A Decoding Procedure for Compress-and-Forward and Quantize-and-Forward Relaying

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***Abstract*— In this paper, a decoding procedure is developed for both the compress-and-forward (CF) and the short message quantize-and-forward (QF-SM) relaying schemes. This procedure is based on exploiting a common feature of the mapping used by both schemes to determine the relay output from the estimate of its input. Using this decoding procedure, it is shown that the CF scheme achieves higher rates than both the conventional CF and the QF-SM schemes. The advantage of the proposed procedure is demonstrated in a Gaussian multicast one-message two-destination relay network and a multicast two-message three-destination network.**

I. Introduction

Compress-and-Forward (CF) is a classical scheme for communicating over relay channels [1]. In its conventional version, the source transmits a new codeword in each time block and the relay uses a pre-designed codebook to generate an estimate of its received signal. The estimation codebook is randomly partitioned into non-overlapping bins. The relay facilitates the decoding at the destination by sending the index of the bin of its received signal in the previous block. The CF scheme was originally proposed for three node channels, but was later extended to channels with multiple relays [2], [3]. Further investigation into its performance with various decoding procedures was provided in [4].

A new relaying scheme, known as noisy network coding (NNC), was proposed in [5]. The philosophy of this scheme resembles, to some extent, that of the CF one. However, there are three differences between the CF scheme and NNC as proposed in [5]. First, in contrast with random binning used in CF, in NNC the relay transmits a codeword that bears a one-to-one correspondence with the estimated codeword. Second, in NNC the source uses repetitive transmission, wherein one long message is encoded over a large number of blocks. This is in contrast with CF, wherein a new short message is transmitted by the source in each time block. Finally, in NNC the signals received at the destination in all time blocks are concatenated and decoded jointly [5]; in CF the decoding is performed on block-by-block basis.

Repetitive encoding of long messages over large blocks incurs significant delay, which renders encoding short messages over smaller blocks more desirable. Variations of the original NNC [5] that use short message encoding were proposed in [6] and [7]. In [6] the rate achieved by short message Quantize-and-forward (QF-SM) is evaluated when either forward or backward block-by-block decoding is used. In contrast, the decoding in [7] is performed using a joint forward block-by-block procedure. It was shown in [6] and [7] that for three node networks (i.e., networks with one relay), the schemes proposed therein achieve the same rate as the original NNC [5]; for larger networks, NNC can achieve higher rates.

The decoding procedures proposed in [6] and [7] for the QF-SM scheme exploit the inherent bijective map between the estimation codewords and the codewords transmitted by the relay. Drawing insight from QF-SM in [6], we develop a novel decoding procedure for the CF and QF-SM schemes. For the QF-SM scheme, our decoding procedure yields the same data rates but is somewhat simpler than the procedure proposed in [6]. For the CF scheme, our decoding procedure uses random binning instead of the mapping used in NNC or QF-SM. It is shown that, for three node channels, this decoding procedure enables the CF scheme to achieve the same rate as the QF-SM scheme in [6]. In addition, it is shown that the constraint imposed by the CF scheme on the cardinality of the relay transmission codebook is less restrictive than its counterpart in the QF-SM scheme. (Similar results pertaining to QF-SM [7] are provided in a more comprehnsive version [8] and are not reported herein for space considerations.)

The advantage of the new decoding procedure over the standard one used in CF decoding is first illustrated by a numerical example of a Gaussian channel with one relay and two destinations. This example shows that the rate achieved by the CF scheme with the modified decoding procedure (CF-M) is the same as that achieved by the QF-SM scheme with the forward decoding procedure proposed in [6] and is higher than the rate achieved by the CF scheme with conventional decoding. Next, we consider a three-destination network in which the relay wishes to transmit an independent message to one of the destinations. We show that, for the Gaussian instance of this network, the CF-M scheme yields a strictly larger rate region than that yielded by the conventional CF and the QF-SM schemes. Detailed proofs of the results reported herein are provided in [8].

Notation: Standard notation will be used throughout. Regular face upper and lower case letters will be used to refer to random variables, and their corresponding realizations. Boldface letters will used to represent length-n sequences.

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II. Achievable rates of CF and QF-SM

We review the achievable rates of the CF scheme with random binning and conventional decoding [9] and the short message QF scheme with mapping and forward decoding [6].

For both CF and QF-SM, the source sends X_1 and the relay and the destination receive Y_1 and Y, respectively. The relay generates an estimate of its received signal, $\hat{Y}_1$, and sends X_2 to facilitate the decoding at the destination; cf. Figure 1. The difference between CF and QF-SM lies in the relationship between $\hat{Y}_1$ and X_2. In CF, potentially multiple codewords $\{\hat{y}_1\}$ are assigned to one bin x_2, whereas in QF-SM, each $\hat{y}_1$ is bijectively mapped to a unique x_2.

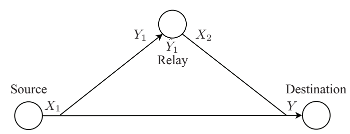

Fig. 1. A three node relay channel.

For the CF scheme, the following result is proved in [9].

Theorem 1 ([9]): The conventional CF scheme achieves

$$\begin{aligned} R_{\text{CF}} = \sup \min\{&I(X_1;\hat{Y}_1, Y|X_2), I(X_1, X_2; Y) \\ &- I(\hat{Y}_1; Y_1|X_1, X_2, Y) + R_0 - I(X_2;Y)\}, \\ &\text{subject to} \quad R_0 \le I(X_2;Y), \end{aligned}$$

where R_0 is the rate of the relay message, and the supremum is taken over the following probability mass functions (pmfs)

$$p(x_1, x_2, y, y_1, \hat{y}_1) = p(x_1)p(x_2)p(y, y_1|x_1, x_2)p(\hat{y}_1|x_2, y_1).$$

□

Remark 1 ([4], [9], [10]): Setting $R_0 = I(X_2;Y)$ maximizes R_{CF}. Hence, for three node relay channels, CF achieves

$$\begin{aligned} R_{\text{CF}} = \sup \min\{&I(X_1;\hat{Y}_1, Y|X_2), \\ &I(X_1, X_2; Y) - I(\hat{Y}_1; Y_1|X_1, X_2, Y)\}. \end{aligned}$$ □

We now consider the QF-SM scheme in [6].

Theorem 2 ([6]): The following rate is achievable by the short message QF scheme in [6].

$$\begin{aligned} R_{\text{QF-SM}} = \sup \min\{&I(X_1;\hat{Y}_1, Y|X_2), \\ &I(X_1, X_2; Y) - I(\hat{Y}_1; Y_1|X_1, X_2, Y)\}, \end{aligned} \tag{1a}$$

$$\text{subject to} \quad I(\hat{Y}_1; Y_1|X_1, X_2, Y) \le I(X_2;Y), \tag{1b}$$

where the supremum is taken over the pmfs of the form in Theorem 1. Achieving $R_{\text{QF-SM}}$ requires R_0 to be equal to $\hat{R}$, where $\hat{R}$ and R_0 are the rates of the relay estimation and transmission codebooks, respectively. □

For three node relay channels, it was shown in [6] that the constraint in (1b) is redundant. See [5]–[7] for results pertaining to NNC and other QF schemes.

III. Insight into the achievability of Theorem 2

In this section we will provide a decoding procedure for achieving the QF-SM rate in Theorem 2. The proposed decoding procedure is slightly different from the one given in [6] and enables us to identify the ostensible advantage of using mapping (in QF-SM) instead of binning (in CF). In fact, adapting the proposed procedure to CF, we will show in the next section that the CF scheme is capable of achieving the same rate in Theorem 2, but with the rate of the relay message, R_0, satisfying a less stringent constraint. An example that exploits this feature to yield a rate gain is given in Section VI.

In the decoding procedure proposed in [6], the source and relay messages are decoded jointly. In contrast, in the proposed procedure the destination uses a successive decoding approach. To describe it, we will use the notation in [6] and the codebook structures used therein. The source and relay messages at time block j are denoted by w_j and v_j and the corresponding codewords are denoted by $\mathbf{x}_1(w_j)$ and $\mathbf{x}_2(v_j)$, respectively. Knowing v_j, the estimation codeword of the relay at the end of block j is denoted by $\hat{\mathbf{y}}_1(u_j|v_j)$, where u_j is the index of the estimation codeword.

A successive decoding procedure

Let $\mathcal{A}_\epsilon^{(n)}$ be the set of length-n jointly ϵ-typical sequences.

