higher complexity, such as quasi-orthogonal space-time block codes (QOSTBCs) [5] and rotation based codes [6]-[10], have been introduced.

While different categories of space-time codes are well-studied, there are only a few papers on designing signal constellations for use with space-time codes in the literature. Indeed, due to their simple encoding and decoding, most STBCs employ the traditional regular lattice based two dimensional (2D) constellations such as rectangular or hexagonal QAM [9]. However, there might be other constellations that provide better performance. One of these is multidimensional constellations.

Multidimensional constellations are signal constellations in which points are mapped to more than two dimensions to allow for increased separation between points in comparison to the widely used 2D constellations [11]. They can be used by projecting the multidimensional constellations onto a set of orthogonal 2D signal spaces, with each projection transmitted independently. For example, a 4D symbol can be transmitted using two 2D symbols. Typically, an OSTBC block carries K symbols, with independent information content carried in each symbol. If we employ multidimensional constellations, each 2D component of a 2K-dimensional constellation can be carried by one of the K different symbols of the OSTBC.

To achieve better performance with multidimensional constellations, it is necessary to carefully optimize the placement of symbol points in the constellation by using knowledge about the distribution of the fading channel. In this paper we present a novel method for this optimization based on minimizing a bound on the error probability. The output of the optimization problem can in general be an irregular constellation. Irregular constellations, as shown in [12] in the context of constellation rearrangement for cooperative relaying, are capable of improving performance in comparison to regular or isometric constellations.

The main contribution of this paper is to present good multidimensional constellations for OSTBC that outperform OSTBC with regular constellations. This technique is intended to operate with systems that provide transmit diversity when there is no channel state information at the transmitter and the number of transmit antennas exceeds the number of receive antennas. We provide an appropriate bound, related to the fading distribution, on the performance of OSTBCs in the most general case of arbitrary irregular multidimensional constellations. We use this bound to design optimized irregular multidimensional constellations for OSTBCs. Regular multidimensional constellations and all 2D constellations (including the labeling aspect) are special cases of this design. When used in OSTBCs, the resulting constellations provide significantly better performance. Simulation results confirm that, to the best of our knowledge, the system performance is better than any of the previously published techniques for the intended operating environment described above with one receive antennas, at low to moderate spectral efficiencies.

The rest of the paper is organized as follows: The system model is described in Section II, the union bound on the probability of error is derived in Section III, the optimization criterion is provided in Section IV, simulation results are reported in Section V, and the conclusions are presented in Section VI.

II. System Model

The system consists of multiple transmit antennas that use STBCs. The data is divided into groups of d bits and accordingly mapped to symbols of different 2D constellations which are projections of a $2K$-dimensional constellation with a modulation order of 2^d. The system is equipped with N_t and N_r antennas at the transmitter and receiver, respectively, and each code block consists of L time slots. Each symbol is transmitted through a slow flat fading channel denoted by the $N_t \times N_r$ matrix $\mathbf{H}$, with elements $h_{ij} = \alpha_{ij} e^{j\phi_{ij}}$ where α_{ij} has a Rayleigh distribution. The system can be described as

$$\mathbf{R} = \mathbf{G}\mathbf{H} + \mathbf{W}, \tag{1}$$

where $\mathbf{R}$ is the received matrix, $\mathbf{G}$ is the $L \times N_t$ transmitted STBC block, and $\mathbf{W}$ is zero-mean complex additive white Gaussian noise (AWGN) with variance $N_0/2$ per dimension. OSTBCs are general structures that can be employed for carrying data orthogonally over slow fading channels. Their simplest form, proposed by Alamouti [3] for two transmit antennas, can be written as

$$\mathbf{G}_0 = \begin{bmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{bmatrix}. \tag{2}$$

