APPENDIX F

MAP PROJECTION COORDINATE SYSTEMS

F.1 INTRODUCTION

Most local surveyors are well served by using map projections such as the state plane coordinate system. These two-dimensional *grid* systems allow surveyors to perform accurate computations over large regions of land using plane surveying computations. They are the basis for the adjustments discussed in Chapters 14 through 16.

Map projections provide a one-to-one mathematical relationship with points on the ellipsoid and those on the mapping surface. There are an infinite number of map projections. Most map projections are defined by a series of mathematical transformations used to convert a point's geodetic coordinates of latitude, ϕ, and longitude, λ, to xy grid coordinates. Some map projections preserve the shape of objects (conformal); others, areas, directions, or distances of lines. However, since Earth is ellipsoidal in shape and a mapping surface is a plane, all map projections introduce some form of distortion to observations. For example, distances and areas are distorted in a conformal map projection.

To reduce the size of these distortions, the developable surface is often made secant to the ellipsoid and the width of the mapping zone is limited in distance. For instance, when the National Geodetic Survey originally designed the state plane coordinate system during the 1930s, the zone widths were limited to 158 miles so that precision between the ellipsoid distance and the grid distance was no worse than 1:10,000. Since most surveys at that time were only accurate to a precision of 1:5000, this was an acceptable limit. However, with today's modern instruments, observations must be reduced properly if survey accuracy is to be preserved in a map projection system.

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All map projections are based on the ellipsoid selected, such as the Geodetic Reference System of 1980 (GRS 80), and defining zone parameters. Typically, the zone parameters define the grid origin (ϕ_0, λ_0); the secant lines of the projection, also known as *standard parallels,* or scale factor, k_0, at the central meridian, λ_0; and the offset distances (E_0, N_b) from the grid origin. Once defined, each map projection has a series of zone constants that are computed using the defining zone parameters. These zone constants are computed only once for each projection. Once the zone constants are computed, the direct and inverse problems can be carried out for any point in the system. The direct problem takes the geodetic coordinates of a point and transforms them into grid coordinates, and the inverse problem takes the grid coordinates of a point and transforms them into geodetic coordinates.

The two primary map projection systems used in the United States are the Lambert Conformal Conic for states that have a long east–west extent and the Transverse Mercator for states that have a long north–south extent. Both map projections are conformal; that is, they preserve angles in infinitesimally small regions about a point. This property is advantageous to surveyors since angles are minimally distorted when using a conformal projection. On the other hand, as shown in Figure F.1, horizontal distances observed must be reduced to the mapping surface to eliminate the distortions of the projection. However, if these reductions are performed properly, the resulting plane computations are as accurate as geodetic computations such as those shown in Chapter 23. In this appendix we look at the mathematics of the Lambert Conformal Conic and Transverse Mercator map projections and demonstrate proper methods in reducing observations before an adjustment.

F.2 MATHEMATICS OF THE LAMBERT CONFORMAL CONIC MAP PROJECTION

The Lambert Conformal Conic map projection was introduced by Johann Lambert in 1772. As its name implies, this map projection uses a cone as its developable surface. The projection is conformal, so angles are preserved but distances are distorted. A Lambert Conformal Conic map projection is defined by two ellipsoidal parameters,[1] grid origin (ϕ_0, λ_0); latitude of the north standard parallel, ϕ_N, and south standard parallel,[2] ϕ_S; false easting, E_0; and false northing, N_b.

[1] Typically, an ellipsoid is defined by the length of its semimajor axis, a, and its flattening factor, f. The first eccentricity is computed as $e = \sqrt{2f - f^2}$. The GRS 80 ellipsoid has defining parameters of a = 6,378,137.0 m and f = 1/298.2572221008.

[2] The standard parallels are the latitudes of the north and south secant lines for the cone on the ellipsoid.

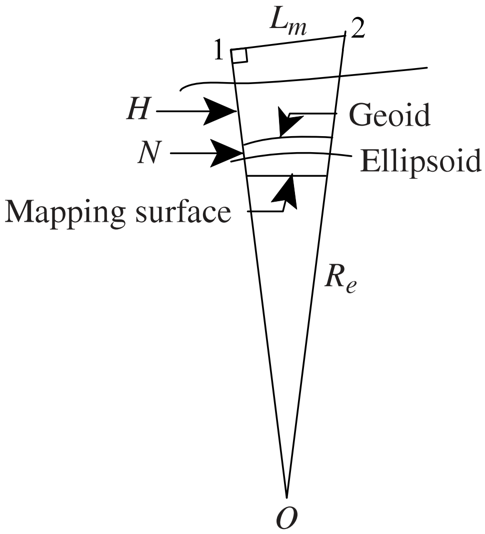

Figure F.1 Reduction of distance to a mapping surface.