- The procedure at the relay is identical to the one in [5]–[7]. At the end of block j the relay selects $\hat{\mathbf{y}}_1(u_j|v_j)$ that is jointly ϵ-typical with $\mathbf{y}_{1,j}$. The relay uses a deterministic map, ϕ_{QF}, from $\hat{\mathbf{y}}_1(u_j|v_j)$ to determine the codeword to be transmitted in the next block, $\mathbf{x}_2(v_{j+1})$.
- The procedure at the destination is successive, rather than joint, and is hence, simpler than the one in [6]. Assume that, at block $j+1$, w_{j-1}, v_{j-1} and u_{j-2} have been successfully recovered by the destination.
  1) Decoding v_j: The decoder constructs a set $\mathcal{S}_u^{(j-1)} = \{u|\big(\mathbf{x}_1(w_{j-1}), \mathbf{x}_2(v_{j-1}), \hat{\mathbf{y}}_1(u|v_{j-1}), \mathbf{y}_{j-1}\big) \in \mathcal{A}_\epsilon^{(n)}\}$. The decoder determines the set $\mathcal{S}_v^{(j)} = \{v|v = \phi_{\text{QF}}(u), u \in \mathcal{S}_u^{(j-1)}\}$. The decoder determines the unique $v_j \in \mathcal{S}_v^{(j)}$ for which $(\mathbf{x}_2(v_j), \mathbf{y}_j) \in \mathcal{A}_\epsilon^{(n)}$. Existence and uniqueness of v_j are guaranteed almost surely, if n is large and if (1b) is satisfied.
  2) Recovering w_j: Using v_j obtained in Step 1, the decoder constructs a set $\mathcal{S}_u'^{(j)} = \{u|\big(\mathbf{x}_2(v_j), \hat{\mathbf{y}}_1(u|v_j), \mathbf{y}_j\big) \in \mathcal{A}_\epsilon^{(n)}, \big(\mathbf{x}_2(v), \mathbf{y}_{j+1}\big) \in \mathcal{A}_\epsilon^{(n)}, v = \phi_{\text{QF}}(u)\}$. The decoder declares that w_j was sent in block j if, for some $u \in \mathcal{S}_u'^{(j)}$, $\big(\mathbf{x}_1(w_j), \hat{\mathbf{y}}_1(u|v_j), \mathbf{x}_{2,j}(v_j), \mathbf{y}_j\big) \in \mathcal{A}_\epsilon^{(n)}$. This step incurs an arbitrarily low error probability if n is sufficiently large and the source transmission rate is no greater than the right hand side of (1a). Note that if $\mathcal{S}_v'^{(j+1)}$ is used to denote the set $\{v|v = \phi_{\text{QF}}(u_j), u_j \in \mathcal{S}_u'^{(j)}\}$, then the bijectivity of ϕ_{QF} implies that $|\mathcal{S}_v'^{(j+1)}| = |\mathcal{S}_u'^{(j)}|$.
- The fact that the above procedure achieves the rate in Theorem 2 can be deduced in an analogous manner to

Appendix I; cf. Remarks 4 and 5 therein.

Remark 2:

i. The decoding procedure exploits that $\hat{\mathbf{y}}_1(u_{j-1}|v_{j-1})$, the estimation codeword at block $j-1$, is bijectively mapped to $\mathbf{x}_2(v_j)$, the relay transmitted codeword in block j.

ii. Step 1 of the decoding procedure effectively determines the intersection of two sets: the set $\mathcal{S}_v^{(j)}$ and the set of $\{v|(\mathbf{x}_2(v), \mathbf{y}_j) \in \mathcal{A}_\epsilon^{(n)}\}$. This partially resembles the technique used in Theorem 6 in [1].

iii. As oppose to Theorem 6 in [1], Step 2 of the procedure does not require the correct decoding of $\hat{\mathbf{y}}_1(u_j|v_j)$. □

In the next section, the proposed procedure will be adapted to CF, but therein the map is not bijective and the procedure is applied in one direction only.

IV. A NEW DECODING PROCEDURE FOR THE CF SCHEME

For distinction from the QF-SM case, let z_j and s_j be respectively the indices of the estimation codeword and the bin sent by the relay at block j.

Unlike the bijective map ϕ_{QF} in QF-SM, random binning in CF does not admit a one-to-one correspondence between z_{j-1} and s_j; each bin contains potentially many estimation codewords. In fact, the binning process corresponds to a surjective map between the estimation codewords and the non-empty bins; i.e., a many-to-one map $\phi_{\text{CF}} : z_{j-1} \mapsto s_j$, unless s_j is empty. We will show that this map suffices for CF to achieve the rate of the short message QF scheme in Theorem 2. Furthermore, the constraint on the rate of the relay message, R_0, in the CF scheme is less restrictive than the corresponding constraint in the QF-SM scheme. We have the following result.

Theorem 3: Let $R_{\text{CF-M}}$ be the rate achieved by the modified CF scheme that uses successive decoding. Then,

$$R_{\text{CF-M}} \geq R_{\text{QF-SM}},$$

where $R_{\text{QF-SM}}$ is given in Theorem 2. □

To arrive at this result, we will show that using successive decoding with CF yields the rate expression in Theorem 2. However, unlike the short message QF scheme, in which $R_0 = \hat{R}$, in the modified CF scheme, the rate of the relay message, R_0, satisfies the more relaxed constraint:

$$R_0 \geq \min\{\hat{R} - I(\hat{Y}_1; Y|X_2), I(X_2; Y)\}. \tag{2}$$

Proof: To prove Theorem 3 we use the codebooks and the encoding procedure in [1]. Assume that at the end of block $j+1$, the destination has obtained w_{j-1} and s_{j-1}.

1) Decoding s_j: This proceeds in three steps. First, the decoder constructs a set $\mathcal{S}_z^{(j-1)} = \{z|(\mathbf{x}_1(w_{j-1}), \mathbf{x}_2(s_{j-1}), \hat{\mathbf{y}}_1(z|s_{j-1}), \mathbf{y}_{j-1}) \in \mathcal{A}_\epsilon^{(n)}\}$. Second, the decoder determines the set $\mathcal{S}_s^{(j)} = \{s|s = \phi_{\text{CF}}(z), z \in \mathcal{S}_z^{(j-1)}\}$. Since ϕ_{CF} is surjective, $|\mathcal{S}_s^{(j)}| \leq |\mathcal{S}_z^{(j-1)}|$. Third, the decoder finds the bin index $s_j \in \mathcal{S}_s^{(j)}$ for which $(\mathbf{x}_2(s_j), \mathbf{y}_j) \in \mathcal{A}_\epsilon^{(n)}$. Existence and uniqueness of s_j is guaranteed almost surely, if n is sufficiently large and $I(\hat{Y}_1; Y_1|X_1, X_2, Y) \leq I(X_2; Y)$, which is the constraint in Theorem 2; cf. Appendix I-A for details.

2) Recovering w_j: Using s_j obtained in Step 1, the decoder constructs a set $\mathcal{S}_z^{\prime(j)} = \{z|(\mathbf{x}_2(s_j), \hat{\mathbf{y}}_1(z|s_j), \mathbf{y}_j) \in \mathcal{A}_\epsilon^{(n)}, (\mathbf{x}_2(s), \mathbf{y}_{j+1}) \in \mathcal{A}_\epsilon^{(n)}, s = \phi_{\text{CF}}(z)\}$. Using $\mathcal{S}_s^{\prime(j+1)}$ to denote the set $\{s|s = \phi_{\text{CF}}(z), z \in \mathcal{S}_z^{\prime(j)})\}$, the decoder declares that w_j was sent in block j if, for some $z \in \mathcal{S}_z^{\prime(j)}$, $(\mathbf{x}_1(w_j), \hat{\mathbf{y}}_1(z|s_j), \mathbf{x}_2(s_j), \mathbf{y}_j) \in \mathcal{A}_\epsilon^{(n)}$. The surjectivity of ϕ_{CF} implies that $|\mathcal{S}_s^{\prime(j+1)}| \leq |\mathcal{S}_z^{\prime(j)}|$. In Appendix I-B, we show that the correct w_j is determined with an arbitrarily high probability if n is sufficiently large, $R_0 \geq \min\{\hat{R} - I(\hat{Y}_1; Y|X_2), I(X_2; Y)\}$, and $R_{\text{CF-M}}$ is less than the right hand side of (1a), i.e.,

$$\begin{aligned} R_{\text{CF-M}} \leq \sup \min\{&I(X_1; \hat{Y}_1, Y|X_2), \\ &I(X_1, X_2; Y) - I(\hat{Y}_1; Y_1|X_1, X_2, Y)\}, \end{aligned} \tag{3}$$

where the supremum is over the pmfs in Theorem 2. ■

Theorem 3 shows that binning suffices for CF to achieve the rate of QF-SM. We make the following remarks.