In Code $\mathbf{G}_0$, data are mapped separately to each constellation point and carried by symbols s_1 and s_2, both of which are independent elements of a 2D constellation, $\mathcal{S}_2$. As described in Section I, to transmit multidimensional constellations using OSTBCs, their 2D components are distributed on OSTBC symbols. By considering s_1 and s_2 used in $\mathbf{G}_0$ as carriers of 2D components of a multidimensional constellation, Alamouti's scheme can be rewritten as

$$\mathbf{G}_1 = \begin{bmatrix} s^{(1)} & s^{(2)} \\ -s^{(2)*} & s^{(1)*} \end{bmatrix}, \tag{3}$$

where $s^{(k)}$ is the kth 2D component for transmission of a multidimensional symbol $\mathbf{s} = [s^{(1)}, s^{(2)}, ..., s^{(K)}]$ with $s \in \mathcal{S}_{2K}$, a $2K$-dimensional constellation. In $\mathbf{G}_1$ data are mapped to two 2D subpoints of a 4D point and the subpoints are carried by $s^{(1)}$ and $s^{(2)}$. As an example, to provide a spectral efficiency of 2 bits per channel-use (bpcu), a 4-QAM constellation should be used for s_1 and s_2 in $\mathbf{G}_0$ whereas a 16-point 4D constellation should be used for $\mathbf{s} = [s^{(1)}, s^{(2)}]$ in $\mathbf{G}_1$.

For the case of four-antenna transmission, the well-known OSTBC mentioned in [13] can be rewritten for multidimensional constellation as

$$\mathbf{G}_2 = \begin{bmatrix} s^{(1)} & s^{(2)} & s^{(3)} & 0 \\ -s^{(2)*} & s^{(1)*} & 0 & s^{(3)} \\ s^{(3)*} & 0 & -s^{(1)*} & s^{(2)} \\ 0 & s^{(3)*} & -s^{(2)*} & -s^{(1)} \end{bmatrix}, \tag{4}$$

and, by dropping the last column of $\mathbf{G}_2$, the corresponding scheme for a three-antenna transmission of a 6D constellation can be written as

$$\mathbf{G}_3 = \begin{bmatrix} s^{(1)} & s^{(2)} & s^{(3)} \\ -s^{(2)*} & s^{(1)*} & 0 \\ s^{(3)*} & 0 & -s^{(1)*} \\ 0 & s^{(3)*} & -s^{(2)*} \end{bmatrix}. \tag{5}$$

By denoting c_i^l as the code symbol transmitted in time slot l from antenna i, the general ML decoding rule in the receiver for the transmission of codeword $\mathbf{c} = c_1^1 c_2^1 ... c_{N_t}^1 ... c_1^L c_2^L ... c_{N_t}^L$ in a $L \times N_t$ space-time block using perfect channel state information can be expressed as the minimization of the following metric over all constellation points:

$$\sum_{l=1}^{L} \sum_{j=1}^{N_r} \left| r_j^l - \sum_{i=1}^{N_t} h_{ij} c_i^l \right|^2. \tag{6}$$

In the above, r_j^l is the received sample on the jth antenna in time slot l. By using the orthogonal structure of OSTBCs, a simplified ML decoder can detect s_k according to

$$\hat{s}_k = \underset{\forall s \in \mathcal{S}_2}{\operatorname{argmin}} \left| P_k - \left(\sum_{i,j} \alpha_{ij}^2 \right) s \right|^2, \tag{7}$$

where

$$P_k = \sum_{j=1}^{N_r} \sum_{l=1}^{L} \sum_{i=1}^{N_t} F_{i,k}^l (r_j^l h_{ij}^*). \tag{8}$$

In (8), $k = 1, 2, ..., K$ shows the index of the different symbols carried by one OSTBC block and $F_{i,k}^l(z)$ can be evaluated as

$$F_{i,k}^l(z) = \begin{cases} z, & \text{if } c_i^l = s_k, \\ z^*, & \text{if } c_i^l = s_k^*, \\ -z, & \text{if } c_i^l = -s_k, \\ -z^*, & \text{if } c_i^l = -s_k^*, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

This simplified decoder can be used for decoding the multidimensional constellations by changing (7) into a summation of decoding of different 2D components of the multidimensional constellations as

$$\hat{\mathbf{s}} = \underset{\forall \mathbf{s} \in \mathcal{S}_{2K}}{\operatorname{argmin}} \sum_{k=1}^{K} \left| P_k - \left(\sum_{ij} \alpha_{ij}^2 \right) s^{(k)} \right|^2. \tag{10}$$

Note that the term P_k should be computed only once for each k, as this decreases the complexity of decoding substantially in comparison to the high performance complex codes such as the perfect codes [8], [9] in which (6) may need to be computed for all points of a constellation.