F.2.1 Zone Constants

A set of three functions is used repeatedly in computations of the Lambert Conformal Conic map projection:

$$W(\phi) = \sqrt{1 - e^2 \sin^2 \phi} \tag{F.1}$$

$$M(\phi) = \frac{\cos \phi}{W(\phi)} \tag{F.2}$$

$$T(\phi) = \sqrt{\frac{1 - \sin \phi}{1 + \sin \phi}\left(\frac{1 + e \sin \phi}{1 - e \sin \phi}\right)^e} \tag{F.3}$$

Using Equations (F.1) through (F.3), the remaining zone constants are defined as

$$w_1 = W(\phi_S) \tag{F.4}$$

$$w_2 = W(\phi_N) \tag{F.5}$$

$$m_1 = M(\phi_S) \tag{F.6}$$

$$m_2 = M(\phi_N) \tag{F.7}$$

$$t_0 = T(\phi_0) \tag{F.8}$$

$$t_1 = T(\phi_S) \tag{F.9}$$

$$t_2 = T(\phi_N) \tag{F.10}$$

$$n = \sin \phi_0 = \frac{\ln m_1 - \ln m_2}{\ln t_1 - \ln t_2} \tag{F.11}$$

$$F = \frac{m_1}{nt_1^n} \tag{F.12}$$

$$R_b = aFt_0^n = \text{radius of the projection} \tag{F.13}$$

F.2.2 Direct Problem

The direct problem takes the geodetic coordinates of latitude, ϕ, and longitude, λ, of a point and transforms them into xy grid coordinates. Often, the y coordinate is called the point's *northing, N,* and the x coordinate its *easting, E*. Thus, given the geodetic coordinates of a point, the northing, y, easting, x, scale factor, k, and convergence angle, γ, of the point are computed as

$$t = T(\phi) \tag{F.14}$$

$$m = M(\phi) \tag{F.15}$$

$$R = aFt^n \tag{F.16}$$

$$\gamma = (\lambda - \lambda_0)n \quad \text{(where western longitude is considered negative)} \tag{F.17}$$

$$E = R \sin \gamma + E_0 \tag{F.18}$$

$$N = R_b - R \cos \gamma + N_b \tag{F.19}$$

$$k = \frac{Rn}{am} \tag{F.20}$$

F.2.3 Inverse Problem

The inverse problem takes a point's northing and easting coordinates and computes its latitude, longitude, scale factor, and convergence angle. For the Lambert Conformal Conic map projection, the equations for the inverse problem are

$$E' = E - E_0 \tag{F.21}$$

$$N' = R_b - (N - N_b) \tag{F.22}$$

$$R = \sqrt{E'^2 + N'^2} \tag{F.23}$$

$$t = \left(\frac{R}{aF}\right)^{1/n} \tag{F.24}$$

$$\gamma = \tan^{-1}\frac{E'}{N'} \tag{F.25}$$

$$\chi = 90° - \tan^{-1}t \tag{F.26}$$

$$\phi = 90° - 2\tan^{-1}\left[t\left(\frac{1 - e\sin\phi}{1 + \sin\phi}\right)^{e/2}\right] \tag{F.27}$$

Repeat Equation (F.27) using χ for ϕ in the first iteration. Iterate until the change in ϕ is insignificant; that is, the change should be less than 0.000005″.

$$\lambda = \frac{\gamma}{n} + \lambda_0 \tag{F.28}$$

$$k = \frac{m_1 t^n}{m t_1^n} \tag{F.29}$$

where m and t are defined in Equations (F.14) and (F.15) using ϕ from Equation (F.27).