Remark 3:

i. Step 1 of the decoding procedure uses the many-to-one map $\phi_{\text{CF}} : z_{j-1} \mapsto s_j$ to recover s_j, without imposing a constraint on R_0. This is in contrast with the procedures in [4], [9], [10], wherein decoding $\mathbf{x}_{2,j}$ does not exploit the map and depends only on finding $(\mathbf{x}_{2,j}, \mathbf{y}_j) \in \mathcal{A}_\epsilon^{(n)}$.

ii. The constraint on R_0 in (2) is less restrictive than the corresponding one in QF-SM, wherein $R_0 = \hat{R}$; restricting R_0 to be equal to $\hat{R}$ makes CF yield the same rate as QF.

iii. When $R_0 > \hat{R}$, the random binning in CF will result in some empty bins. In Appendix I, it is shown that using the successive decoding procedure, these bins do not contribute to the error probability. □

In [8], we modify the CF scheme to yield the same rate of the short message QF with backward decoding in [6].

V. A ONE-MESSAGE TWO-DESTINATION EXAMPLE

In this section, we apply Theorem 3 to the Gaussian multicast relay network in Figure 2. In this network, a source S is assisted by a relay R and wishes to send a common message to two destinations D_1 and D_2. The noise and received signal at destination D_i are denoted by Z_{D_i} and Y_{D_i}, $i = 1, 2$, respectively, where Z_1, Z_{D_1} and Z_{D_2} are Gaussian and statistically independent with unit variance. The source and relay use Gaussian codebooks with average power constraints. The signal-to-noise ratio (SNR) of the S-R, S-D_i and R-D_i are denoted by $\gamma_{\text{SR}}, \gamma_{\text{SD}_i}$ and γ_{RD_i}, $i = 1, 2$, respectively. The relay estimation noise variance [11] is denoted by γ'.

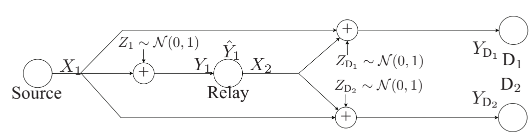

Fig. 2. Gaussian multicast relay channel with 2 destinations.

Using the CF decoding procedure in [9], the rate of the relay message, R_0, must satisfy $R_0 \le \min_{i=1,2}\{I(X_2; Y_{\mathrm{D}_i})\}$. The following rate is achievable with Gaussian codebooks.

$$\begin{aligned} R_{\mathrm{CF}} = \max_{\gamma', R_0} \min_{i=1,2} \min\big\{&\mathcal{C}(\gamma_{\mathrm{SD}_i} + \gamma_{\mathrm{RD}_i}) - \mathcal{C}(1/\gamma') + R_0 \\ &- \mathcal{C}\big(\gamma_{\mathrm{RD}_i}/(1+\gamma_{\mathrm{SD}_i})\big), \mathcal{C}\big(\gamma_{\mathrm{SD}_i} + \gamma_{\mathrm{SR}}/(1+\gamma')\big)\big\}, \\ \text{subject to} \quad & R_0 \le \min_{i=1,2}\big\{\mathcal{C}\big(\gamma_{\mathrm{RD}_i}/(1+\gamma_{\mathrm{SD}_i})\big)\big\}, \end{aligned} \tag{4}$$

where $\mathcal{C}(x) = \frac{1}{2}\log(1+x)$.

The CF decoding procedure in Section IV requires $I(\hat{Y}_1; Y_1|X_1, X_2, Y_{\mathrm{D}_i}) \le I(X_2; Y_{\mathrm{D}_i})$, $i = 1, 2$, which yields

$$\begin{aligned} R_{\text{CF-M}} = \max_{\gamma'} \min_{i=1,2} \min\big\{&\mathcal{C}\big(\gamma_{\mathrm{SD}_i} + \gamma_{\mathrm{SR}}/(1+\gamma')\big), \\ &\mathcal{C}(\gamma_{\mathrm{SD}_i} + \gamma_{\mathrm{RD}_i}) - \mathcal{C}(1/\gamma')\big\}, \\ \text{subject to} \quad & \gamma' \ge \max_{i=1,2}\big\{(1+\gamma_{\mathrm{SD}_i})/(\gamma_{\mathrm{RD}_i})\big\}. \end{aligned} \tag{5}$$

Figure 3 shows the rate of the short message QF scheme and the rates resulting from performing the optimization in (4) and (5) for various γ_{SD_2} when γ_{SR} =10dB, γ_{SD_1} =0dB, $\gamma_{\mathrm{SD}_2} = \gamma_{\mathrm{RD}_1}$ =5dB. It can be seen that using the proposed

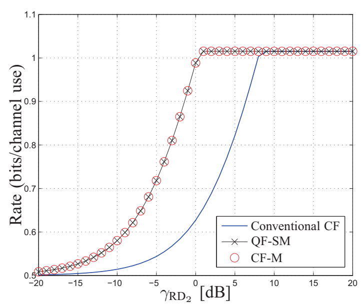

Fig. 3. Achievable rate of the Gaussian multicast relay channel in Figure 2.

decoding procedure with CF yields the same rate as QF-SM with forward decoding [6] and can yield a significant gain over CF with standard decoding at low-to-moderate SNRs. Note that, for QF, $R_0 = \hat{R}$, whereas for CF

$$R_0 \ge \max_{i=1,2}\Big\{\min\Big\{\hat{R} - \mathcal{C}\Big(\frac{1}{\gamma'} + \frac{\gamma_{\mathrm{SR}}\gamma'^{-1}}{1+\gamma_{\mathrm{SD}_i}}\Big), \mathcal{C}\Big(\frac{\gamma_{\mathrm{RD}_i}}{1+\gamma_{\mathrm{SD}_i}}\Big)\Big\}\Big\}.$$

VI. A Two-Message Three-Destination Example

In this section, we will provide an example that exposes the advantage of the proposed CF-M over QF-SM and conventional CF. The key philosophy that underlies the structure of this example is to construct a setup in which the rate that can be reliably communicated to one of the nodes is reduced by increasing the rate at which the relay sends the bin index to other nodes in the network, i.e., R_0. Since R_0 in CF-M is lower than the corresponding rate in QF-SM, it can be shown that, in this scenario, CF-M can achieve higher rates.

We consider a communication network wherein a source S wishes to send a common message to two destinations D_1 and D_2. Node R acts as a relay to assist S, and in addition, R acts as a source that wishes to send an independent message to a third destination D_3. There is no direct link between S and D_3 and they do not wish to communicate with each other. An abstract version of this network is depicted in Figure 4.

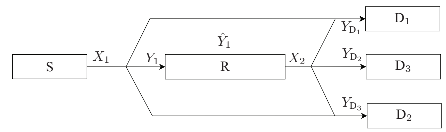

Fig. 4. A two-message three-user network.

A. Application of CF-M Relaying

For the network depicted in Figure 4, the CF-M scheme can be combined with standard superposition coding [12] to yield the rate region given in the following theorem. In this theorem, $R_1^{\text{CF-M}}$ denotes the rate of the common message sent from S to D_1 and D_2, and $R_2^{\text{CF-M}}$ denotes the rate of the independent message sent from R to D_3.

Theorem 4: Consider the two-message three-user channel in Figure 4 $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1, y_{\mathrm{D}_1}, y_{\mathrm{D}_2}, y_{\mathrm{D}_3}|x_1, x_2), \mathcal{Y}_1 \times \mathcal{Y}_{\mathrm{D}_1} \times \mathcal{Y}_{\mathrm{D}_2} \times \mathcal{Y}_{\mathrm{D}_3})$. For any fixed pmf satisfying

$$\begin{aligned} &p(u, x_1, x_2, \hat{y}_1, y_1, y_{\mathrm{D}_1}, y_{\mathrm{D}_2}, y_{\mathrm{D}_3}) = p(u)p(x_1)p(x_2|u) \\ &\times p(\hat{y}_1|u, y_1)p(y_1, y_{\mathrm{D}_1}, y_{\mathrm{D}_2}|x_1, x_2, y_{\mathrm{D}_3})p(y_{\mathrm{D}_3}|x_2), \text{and} \end{aligned} \tag{6}$$

$$I(\hat{Y}_1; Y_1|X_1, X_2, Y_{\mathrm{D}_i}) \le I(U; Y_{\mathrm{D}_i}), \qquad i = 1, 2, \tag{7}$$

the rate vector $(R_1^{\text{CF-M}}, R_2^{\text{CF-M}})$ is achievable, where

$$\begin{aligned} R_1^{\text{CF-M}} \le \min_{i=1,2} \min\{&I(X_1; \hat{Y}_1, Y_{\mathrm{D}_i}|U), \\ &I(X_1, U; Y_{\mathrm{D}_i}) - I(\hat{Y}_1; Y_1|X_1, U, Y_{\mathrm{D}_i})\} \end{aligned} \tag{8a}$$

$$R_2^{\text{CF-M}} \le \min\{I(X_2; Y_{\mathrm{D}_3}|U), I(X_2; Y_{\mathrm{D}_3}) - R_0^{\text{CF-M}}\}, \tag{8b}$$

$$R_0^{\text{CF-M}} \ge \max_{i=1,2} \min\{I(\hat{Y}_1; Y_1|U) - I(\hat{Y}_1; Y_{\mathrm{D}_i}|U), I(U; Y_{\mathrm{D}_i})\}. \tag{8c}$$

□

In this theorem, U is the auxiliary random variable representing the bin index of $\hat{Y}_1$ and used to assist D_1 and D_2. The relay transmits the codeword X_2, which is constructed by superimposing the incremental information intended for D_3 on U; in the absence of D_3, $X_2 = U$.