III. An Upper Bound on The Block Error Rate

An upper bound for the error performance of the scheme, based on the constellation points, is derived in this section. This bound is used for finding the optimized constellations. Based on the orthogonal structure of the OSTBC, its pairwise error probability can be expressed as [13]

$$P(\mathbf{s} \to \hat{\mathbf{s}} \mid \mathbf{H}) = \mathrm{Q}\left(\sqrt{ \left(\frac{\sum_{j=1}^{N_r} \sum_{i=1}^{N_t} \alpha_{ij}^2}{2N_0} \right) \sum_{k=1}^{K} \left| s^{(k)} - \hat{s}^{(k)} \right|^2 } \right), \tag{11}$$

where $Q(\cdot)$ is the Gaussian tail function. By using the Chernoff bound, (11) can be upper bounded as

$$P(\mathbf{s} \to \hat{\mathbf{s}} \mid \mathbf{H}) \leq \exp\left(-\left(\frac{\sum_{j=1}^{N_r}\sum_{i=1}^{N_t}\alpha_{ij}^2}{4N_0}\right)\sum_{k=1}^{K}\left|s^{(k)}-\hat{s}^{(k)}\right|^2\right) = \prod_{j=1}^{N_r}\prod_{i=1}^{N_t}\exp\left(-\left(\frac{\alpha_{ij}^2}{4N_0}\right)\sum_{k=1}^{K}\left|s^{(k)}-\hat{s}^{(k)}\right|^2\right). \tag{12}$$

By considering the exponential distribution of α_{ij}^2, the pairwise error probability (11) can be upper bounded further as

$$P(\mathbf{s} \to \hat{\mathbf{s}}) \leq \frac{4^{N_t N_r}}{\prod_{j=1}^{N_r}\prod_{i=1}^{N_t}\left(\bar{\gamma}_{ij}\sum_{k=1}^{K}\left|s^{(k)}-\hat{s}^{(k)}\right|^2+4\right)}, \tag{13}$$

where $\bar{\gamma}_{ij} = E[|\alpha_{ij}|^2]/N_0$ is the average SNR of each path. This in turn can be bounded by

$$P(\mathbf{s} \to \hat{\mathbf{s}}) \leq \frac{4^{N_t N_r}}{\left(\prod_{j=1}^{N_r}\prod_{i=1}^{N_t}\bar{\gamma}_{ij}\right)\left(\sum_{k=1}^{K}\left|s^{(k)}-\hat{s}^{(k)}\right|^2\right)^{N_t N_r}}. \tag{14}$$

From (14), the corresponding union bound on the symbol error rate (SER) can be written as

$$P_s \leq \frac{1}{2^d}\sum_{s\in\mathcal{S}_{2K}}\sum_{\hat{s}\in\mathcal{S}_{2K}\setminus s}\frac{4^{N_t N_r}}{\bar{\gamma}\left(\sum_{k=1}^{K}\left|s_v^{(k)}-s_{v'}^{(k)}\right|^2\right)^{N_t N_r}}, \tag{15}$$

where $\bar{\gamma}$ is defined as

$$\bar{\gamma} = \prod_{j=1}^{N_r}\prod_{i=1}^{N_t}\bar{\gamma}_{ij}. \tag{16}$$

Since in each space-time block of the proposed scheme, only one symbol from the multidimensional constellation is transmitted, the SER and block error rate (BLER) of the STBC block are identical. Therefore, the above bound can be used for finding the locally optimum constellations for minimizing the BLER.