F.3 MATHEMATICS OF THE TRANSVERSE MERCATOR

The Transverse Mercator map projection uses a cylinder as its developable surface. It preserves scale in a north–south direction and thus is good for regions with a long north–south extent. This projection was proposed by Johann Lambert, but the mathematics for an ellipsoid were not solved until the early twentieth century. In many countries, this projection is also known as the Gauss–Krüger map projection. The most famous Transverse Mercator map projection is the Universal Transverse Mercator (UTM) developed by the National Geospatial-Information Agency to provide a worldwide mapping system from 80° south latitude to 80° north latitude. This map projection is defined by two ellipsoidal parameters,[3] grid origin (ϕ_0,λ_0), scale factor, k_0, at the central meridian, λ_0, false easting, E_0, and false northing, N_b.

There are 60 zones in the Universal Transverse Mercator map projection, each nominally 6° wide. Each zone overlaps its neighboring zones by 30′. The central meridian, λ_0, for each zone is assigned a false easting, E_0, of 500,000 m. The false northing, N_b, is 0.000 m in the northern hemisphere and 10,000,000.000 m in the southern hemisphere. The scale factor at the central meridian, k_0, is 0.9996, which yields a distance precision of 1:2500. The central meridians (λ_0) for each zone start at 177° west longitude and with a few exceptions, proceeds easterly by 6° for each subsequent zone. The grid origins are at 0° and λ_0.

F.3.1 Zone Constants

The Transverse Mercator map projection use the following defining functions:

$$C(\phi) = e'^2 \cos^2\phi \tag{F.30}$$

$$T(\phi) = \tan \phi \tag{F.31}$$

$$\begin{aligned} M(\phi) = a\Bigg[&\left(1 - \frac{e^2}{4} - \frac{3e^4}{64} - \frac{5e^6}{256}\right)\phi - \left(\frac{3e^2}{8} + \frac{3e^4}{32} + \frac{45e^6}{1024}\right)\sin 2\phi \\ &+ \left(\frac{15e^4}{256} + \frac{45e^6}{1024}\right)\sin 4\phi - \left(\frac{35e^6}{3072}\right)\sin 6\phi\Bigg] \end{aligned} \tag{F.32}$$

where e is the first eccentricity of the ellipse as defined in Equation (17.5) and e' is defined as

$$\begin{aligned} b &= a(1 - f) \\ e' &= \frac{\sqrt{a^2 - b^2}}{b} = \sqrt{\frac{1 - e^2}{e^2}} \end{aligned} \tag{F.33}$$

$$m_0 = M(\phi_0) \tag{F.34}$$

[3]The Universal Transverse Mercator (UTM) uses the WGS 84 ellipsoid, defined in Chapter 17.

F.3.2 Direct Problem

The equations in the Transverse Mercator for the direct problem are

$$m = M(\phi) \tag{F.35}$$

$$t = T(\phi) \tag{F.36}$$

$$c = C(\phi) \tag{F.37}$$

$$A = (\lambda - \lambda_0)\cos\phi \quad \text{where western longitudes are negative} \tag{F.38}$$

$$\begin{aligned} E = k_0 R_N \bigg[& A + (1 - t + c)\frac{A^3}{6} \\ & + (5 - 18t + t^2 + 72c - 58e'^2)\frac{A^5}{120}\bigg] + E_0 \end{aligned} \tag{F.39}$$

$$\begin{aligned} N = k_0 \bigg\{ & m - m_0 + R_N \tan\phi \bigg[\frac{A^2}{2} + (5 - t + 9c + 4c^2)\frac{A^4}{24} \\ & + (61 - 58t + t^2 + 600c - 300e'^2)\frac{A^6}{720}\bigg]\bigg\} + N_b \end{aligned} \tag{F.40}$$

where R_N is the radius in the prime vertical as defined by N in Equation (17.6).

$$c_2 = \frac{1 + 3c + 2c^2}{3} \qquad c_3 = \frac{2 - \tan^2\phi}{15} \tag{F.41}$$

$$\gamma = A\tan\phi[1 + A^2(c_2 + c_3A^2)]$$

$$\begin{aligned} k = k_0 \bigg[& 1 + (1 + c)\frac{A^2}{2} + (5 - 4t + 42c + 13c^2 - 23e'^2)\frac{A^4}{24} \\ & + (61 - 148t + 16t^2)\frac{A^6}{720}\bigg] \end{aligned} \tag{F.42}$$