Proof: For brevity, we provide a sketch of the proof.

Codebook:: Let m_1 be the common message intended for D_1 and D_2, and m_2 be the independent message intended for D_3. Let $\mathbf{u}$, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\hat{\mathbf{y}}_1$ be length-n vectors with independent and identically distributed (i.i.d) random entries generated with the fixed distributions $p(\mathbf{u}) = \prod_{i=1}^n p(u_i)$, $p(\mathbf{x}_1) = \prod_{i=1}^n p(x_{1i})$, $p(\mathbf{x}_2|\mathbf{u}) = \prod_{i=1}^n p(x_{2i}|u_i)$ and $p(\hat{\mathbf{y}}_1|\mathbf{u}) = \prod_{i=1}^n p(\hat{y}_{1i}|u_i)$, respectively. The distributions $p(\mathbf{u})$ and $p(\mathbf{x}_1)$ are used to generate $2^{nR_1^{\text{CF-M}}}$ random sequences $\{\mathbf{x}_1(m_1)\}_{m_1=1}^{2^{nR_1^{\text{CF-M}}}}$ and $2^{nR_0^{\text{CF-M}}}$ random sequences $\{\mathbf{u}(s)\}_{s=1}^{2^{nR_0^{\text{CF-M}}}}$ independently. For each $s \in \{1, \ldots, 2^{nR_0^{\text{CF-M}}}\}$, the distributions $p(\hat{\mathbf{y}}_1|\mathbf{u})$ and $p(\mathbf{x}_2|\mathbf{u})$ are respectively used to generate $2^{n\hat{R}}$ random sequences $\{\hat{\mathbf{y}}_1(z|s)\}_{z=1}^{2^{n\hat{R}}}$ and $2^{nR_2^{\text{CF-M}}}$ random sequences $\{\mathbf{x}_2(m_2|s)\}_{m_2=1}^{2^{nR_2^{\text{CF-M}}}}$ independently, where $\hat{R} \ge I(\hat{Y}_1; Y_1|U)$ [1]. This resembles superposition coding [12] in that m_2, the message for D_3, is superimposed on s, the bin index of the message for D_1 and D_2.

Random Binning: The set $\{1, 2, \cdots, 2^{n\hat{R}}\}$ is randomly partitioned into $2^{nR_0^{\text{CF-M}}}$ cells $\{S_s\}_{s=1}^{2^{nR_0^{\text{CF-M}}}}$.

Encoding: In block j, node S wishes to send message $m_{1,j}$ to D_1 and D_2, and node R wishes to send message $m_{2,j}$ to D_3. Upon receiving $\mathbf{y}_{1,j-1}$ at the end of block $j-1$, node R finds an index z_{j-1} such that $(\hat{\mathbf{y}}_1(z_{j-1}|s_{j-2}), \mathbf{u}(s_{j-2}), \mathbf{y}_{1,j-1}) \in \mathcal{A}_\epsilon^{(n)}$. Subsequently, node R finds the index, s_{j-1}, of the bin containing z_{j-1}. Node R selects the codeword $\mathbf{x}_2(m_{2,j}|s_{j-1})$. The codeword pair $(\mathbf{x}_1(m_{1,j}), \mathbf{x}_2(m_{2,j}|s_{j-1}))$ will be sent in block j. In comparison, in the absence of node D_3 and message $m_{2,j}$, as in the previous example, node R transmits $\mathbf{x}_2(s_{j-1})$.

Decoding and Achievability: The broadcast nature of the network implies that the destinations D_i, $i = 1, 2, 3$, decode their received signals independently.

The decoding procedure for D_1 and D_2 is analogous to that in Section IV, and can be summarized as follows. The probability that either D_1 or D_2 decodes s_j erroneously can be made arbitrarily small if n is chosen to be sufficiently large and if $I(\hat{Y}_1; Y_1|X_1, X_2, Y_{\text{D}_i}) \leq I(U; Y_{\text{D}_i})$, for $i = 1, 2$. Subject to these constraints, the probability that either D_1 or D_2 decodes $m_{1,j}$ erroneously can be made arbitrarily small if $R_0^{\text{CF-M}} \geq \max_{i=1,2} \min\{I(\hat{Y}_1; Y_1|U) - I(\hat{Y}_1; Y_{\text{D}_i}|U), I(U; Y_{\text{D}_i})\}$ and $R_1^{\text{CF-M}} \leq \min_{i=1,2} \min\{I(X_1; \hat{Y}_1, Y_{\text{D}_1}|U), I(X_1, U; Y_{\text{D}_i}) - I(\hat{Y}_1; Y_1|X_1, U, Y_{\text{D}_i})\}$.

To recover $m_{2,j}$, D_3 uses $\mathbf{y}_{\text{D}_3}$ to find the unique index $\hat{m}_{2,j}$ such that $(\mathbf{x}_2(\hat{m}_{2,j}|\hat{s}_{j-1}), \mathbf{u}(\hat{s}_{j-1}), \mathbf{y}_{\text{D}_{3,j-1}}) \in \mathcal{A}_\epsilon^{(n)}$, for some $\hat{s}_{j-1} \in \{1, \ldots, 2^{nR_0^{\text{CF-M}}}\}$. Note that, in this process, D_3 uses the structure of $\mathcal{U}$, but does not attempt to decode $\mathbf{u}(s_{j-1})$. In other words, $\hat{s}_{j-1}$ obtained by D_3 may not be unique. Hence, we consider two case: $\hat{s}_{j-1} = s_{j-1}$, and $\hat{s}_{j-1} \neq s_{j-1}$.

When $\hat{s}_{j-1} = s_{j-1}$ and $R_2^{\text{CF-M}} \leq I(X_2; Y_{\text{D}_3}|U)$, the probability that $\hat{m}_{2,j} \neq m_{2,j}$ can be made arbitrarily small by choosing n to be sufficiently large.

When $\hat{s}_{2,j-1} \neq s_{2,j-1}$, the probability that $\hat{m}_{2,j} \neq m_{2,j}$ such that $(\mathbf{x}_2(\hat{m}_{2,j}|\hat{s}_{j-1}), \mathbf{u}(\hat{s}_{j-1}), \mathbf{y}_{\text{D}_{3,j}}) \in \mathcal{A}_\epsilon^{(n)}$, for a particular $\hat{s}_{2,j-1} \neq s_{2,j-1}$ and $\hat{m}_{2,j} \neq m_{2,j}$, is asymptotically bounded by $2^{-nI(X_2, U; Y_{\text{D}_3})} = 2^{-nI(X_2; Y_{\text{D}_3})}$. Adding over all possible $\hat{s}_{2,j-1}$ and $\hat{m}_{2,j}$ yields that the probability of decoding error is asymptotically bounded by $2^{n(R_2^{\text{CF-M}} + R_0^{\text{CF-M}} - I(X_2, U; Y_{\text{D}_3}))}$. Hence, the probability of error can be made arbitrarily small if n is large and $R_2^{\text{CF-M}} \leq I(X_2; Y_{\text{D}_3}) - R_0^{\text{CF-M}}$.

Combining the cases of $\hat{s}_{2,j-1} = s_{2,j-1}$ and $\hat{s}_{2,j-1} \neq s_{2,j-1}$, it can be seen that the probability of error at D_3 can be made arbitrarily small if n is large and $R_2^{\text{CF-M}}$ satisfies (8b). ∎

The proof of this theorem relies essentially on combining the principles of superposition coding with the modified CF relaying. For comparison, we consider combining superposition coding with QF-SM with forward decoding [6] for the same network scenario in Figure 4.

B. Application of QF-SM Relaying

To apply QF-SM with forward decoding [6], the codebooks are generated in essentially the same way as in the case of CF-M relaying, but with the additional restriction $R_0^{\text{QF-SM}} = \hat{R}$.