IV. Optimization Criterion

The use of multidimensional constellations in orthogonal codes provides an opportunity for optimizing constellation points. For finding optimized constellations, the union bound (15) on the BLER is minimized. To improve the performance by maximizing the shaping gain instead of increasing the power, the only constraint used in optimization of multidimensional constellations is that the average power of the constellation points is limited to one. The problem is generally nonlinear but can be solved with a general global optimization solver. For optimization, the "fmincon" solver in the MATLAB optimization toolbox was used. A sample of two 16-point optimized 2D constellations as components of a 4D constellation is sketched in Fig. 1. In this figure, each 2D constellation point represents two dimensions of a 4D constellation point and points with the same label are indicated with the same marker and color. This also shows that constellation points in this scheme can be irregularly placed anywhere in the signal space (within the constraint of average energy) which provides more degrees of freedom for optimization. The optimization problem is to find $s_v^{(k)} \in \mathbb{C}$ for all $k \in \{1, ..., K\}$ and $v \in \{1, ..., 2^d\}$ that will

$$\begin{aligned} \text{minimize} \quad & \sum_{v=1}^{2^d}\sum_{\substack{v'=1\\ v'\neq v}}^{2^d}\frac{C}{\left(\sum_{k=1}^{K}\left|s_v^{(k)}-s_{v'}^{(k)}\right|^2\right)^{N_t N_r}}, \\ \text{subject to} \quad & \frac{1}{2^d K}\sum_{v=1}^{2^d}\sum_{k=1}^{K}\left|s_v^{(k)}\right|^2 \leq 1, \end{aligned} \tag{17}$$

where $C = 4^{N_t N_r}/2^d\bar{\gamma}$, which does not effect the optimization. To initiate the "fmincon" solver with a good starting point, all 2D constellations are initially selected from rectangular QAM constellations and, by keeping the position of the points constant, their labels are optimized. We use either an exhaustive search for small constellations, a branch-and-bound method for medium-size ones and a random search for large constellations to find either a globally or locally optimized solution for the labeling problem. In the next step, "fmincon" provides irregular constellations distributed in the 2K-dimensional space. Note that (17) does not depend on $\bar{\gamma}$ and therefore the output of the optimization is an SNR-independent constellation.

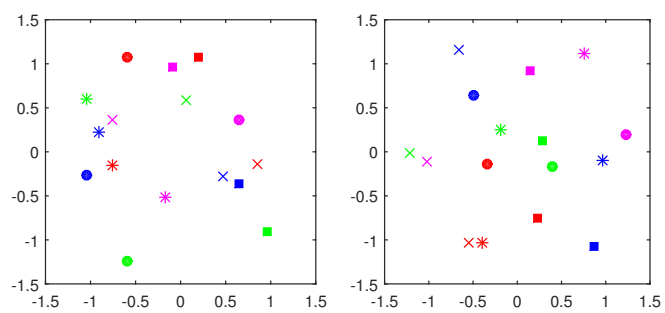

Figure 1. Sample of optimized constellations used for the proposed scheme $\mathbf{G}_1$.

V. Performance Evaluation

As indicated in Section IV, by using the "fmincon" solver started from several initial points multidimensional constellations were optimized to provide the locally optimal constellations and the constellations with the lowest BLER bound value were selected for performance evaluation. The channel is modeled as experiencing uncorrelated Rayleigh fading with AWGN. As the baseline, OSTBCs with QAM constellations are compared against the proposed scheme. Furthermore, the scheme is compared with the Golden code [8] and the algebraic

MISO code in [10] that we refer to as the "Oggier code". The constellations used with the Golden code for 2 bpcu and 4 bpcu are BPSK and QPSK, respectively, and with the Oggier code for 1 bpcu is BPSK. Furthermore, the scheme has been compared with the QOSTBC in [5] with QPSK for a spectral efficiency of 2 bpcu.

In Fig. 2 the BLER performance of the Golden code, Alamouti's OSTBC $\mathbf{G}_0$ and the proposed scheme $\mathbf{G}_1$, all with 2 bpcu, are compared. This result shows that the proposed scheme outperforms OSTBC by 0.4 dB in a 2×1 configuration and by 0.5 dB in a 2×2 configuration at a BLER of 10^{-4}. Furthermore, the Golden code in a 2×2 configuration shows a BLER worse than OSTBC. Indeed, most algebraic codes are designed for high rates and therefore show poor performance at low rates since all their degrees of freedom are not exploited well.