Corollary 1: For any pmf of the form in (6) satisfying the condition in (7), all the rate vectors $(R_1^{\text{QF-SM}}, R_2^{\text{QF-SM}})$ satisfying the following constraints are achievable.

$$\begin{aligned} R_1^{\text{QF-SM}} \leq \min_{i=1,2} \min\{&I(X_1; \hat{Y}_1, Y_{\text{D}_i}|U), \\ &I(X_1, U; Y_{\text{D}_1}) - I(\hat{Y}_1; Y_1|X_1, U, Y_{\text{D}_i})\} \end{aligned} \tag{9a}$$

$$\begin{aligned} R_2^{\text{QF-SM}} \leq \min\{&I(X_2; Y_{\text{D}_3}|U), \\ &I(X_2; Y_{\text{D}_3}) - I(\hat{Y}_1; Y_1|X_2, Y_{\text{D}_3})\}. \end{aligned} \tag{9b}$$

Proof: The decoding procedure of $m_{1,j}$ at D_1 and D_2 is analogous to that in Section V.

The decoding procedure of $m_{2,j}$ at D_3 is analogous to that provided in Section II. D_3 chooses $\hat{z}_{j-1}$ if $(\hat{\mathbf{y}}_1(\hat{z}_{j-1}), \mathbf{x}_2(m_{2,j-1}|s_{j-2}), \mathbf{u}(s_{j-2}), \mathbf{y}_{\text{D}_{3,j-1}}) \in \mathcal{A}_\epsilon^{(n)}$. There are two possibilities: $\hat{z}_{j-1} = z_{j-1}$ or $\hat{z}_{j-1} \neq z_{j-1}$. If $\hat{z}_{j-1} = z_{j-1}$, the bijectivity of ϕ_{QF} yields $\hat{s}_{j-1} = sj - 1$, which implies that if $R_2^{\text{QF-SM}} \leq I(X_2; Y_{\text{D}_3}|U)$, the probability that $\hat{m}_{2,j} \neq m_{2,j}$ can be made arbitrarily small by choosing n to be large. We now consider the second possibility. The probability that $\hat{z}_{j-1} \neq z_{j-1}$ is asymptotically bounded by $2^{-nI(\hat{Y}_1; X_2, Y_{\text{D}_3}|U)}$. In this case, $\hat{s}_{j-1} \neq s_{j-1}$, and a decoding error occurs if, for this $\hat{s}_{j-1}$, D_3 finds $\hat{m}_{2,j} \neq m_{2,j}$ such that $(\mathbf{x}_2(\hat{m}_{2,j}|\hat{s}_{j-1}), \mathbf{u}(\hat{s}_{j-1}), \mathbf{y}_j) \in \mathcal{A}_\epsilon^{(n)}$. The probability of finding such an $\hat{m}_{2,j}$ is asymptotically bounded by $2^{-nI(X_2, U; Y_{\text{D}_3})} = 2^{-nI(X_2; Y_{\text{D}_3})}$. Hence, adding over all possible $\hat{s}_{2,j-1}$ and $\hat{z}_{j-1}$ yields that the probability of decoding error is asymptotically bounded by $2^{n(R_2^{\text{QF-SM}} + \hat{R} - I(X_2; Y_{\text{D}_3}) - I(\hat{Y}_1; X_2, Y_{\text{D}_3}|U))}$, whereupon it can be made arbitrarily small if n is large and $R_2^{\text{QF-SM}} \leq I(X_2; Y_{\text{D}_3}) + I(\hat{Y}_1; X_2, Y_{\text{D}_3}|U) - \hat{R}$. Using the fact that $\hat{R} \geq I(\hat{Y}_1; Y_1|U)$, we have $R_2^{\text{QF-SM}} \leq I(X_2; Y_{\text{D}_3}) - I(\hat{Y}_1; Y_1|X_2, Y_{\text{D}_3})$. ∎

A comparison between the rates achieved by CF-M and QF-SM is provided in the following corollary.

Corollary 2: For the network in Figure 4, the region of rates achieved by CF-M contains the region of rates achieved by QF-SM; i.e., for every $R_1^{\text{CF-M}} = R_1^{\text{QF-SM}}$, $R_2^{\text{CF-M}} \geq R_2^{\text{QF-SM}}$.

Proof: First, we note that, CF-M and QF-SM achieve the same rate for the message intended to D_1 and D_2 as shown in Section V, i.e., for any given pmf of the form in (6) satisfying the condition in (7), $R_1^{\text{CF-M}} = R_1^{\text{QF-SM}}$.

To prove the corollary, we note that, for a given bin index (represented by U), X_2 represents the incremental information intended for D_3 and is independent of $\hat{Y}_1$; i.e., $I(\hat{Y}_1; X_2|U) = 0$. Next, we note that, conditioned on X_2, Y_{D_3} is independent of $\hat{Y}_1$ and Y_1. Using this in (9b) yields

$$\begin{aligned} I(\hat{Y}_1; Y_1|X_2, Y_{\text{D}_3}) &= I(\hat{Y}_1; Y_1|X_2) \\ &= I(\hat{Y}_1; X_2|U) + I(\hat{Y}_1; Y_1|X_2, U) \\ &= I(\hat{Y}_1; Y_1, X_2|U) \\ &= I(\hat{Y}_1; Y_1|U). \end{aligned}$$

Using this in (9b) yields

$$R_2^{\text{QF-SM}} \le \min\{I(X_2;Y_{\mathrm{D}_3}|U), I(X_2;Y_{\mathrm{D}_3}) - I(\hat{Y}_1;Y_1|U)\}. \quad (10)$$

Comparing this with (8b) and (8c), it can be seen that $R_2^{\text{CF-M}} \ge R_2^{\text{QF-SM}}$, which completes the proof of the corollary. ■

C. Application of Conventional CF Relaying

For comparison, we now provide the expressions for the rates achievable by the conventional CF scheme. For this scheme, the codebooks are generated in an analogous manner to the CF-M scheme, but with the decoding procedure in [9] used at D_1 and D_2 and the decoding procedure in Theorem 4 used at D_3. Using these codebooks and decoding procedures, it is straightforward to prove the following result.

Corollary 3: For any pmf of the form in (6), the rate vectors $(R_1^{\text{CF}}, R_2^{\text{CF}})$ satisfying the following constraints are achievable.

$$\begin{aligned} R_1^{\text{CF}} &\le \min\{I(X_1;\hat{Y}_1, Y_{\mathrm{D}_i}|U), I(X_1, U; Y_{\mathrm{D}_i}) \\ &\qquad - I(\hat{Y}_1;Y_1|X_1,U,Y_{\mathrm{D}_i}) + R_0^{\text{CF}} - I(U;Y_{\mathrm{D}_i})\}, \\ R_2^{\text{CF}} &\le \min\{I(X_2;Y_{\mathrm{D}_3}|U), I(X_2;Y_{\mathrm{D}_3}) - R_0^{\text{CF}}\}, \\ R_0^{\text{CF}} &\le \min_{i=1,2} \min\{I(U;Y_{\mathrm{D}_i})\}. \qquad \square \end{aligned}$$

D. Applying CF, QF-SM and CF-M in A Gaussian Network

It is instructive to compare the performance of the CF, QF-SM and CF-M schemes in the case when each link in Figure 4 is an additive white Gaussian channel with identically distributed statistically independent zero mean unit variance Gaussian noises Z_1 at R, and Z_{D_i} and at D_i, $i = 1, 2, 3$, respectively. This case is shown in Figure 5. As before, S and R use Gaussian codebooks with average power constraints. For constructing these codebooks, we use $\alpha_0 \in [0, 1]$ to represent the fraction of power that node R allocates to transmit the bin index s and $\alpha_1 = 1 - \alpha_0$ to represent the fraction of power that node R allocates to transmit its own message index m_2. The SNR of the S-R, S-D_i and R-D_i links are denoted by $\gamma_{\text{SR}}, \gamma_{\text{SD}_i}$ and γ_{RD_i}, $i = 1, 2, 3$, respectively. The relay estimation noise variance [11] is denoted by γ'.

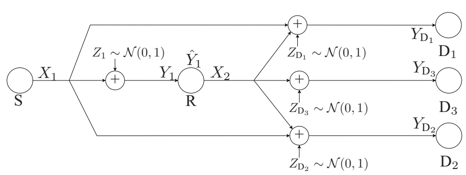

Fig. 5. A Gaussian network with two messages and three destinations.

Using the codebook structure and Corollary 3, we have for any $\alpha_0 \in [0, 1]$ and

$$R_0^{\text{CF}} \le \min_{i=1,2}\{\mathcal{C}\big(\alpha_0\gamma_{\text{RD}_i}/(1+\gamma_{\text{SD}_i}+(1-\alpha_0)\gamma_{\text{RD}_i})\big)\},$$

conventional CF [9] achieves the following rate pair

$$\begin{aligned} R_1^{\text{CF}} = \min_{i=1,2}\Big\{&\mathcal{C}\Big(\frac{\gamma_{\text{SD}_i}+\alpha_0\gamma_{\text{RD}_i}}{1+(1-\alpha_0)\gamma_{\text{RD}_i}}\Big) - \mathcal{C}(1/\gamma') + R_0^{\text{CF}} \\ &- \mathcal{C}\Big(\frac{\alpha_0\gamma_{\text{RD}_i}}{(1+\gamma_{\text{SD}_i}+(1-\alpha_0)\gamma_{\text{RD}_i})}\Big), \\ &\mathcal{C}\Big(\frac{\gamma_{\text{SD}_i}}{1+(1-\alpha_0)\gamma_{\text{RD}_i}} + \frac{(1-\alpha_0)\gamma_{\text{SR}}\gamma_{\text{RD}_i}+\gamma_{\text{SR}}}{(1+\gamma')(1+(1-\alpha_0)\gamma_{\text{RD}_i})}\Big)\Big\}, \\ R_2^{\text{CF}} = \min\big\{&\mathcal{C}\big((1-\alpha_0)\gamma_{\text{RD}_3}\big), \mathcal{C}\big(\gamma_{\text{RD}_3}\big) - R_0^{\text{CF}}\big\}. \end{aligned}$$

For the same codebook structure and

$$\gamma' \ge \max_{i=1,2}\big\{(1+\gamma_{\text{SD}_i}+(1-\alpha_0)\gamma_{\text{RD}_i})/(\alpha_0\gamma_{\text{RD}_i})\big\}, \quad (11)$$

the following rate pair $(R_1^{\text{QF-SM}}, R_2^{\text{QF-SM}})$ is achievable by the QF-SM scheme in [6].

$$\begin{aligned} R_1^{\text{QF-SM}} = \min_{i=1,2}\Big\{&\mathcal{C}\Big(\frac{\gamma_{\text{SD}_i}+\alpha_0\gamma_{\text{RD}_i}}{1+(1-\alpha_0)\gamma_{\text{RD}_i}}\Big) - \mathcal{C}(1/\gamma'), \\ &\mathcal{C}\Big(\frac{\gamma_{\text{SD}_i}}{1+(1-\alpha_0)\gamma_{\text{RD}_i}} + \frac{(1-\alpha_0)\gamma_{\text{SR}}\gamma_{\text{RD}_i}+\gamma_{\text{SR}}}{(1+\gamma')(1+(1-\alpha_0)\gamma_{\text{RD}_i})}\Big)\Big\}, \\ R_2^{\text{QF-SM}} = \min_{i=1,2}\big\{&\mathcal{C}\big((1-\alpha_0)\gamma_{\text{RD}_3}\big), \mathcal{C}\big(\gamma_{\text{RD}_3}\big) - \mathcal{C}\big((1+\gamma_{\text{SR}})/\gamma'\big)\big\}. \end{aligned}$$

Similarly, for CF-M, for any γ' satisfying (11) and

$$\begin{aligned} R_0^{\text{CF-M}} \ge \max_{i=1,2}\min\Big\{&\mathcal{C}\Big(\frac{\alpha_0\gamma_{\text{RD}_i}}{1+\gamma_{\text{SD}_i}+(1-\alpha_0)\gamma_{\text{RD}_i}}\Big), \\ &\mathcal{C}\Big(\frac{1+\gamma_{\text{SD}_i}+\gamma_{\text{SR}}+(1-\alpha_0)\gamma_{\text{RD}_i}(1+\gamma_{\text{SR}})}{\gamma'(1+\gamma_{\text{SD}_i}+(1-\alpha_0)\gamma_{\text{RD}_i})}\Big)\Big\}, \end{aligned}$$

Theorem 4 yields

$$\begin{aligned} R_1^{\text{CF-M}} &= R_1^{\text{QF-SM}}, \\ R_2^{\text{CF-M}} &= \min\big\{\mathcal{C}\big((1-\alpha_0)\gamma_{\text{RD}_3}\big), \mathcal{C}\big(\gamma_{\text{RD}_3}\big) - R_0^{\text{CF-M}}\big\}. \end{aligned}$$

Similar to Corollary 2, the codebook construction for the Gaussian case implies that $I(\hat{Y}_1; X_2|U) = 0$, which further implies that the region of rates achieved by the CF-M scheme in this case contains the corresponding region of the QF-SM scheme. In Figure 6, the rate regions of the conventional CF and the CF-M and QF-SM schemes are plotted for an instance in which $\gamma_{\text{SR}} = \gamma_{\text{RD}_2} = 15$ dB, $\gamma_{\text{SD}_1} = \gamma_{\text{SD}_2} = \gamma_{\text{RD}_3} = 5$ dB and $\gamma_{\text{RD}_1} = 0$ dB. For this instance, it can be seen from the figure that both the conventional CF and the QF-SM rate regions are properly contained in the CF-M rate region. For instance, comparing the sum rates achieved by the considered schemes, it can be seen that the maximum sum rates of the conventional CF and the QF-SM schemes are 1.1641 and 1.1991 bits per channel use (bpcu), respectively, whereas the corresponding sum rate of the CF-M scheme is 1.2886 bpcu. For the conventional CF, QF-SM and the CF-M scheme, the maximum sum rate is achieved with $\alpha_0 = 1$ and $\gamma' = 35.7851$, $\alpha_0 = 0.95$ and $\gamma' = 12.5926$, and $\alpha_0 = 0.86$ and $\gamma' = 11.3829$, respectively.

VII. Conclusion

Short message QF (QF-SM) and CF are more practical than Noisy Netwrok Coding (NNC) as they incur a significantly less delay and decoding complexity. In this paper we provided a new forward decoding procedure that can be used in both QF-SM and CF and that enables CF to yield higher achievable rates than QF-SM. To illustrate the advantage of

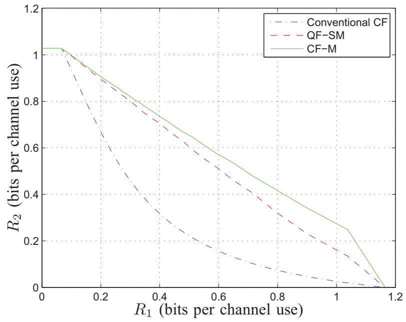

Fig. 6. Rate region for the network in Figure 5. ($\gamma_{\text{SR}} = 15$ dB, $\gamma_{\text{SD}_1} = 5$ dB, $\gamma_{\text{RD}_1} = 0$ dB, $\gamma_{\text{SD}_2} = 5$ dB, $\gamma_{\text{RD}_2} = 15$ dB, $\gamma_{\text{RD}_3} = 5$ dB.)

this procedure, we applied the conventional CF, the QF-SM and the CF schemes with modified decoding (CF-M) in two network scenarios, each with one source and one relay. In the first scenario there are two destinations, whereas in the second scenario, there are three destinations, and the relay has its own message to send to one of them. In the Gaussian case of the first scenario, QF-SM and CF-M achieve the same rate, which is higher than that achieved by the conventional CF. In the second scenario, CF-M is shown to achieve strictly higher rates than both conventional CF and QF-SM.

Appendix I
Probability of Error Analysis of Theorem 3

Let the source, relay estimation and transmission codewords be indexed by $m_\ell = 1$, $z_\ell = 1$ and $s_\ell = 1$ for blocks $\ell = j-1, j, j+1$, respectively. The codewords at different time blocks will be distinguished by the block index; e.g., $\mathbf{x}_{1,j}$ will be used to denote $\mathbf{x}_1(w_j)$. Assume that, at the end of block $j+1$, the destination has correctly decoded s_{j-1} and w_{j-1}, but z_{j-1} is not decoded. We will bound the probability of erroneous decoding of s_j and w_j. For $\epsilon > 0$, we will use $\delta_i = \delta_i(\epsilon)$ to be such that $\delta_i(\epsilon) \searrow 0$ as $\epsilon \to 0$, $i = 1, \ldots, 9$.