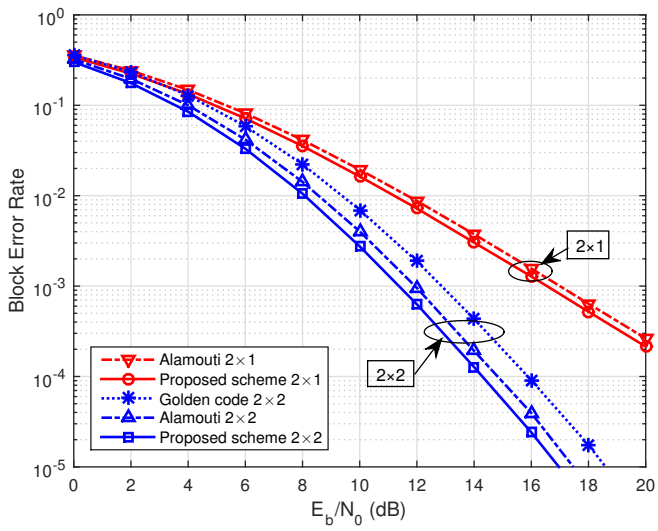

Figure 2. BLER Comparison of the proposed scheme, OSTBC and Golden code for 2 bpcu.

Fig. 3 shows the performance comparison of the proposed scheme $\mathbf{G}_1$ with OSTBC in 2×1 and 2×2 configurations and the Golden code 2×2 for 4 bpcu. The outcome indicates that the proposed scheme works better than OSTBC as one of the best codes for the 2×1 configuration, by 0.9 dB at a BLER of 10^{-3}. Furthermore, for the 2×2 antenna configuration, it is 0.9 dB better than OSTBC and only 0.5 dB worse than the Golden code at 10^{-4}. Note that the Golden code 2×2 outperforms the proposed scheme since it benefits from more degrees of freedom, but it also has four times more complexity in terms of complex multiplications in each search for ML decoding even though the number of searches is the same as that in the proposed scheme.

Fig. 4 shows the error performance of the proposed schemes $\mathbf{G}_2$ and $\mathbf{G}_3$ for 3×1 and 4×1 antenna configurations, respectively. The proposed scheme outperforms OSTBC by 1.5 dB in 3×1 and in 4×1 configurations at a BLER of 10^{-4}. Furthermore, the BLER comparison of the proposed scheme and OSTBC in 3×2 and 4×2 configurations in Fig. 5 shows 1.7 dB and 1.8 dB improvement at 10^{-4}, respectively. To compare the proposed scheme with algebraic codes, the

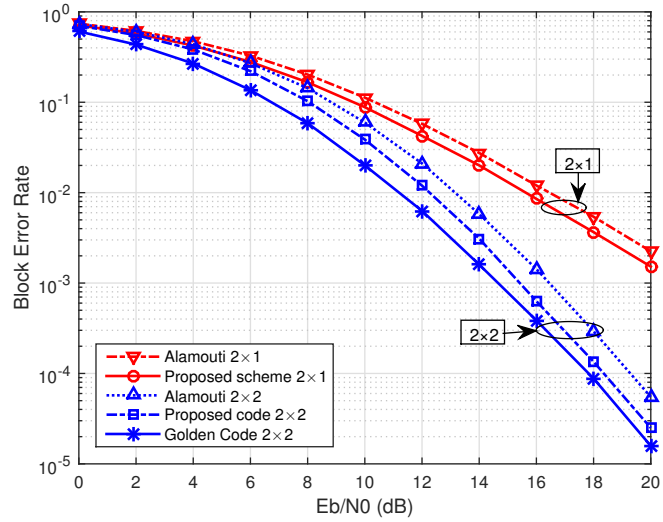

Figure 3. BLER Comparison of the proposed scheme, OSTBC and Golden code for 4 bpcu.

recently designed MISO code in [10] (the "Oggier code") that can support lower rates was tested; similar to the Golden code in Fig. 2, its BLER is worse than the corresponding OSTBC.