A. Probability of Erroneous Decoding of s_j

The probability of error $P(\mathcal{E}_s) = P(\bigcup_{i=1}^3 \mathcal{E}_i)$, where

$$\begin{aligned}
\mathcal{E}_1 &= \{(\mathbf{x}_{2,j}(1), \mathbf{Y}_j) \notin \mathcal{A}_\epsilon^{(n)} \cup (\mathbf{x}_{1,j-1}(1), \hat{\mathbf{y}}_{1,j-1}(1|1),\\
&\qquad \mathbf{x}_{2,j-1}(1), \mathbf{Y}_{j-1}) \notin \mathcal{A}_\epsilon^{(n)}\},\\
\mathcal{E}_2 &= \{(\hat{\mathbf{Y}}_{1,j-1}(z|1), \mathbf{x}_{2,j-1}(1), \mathbf{Y}_{1,j-1}) \notin \mathcal{A}_\epsilon^{(n)}, \text{for all } z\},\\
\mathcal{E}_3 &= \{(\mathbf{x}_{1,j-1}(1), \hat{\mathbf{Y}}_{1,j-1}(\hat{z}|1), \mathbf{x}_{2,j-1}(1), \mathbf{Y}_{j-1}) \in \mathcal{A}_\epsilon^{(n)},\\
&\qquad \cap (\mathbf{X}_{2,j}(\hat{s}), \mathbf{Y}_j) \in \mathcal{A}_\epsilon^{(n)}, \text{for some } \hat{s} = \phi_{\text{CF}}(\hat{z}) \neq 1\}.
\end{aligned}$$

Note that in the last event $\hat{s} \neq 1$ and hence, $\hat{z} \neq 1$.

We have $P(\mathcal{E}_s) \leq \sum_{i=1}^3 P(\mathcal{E}_i)$ by the union bound. Using the conditional joint typicality and covering lemmas in [9, Part I], the probabilities $P(\mathcal{E}_1), P(\mathcal{E}_2) \to 0$, if $n \to \infty$, and

$$\hat{R} \geq I(\hat{Y}_1; Y_1|X_2) + \delta_1, \tag{12}$$

$$\begin{aligned}
P(\mathcal{E}_3) &\leq \sum_{\hat{z}=\phi_{\text{CF}}^{-1}(\hat{s}), \hat{s}\neq 1} P\big((\mathbf{X}_{2,j}(\hat{s}), \mathbf{Y}_j) \in \mathcal{A}_\epsilon^{(n)} \cap\\
&\qquad (\mathbf{x}_{1,j-1}(1), \hat{\mathbf{Y}}_{1,j-1}(\hat{z}|1), \mathbf{x}_{2,j-1}(1), \mathbf{Y}_{j-1}) \in \mathcal{A}_\epsilon^{(n)}\big)\\
&= \sum_{\hat{z}=\phi_{\text{CF}}^{-1}(\hat{s}), \hat{s}\neq 1} P\big((\mathbf{X}_{2,j}(\hat{s}), \mathbf{Y}_j) \in \mathcal{A}_\epsilon^{(n)}\big) P\big(\mathbf{x}_{1,j-1}(1),\\
&\qquad \hat{\mathbf{Y}}_{1,j-1}(\hat{z}|1), \mathbf{x}_{2,j-1}(1), \mathbf{Y}_{j-1}) \in \mathcal{A}_\epsilon^{(n)}\big) \quad (13)\\
&\leq \sum_{\hat{z}=\phi_{\text{CF}}^{-1}(\hat{s}), \hat{s}\neq 1} \sum_{\mathbf{y}} p(\mathbf{y}) P\big((\mathbf{X}_2(\hat{s}), \mathbf{y}) \in \mathcal{A}_\epsilon^{(n)} \big| \mathbf{y}\big) \sum_{(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) \in \mathcal{A}_\epsilon^{(n)}}\\
&\qquad p(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) P\big((\mathbf{x}_1, \mathbf{x}_2, \hat{\mathbf{Y}}_1, \mathbf{y}) \in \mathcal{A}_\epsilon^{(n)} \big| \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}\big),
\end{aligned}$$

where the equality follows from the memorylessness of the channel. Using the surjectivity of ϕ_{CF},

$$\begin{aligned}
P\big((\mathbf{X}_2, \mathbf{y}) \in \mathcal{A}_\epsilon^{(n)} \big| \mathbf{y}\big) &= \sum_{\mathbf{x}_2 \in \mathcal{A}_\epsilon^{(n)}(\mathbf{X}_2|\mathbf{y})} p(\mathbf{x}_2),\\
P\big((\mathbf{x}_1, \mathbf{x}_2, \hat{\mathbf{Y}}_1, \mathbf{y}) \in \mathcal{A}_\epsilon^{(n)} \big| \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}\big) &= \sum_{\hat{\mathbf{y}}_1 \in \mathcal{A}_\epsilon^{(n)}(\hat{\mathbf{Y}}_1|\mathbf{x}_1, \mathbf{x}_2, \mathbf{y})} p(\hat{\mathbf{y}}_1|\mathbf{x}_2).
\end{aligned}$$

and the properties of jointly typical sequences [12], we have

$$P(\mathcal{E}_3) \leq 2^{n(\hat{R} - I(X_2;Y) - I(\hat{Y}_1; X_1, Y|X_2) + \delta_2)}.$$

Hence, if $\hat{R} \leq I(X_2;Y) + I(\hat{Y}_1; X_1, Y|X_2) - \delta_2$, the probability of error in decoding s_j tends to 0 as $n \to \infty$.

Remark 4: Analogous error events can be defined for Step 1 of the QF decoding scheme in Section III. Since therein ϕ_{QF} is bijective, the probability of events corresponding to $\mathcal{E}_3$ can be bounded similarly resulting in the same constraint on $\hat{R}$. □

B. Probability of Erroneous Decoding of w_j

Since the probability of decoding s_j incorrectly can be made arbitrarily small, we can assume that s_j is correctly decoded. To bound the probability of decoding w_j incorrectly, we have $P(\mathcal{E}_w) \leq \sum_{i=1}^4 P(\mathcal{E}_i)$, where

$$\begin{aligned}
\mathcal{E}_1 &= \{(\mathbf{x}_{1,j}(1), \mathbf{x}_{2,j}(1), \hat{\mathbf{y}}_{1,j}(1|1), \mathbf{Y}_j) \notin \mathcal{A}_\epsilon^{(n)}\\
&\qquad \cup (\mathbf{x}_{2,j+1}(1), \mathbf{Y}_{j+1}) \notin \mathcal{A}_\epsilon^{(n)}\},\\
\mathcal{E}_2 &= \{(\mathbf{X}_{1,j}(\hat{w}), \mathbf{x}_{2,j}(1), \hat{\mathbf{y}}_{1,j}(1|1), \mathbf{Y}_j) \in \mathcal{A}_\epsilon^{(n)}, \hat{w} \neq 1\},\\
\mathcal{E}_3 &= \{(\mathbf{X}_{1,j}(\hat{w}), \mathbf{x}_{2,j}(1), \hat{\mathbf{Y}}_{1,j}(\hat{z}|1), \mathbf{Y}_j) \in \mathcal{A}_\epsilon^{(n)}\\
&\qquad \cap (\mathbf{X}_{2,j+1}(\hat{s}), \mathbf{Y}_{j+1}) \in \mathcal{A}_\epsilon^{(n)}, \text{for some}\\
&\qquad \hat{w} \neq 1, \hat{z} \neq 1, \hat{s} = \phi_{\text{CF}}(\hat{z}) \neq 1, z \in \mathcal{S}_z'^{(j)}\},\\
\mathcal{E}_4 &= \{(\mathbf{X}_{1,j}(\hat{w}), \mathbf{x}_{2,j}(1), \hat{\mathbf{Y}}_{1,j}(\hat{z}|1), \mathbf{Y}_j) \in \mathcal{A}_\epsilon^{(n)},\\
&\qquad \text{for some } \hat{w} \neq 1, \hat{z} \neq 1, \hat{s} = \phi_{\text{CF}}(\hat{z}) = 1, z \in \mathcal{S}_z'^{(j)}\},
\end{aligned}$$

where $\mathcal{S}_z'^{(j)}$ is defined in Section IV. Using the conditional typicality lemma [9], $P(\mathcal{E}_1) \to 0$ as $n \to \infty$. Furthermore,

$$\begin{aligned}
P(\mathcal{E}_2) &\leq \sum_{\hat{w}\neq 1} \sum_{(\mathbf{x}_2, \hat{\mathbf{y}}_1, \mathbf{y}) \in \mathcal{A}_\epsilon^{(n)}} p(\mathbf{x}_2, \hat{\mathbf{y}}_1, \mathbf{y})\\
&\qquad \times P\big((\mathbf{X}_1, \mathbf{x}_2, \hat{\mathbf{y}}_1, \mathbf{y}) \in \mathcal{A}_\epsilon^{(n)} \big| \mathbf{x}_2, \hat{\mathbf{y}}_1, \mathbf{y}\big)\\
&= \sum_{\hat{w}\neq 1} \sum_{(\mathbf{x}_2, \hat{\mathbf{y}}_1, \mathbf{y}) \in \mathcal{A}_\epsilon^{(n)}} p(\mathbf{x}_2, \hat{\mathbf{y}}_1, \mathbf{y}) \sum_{\mathbf{x}_1 \in \mathcal{A}_\epsilon^{(n)}(\mathbf{X}_1|\mathbf{x}_2, \hat{\mathbf{y}}_1, \mathbf{y})} p(\mathbf{x}_1)\\
&\leq 2^{n(R_{\text{CF-M}} - I(X_1; \hat{Y}_1, Y|X_2) + \delta_3)}.
\end{aligned}$$