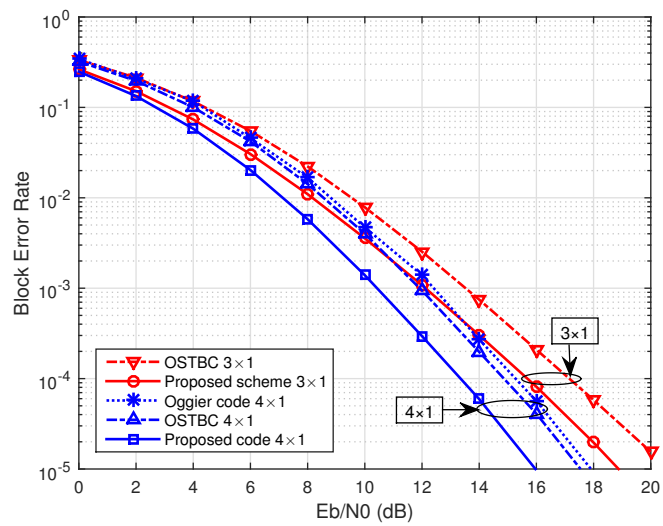

Figure 4. BLER Comparison of the proposed scheme, OSTBC and Oggier code for 1 bpcu, with $N_r = 1$.

Finally, Fig. 6 shows the performance of the proposed schemes $\mathbf{G}_2$ and $\mathbf{G}_3$ in comparison to OSTBC and QOSTBC 4×1 and 4×2 for 2 bpcu. The results show that the scheme outperforms OSTBC by around 4 dB at 10^{-3} and also outperforms QOSTBC 4×1 and 4×2 by 0.8 dB and 0.4 dB at 10^{-4}, respectively. Note that the improvement in comparison to OSTBC or QOSTBC is achieved at the expense of more decoding complexity. In the case of QOSTBC, joint pairwise decoding results in a lower number of searches, but each search is more complex than the proposed scheme.

By using the multidimensional constellations with OSTBC, high performance improvements can be achieved at the ex-

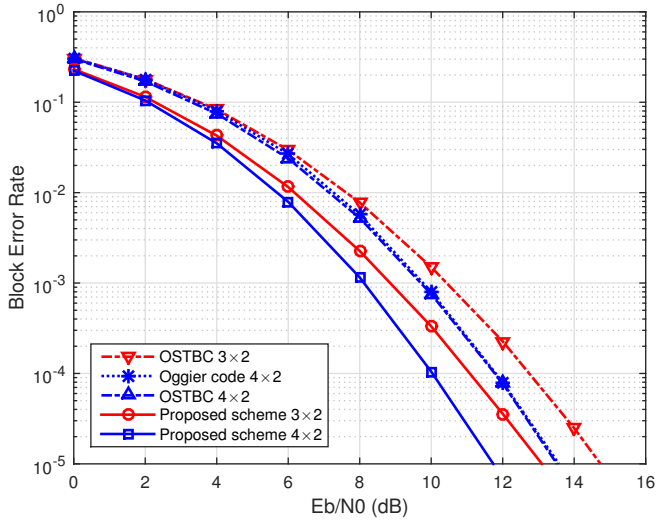

Figure 5. BLER Comparison of the proposed scheme, OSTBC and Oggier code for 1 bpcu, with $N_r = 2$.

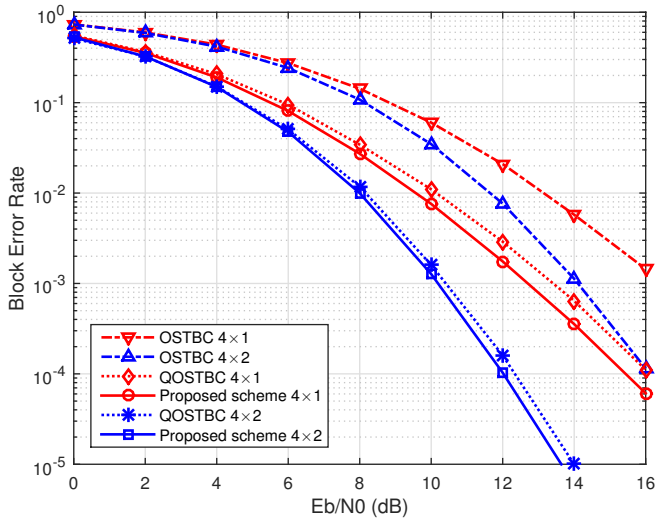

Figure 6. BLER Comparison of the proposed scheme, OSTBC and QOSTBC for 2 bpcu.

pense of increasing the complexity of decoding. By increasing the dimensionality of the constellation, there is more space for points to be further apart and this improves the performance even though the size of the employed OSTBC, including the number of antennas and the number of time slots, may be increased. Indeed, this is the main reason that schemes $\mathbf{G}_2$ and $\mathbf{G}_3$ provide more gain in comparison to $\mathbf{G}_1$.

VI. Conclusions

By employing irregular multidimensional constellations for transmission of the OSTBCs symbols, the BLER can be improved in comparison to regular 2D QAM constellations. To achieve high performance, these constellations should be optimized by using foreknowledge about the distribution of fading channel coefficients. The proposed scheme works well for low to moderate spectral efficiencies values where all the degrees of freedom of algebraic codes cannot be fully exploited. Even though the algebraic codes may provide better performance at higher spectral efficiencies, the proposed scheme provides a tradeoff between decoding complexity and performance. Furthermore, the scheme outperforms QOSTBCs, which are among the best codes for the $N_T \times 1$ antenna configuration but this improvement is achieved at the expense of higher ML decoding complexity.

VII. Acknowledgment

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References

[1] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. Inf. Theory*, vol. 44, no. 2, pp. 744-765, Mar. 1998.

[2] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inf. Theory*, vol. 45, no. 5, pp. 1456-1467, Jul. 1999.

[3] S. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. Sel. Areas Commun.*, vol. 16, no. 8, pp. 1451-1458, Oct.1998.

[4] *LTE; Evolved Universal Terrestrial Radio Access (E-UTRA); Physical channels and modulation*, 3GPP TS 36.211 version 12.4.0 release 12, Feb. 2015.

[5] H. Jafarkhani, "A quasi-orthogonal space-time block code," *IEEE Trans. Commun.*, vol. 49, no. 1, pp. 1-4, Jan. 2001.

[6] H. Yao and G. W. Wornell, "Achieving the full MIMO diversity-multiplexing frontier with rotation-based space-time codes," in *Proc. Allerton Conf. Commun., Control, and Computing*, Monticello, IL, Oct. 2003.

[7] H. El Gamal and M. O. Damen, "Universal space-time coding," *IEEE Trans. Inf. Theory*, vol. 49, pp. 1097-1119, May 2003.

[8] J.-C. Belfiore, G. Rekaya, and E. Viterbo, "The golden code: A 2×2 full-rate space-time code with nonvanishing determinants," *IEEE Trans. Inf. Theory*, vol. 51, no. 4, pp. 1432-1436, Apr. 2005.

[9] F. Oggier, G. Rekaya, J.-C. Belfiore, and E. Viterbo, "Perfect space-time block codes," *IEEE Trans. Inf. Theory*, vol. 52, no. 9, pp. 3885-3902, Sept. 2006.

[10] F. Oggier, C. Hollanti, and R. Vehkalahti, "An algebraic MIDO-MISO code construction," in *Proc. 8th International Conf. on Signal Process. and Commun.*, Bangalore, India, Jul. 2010.

[11] A. Gersho and V. Lawrence, "Multidimensional signal constellations for voiceband data transmission," *IEEE J. Sel. Areas Commun.*, vol. 2, no. 5, pp. 687-702, Sept. 1984.

[12] A. Bin Sediq, P. Djukic, H. Yanikomeroglu, and J. Zhang, "Optimized nonuniform constellation rearrangement for cooperative relaying," *IEEE Trans. Veh. Technol.*, vol. 60, no. 5, pp. 2340-2347, Jun. 2011.

[13] H. Jafarkhani, *Space-Time Coding: Theory and Practice*, Cambridge Univ. Press, 2005.