Hence, $P(\mathcal{E}_2)$ tends to 0 if n is sufficiently large and

$$R_{\text{CF-M}} \leq I(X_1; \hat{Y}_1, Y|X_2) - \delta_3. \tag{14}$$

We now consider $P(\mathcal{E}_3)$.

$$P(\mathcal{E}_3) \leq \sum_{\hat{w}\neq 1} \sum_{\hat{z}=\phi_{\text{CF}}^{-1}(\hat{s}),\hat{s}\neq 1} P\Big(\big(\mathbf{X}_{2,j+1}(\hat{s}), \mathbf{Y}_{j+1}\big) \in \mathcal{A}_\epsilon^{(n)} \cap \big(\mathbf{X}_{1,j}(\hat{w}), \mathbf{x}_{2,j}(1), \hat{\mathbf{Y}}_{1,j}(\hat{z}|1), \mathbf{Y}_j\big) \in \mathcal{A}_\epsilon^{(n)}\Big)$$

$$= \sum_{\hat{w}\neq 1} \sum_{\hat{z}=\phi_{\text{CF}}^{-1}(\hat{s}),\hat{s}\neq 1} P\Big(\big(\mathbf{X}_{2,j+1}(\hat{s}), \mathbf{Y}_{j+1}\big) \in \mathcal{A}_\epsilon^{(n)}\Big) \times P\Big(\big(\mathbf{X}_{1,j}(\hat{w}), \mathbf{x}_{2,j}(1), \hat{\mathbf{Y}}_{1,j}(\hat{z}|1), \mathbf{Y}_j\big) \in \mathcal{A}_\epsilon^{(n)}\Big) \quad (15)$$

$$= \sum_{\hat{w}\neq 1} \sum_{\hat{z}=\phi_{\text{CF}}^{-1}(\hat{s}),\hat{s}\neq 1} \sum_{(\mathbf{x}_2,\mathbf{y})\in\mathcal{A}_\epsilon^{(n)}} p(\mathbf{x}_2,\mathbf{y}) \times P\big(\mathbf{x}_2, \hat{\mathbf{Y}}_1, \mathbf{y}) \in \mathcal{A}_\epsilon^{(n)} | \mathbf{x}_2, \mathbf{y}\big) \times P\big((\mathbf{X}_1, \mathbf{x}_2, \hat{\mathbf{y}}_1, \mathbf{y}) \in \mathcal{A}_\epsilon^{(n)} | \mathbf{x}_2, \hat{\mathbf{y}}_1, \mathbf{y}\big) \times \sum_{\mathbf{y}} p(\mathbf{y}) \sum_{\mathbf{x}_2\in\mathcal{A}_\epsilon^{(n)}(\mathbf{X}_2|\mathbf{y})} P\big((\mathbf{X}_2, \mathbf{y}) \in \mathcal{A}_\epsilon^{(n)} | \mathbf{y}\big)$$

$$\leq 2^{n\left(R_{\text{CF-M}} + \hat{R} - I(\hat{Y}_1;Y|X_2) - I(X_1;\hat{Y}_1,Y|X_2) - I(X_2;Y) + \delta_3\right)},$$

where in writing in (15) we have used the memorylessness of the channel. The last inequality follows from the construction of the codebooks, the surjectivity of ϕ_{CF}, and the properties of jointly typical sequences. Thus, $P(\mathcal{E}_3) \to 0$ if $n \to \infty$ and

$$R_{\text{CF-M}} \leq I(\hat{Y}_1;Y|X_2) + I(X_1;\hat{Y}_1,Y|X_2) - \hat{R} + I(X_2;Y) - \delta_3. \quad (16)$$

Invoking (12), yields the following constraint.

$$R_{\text{CF-M}} \leq I(X_1,X_2;Y) - I(\hat{Y}_1;Y_1|X_1,X_2,Y) - \delta_4. \quad (17)$$

Next, we bound $P(\mathcal{E}_4)$. Let $\mathcal{S}''_{s=1} = \{z|\phi_{\text{CF}}(z) = 1\}$. Then,

$$P(\mathcal{E}_4) \leq \sum_{\hat{w}\neq 1} \sum_{\hat{z}\neq 1,\phi_{\text{CF}}(\hat{z})=1} \sum_{(\mathbf{x}_2,\mathbf{y})\in\mathcal{A}_\epsilon^{(n)}} p(\mathbf{x}_2,\mathbf{y}) \times P\big((\hat{\mathbf{Y}}_1, \mathbf{x}_2, \mathbf{y}) \in \mathcal{A}_\epsilon^{(n)} | \mathbf{x}_2, \mathbf{y}\big) \times P\big((\mathbf{X}_1, \mathbf{x}_2, \hat{\mathbf{y}}_1, \mathbf{y}) \in \mathcal{A}_\epsilon^{(n)} | \mathbf{x}_2, \hat{\mathbf{y}}_1, \mathbf{y}\big)$$

$$\leq 2^{nR_{\text{CF-M}}} |\mathcal{S}''_{s=1}| 2^{-n(I(\hat{Y}_1;Y|X_2) + I(X_1;\hat{Y}_1,Y|X_2) - \delta_5)}.$$

For $R_0 < \hat{R}$, $|\mathcal{S}''_{s=1}| \leq 2^{n\left(\hat{R} - R_0 + \delta_6\right)}$. Hence, $P(\mathcal{E}_4) \leq 2^{n\left(R_{\text{CF-M}} + \hat{R} - I(\hat{Y}_1;Y|X_2) - I(X_1;\hat{Y}_1,Y|X_2) - R_0 + \delta_7\right)}$, which implies that $P(\mathcal{E}_4) \to 0$ if $n \to \infty$ and

$$R_{\text{CF-M}} \leq I(\hat{Y}_1;Y|X_2) + I(X_1;\hat{Y}_1,Y|X_2) - \hat{R} + R_0 - \delta_7, \quad (18)$$

For $R_0 \geq \hat{R}$, $|\mathcal{S}''_{s=1}| \leq 2^{n\delta_6}$, and hence, $P(\mathcal{E}_4) \leq 2^{n\left(R_{\text{CF-M}} - I(X_1;Y|X_2) - I(\hat{Y}_1;X_1,Y|X_2) + \delta_7\right)}$, which implies that $P(\mathcal{E}_4) \to 0$ if $n \to \infty$ and

$$R_{\text{CF-M}} \leq I(\hat{Y}_1;Y|X_2) + I(X_1;\hat{Y}_1,Y|X_2) - \delta_7. \quad (19)$$

Now, we observe that the right hand side of (19) is greater than or equal to that of (14). This implies that, when $R_0 \geq \hat{R}$, $P(\mathcal{E}_4) \leq P(\mathcal{E}_2)$ and the constraint in (19) is redundant.

For the case of $R_0 < \hat{R}$, we compare (18) with (14). This comparison shows that if

$$R_0 \geq \hat{R} - I(\hat{Y}_1;Y|X_2) + \delta_8, \quad (20)$$

$P(\mathcal{E}_4) \leq P(\mathcal{E}_2)$ and the constraint in (18) is redundant.

Now, comparing the constraint in (16) pertaining to $P(\mathcal{E}_3)$ with the one in (18), it can be seen that when

$$R_0 \geq I(X_2;Y) + \delta_9, \quad (21)$$

$P(\mathcal{E}_4) \leq P(\mathcal{E}_3)$, and the constraint in (18) is redundant.

Since (20) and (21) imply that $P(\mathcal{E}_4) \leq \max\{P(\mathcal{E}_2), P(\mathcal{E}_3)\} \to 0$, it follows that choosing R_0 to satisfy (2) guarantees that $P(\mathcal{E}_4) \to 0$. Combining this with (14) and (17) yields the statement of Theorem 3.

Remark 5: Analogous error events can be defined for Step 2 of the QF decoding scheme in Section III. Since therein ϕ_{QF} is bijective, the error events in the QF decoding scheme can be bounded similarly. However, in that case $|\mathcal{S}''_{s=1}| = 1$. Combining (14) and (17) proves the achievability of the rate in Theorem 2 with successive decoding. $\square$

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