F.3.3 Inverse Problem

The equations in the Transverse Mercator for the inverse problem are

$$E' = E - E_0 \tag{F.43}$$

$$N' = N - N_b \tag{F.44}$$

$$e_1 = \frac{1 - \sqrt{1 - e^2}}{1 + \sqrt{1 - e^2}} \tag{F.45}$$

$$m = m_0 + \frac{N'}{k_0} \tag{F.46}$$

$$\chi = \frac{m}{a\left(1 - e^2/4 - 3e^4/64 - 5e^6/256\right)} \tag{F.47}$$

The *foot-point latitude* is

$$\begin{aligned}\phi_f = \chi &+ \left(\frac{3e_1}{2} - \frac{27e_1^3}{32}\right) \sin 2\chi + \left(\frac{21e_1^2}{16} - \frac{55e_1^4}{32}\right) \sin 4\chi \\ &+ \frac{151e_1^3}{96} \sin 6\chi + \frac{1097e_1^4}{512} \sin 8\chi\end{aligned} \tag{F.48}$$

Using the foot-point latitude and functions defined in Section F.3.1 and Equation (23.16) yields

$$c_1 = C(\phi_f) \tag{F.49}$$

$$t_1 = T(\phi_f) \tag{F.50}$$

$$N_1 = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi_f}} \tag{F.51}$$

$$M_1 = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi_f)^{3/2}} \tag{F.52}$$

$$D = \frac{E'}{N_1 k_0} \tag{F.53}$$

$$\begin{aligned}B = \frac{D^2}{2} &- (5 + 3t_1 + 10c_1 - 4c_1^2 - 9e')\,\frac{D^4}{24} \\ &+ (61 + 90t_1 + 298c_1 + 454t_1^2 - 252e' - 3c_1^2)\,\frac{D^6}{720}\end{aligned} \tag{F.54}$$

$$\phi = \phi_f - \frac{N_1 \tan \phi_f}{M_1} B \tag{F.55}$$

$$\lambda = \lambda_0 + \frac{D - (1 - 2t_1 + c_1)(D^3/6) + (5 - 2c_1 + 28t_1 - 3c_1^2 + 8e'^2 + 24t_1^2)(D^5/120)}{\cos \phi_f} \tag{F.56}$$

Note that Equations (F.41) and (F.42) can be used to compute the convergence angle γ and scale factor k for the point.

F.4 REDUCTION OF OBSERVATIONS

Most often, the grid coordinates of a point are known prior to the survey and all that is needed is to reduce the observations to the mapping surface. The basic principle to bear in mind is that *grid computations should only be performed with grid observations*. Since the two map projections discussed previously are conformal, observed distances must be reduced to the mapping surface. Similarly, geodetic and astronomical directions must be converted to their grid equivalents.

As discussed in this section, conformality implies that the angles will be only slightly distorted. As will be shown, the *arc-to-chord* correction is applied directions and angles when the sight distances are long. For example, in the state plane coordinate system, this correction should be considered for angles whose sight distances are greater than 8 km. In this section, proper reduction of distance, direction, and angle observations is discussed.

F.4.1 Reduction of Distances

As shown in Figure F.1, an observed horizontal distance must be reduced to the mapping surface. This reduction usually involves using the *grid factor*. The grid factor is the product of the *elevation factor,* which reduces the observed distance to the ellipsoid, and a scale factor (k), which reduces the ellipsoidal distance to the mapping surface.

There are several procedures for reducing an observed distance to the ellipsoid, the most precise being a geodetic reduction. However, surveyed lengths typically contain only five or six significant figures. Thus, less strict methods can be applied to these short lengths. The elevation factor is computed as

$$\text{EF} = \frac{R_e}{R_e + H + N} = \frac{R_e}{R_e + h} \tag{F.57}$$

In Equation (F.57), R_e is the radius of the Earth, H the orthometric height, N the geoidal height, and h the geodetic height. All of these parameters are determined at the observation station. The relationship between the geodetic height, h, and orthometric height, H, is

$$h = H + N \tag{F.58}$$

In Equation (F.57), the radius in the azimuth of the line should be used for R_e. Again since surveyors observe short distances typically, an average radius of the Earth of 6,371,000 m can be used in computing EF. These approximations are demonstrated in Example F.1.

In a map projection system, the scale factor computed using Equation (F.20), (F.29), or (F.42) is for a point. Generally, the scale factor changes continuously along the length of the line. Thus, a weighted mean using two endpoints of the line (k_1 and k_2) and midpoint (k_m) is a logical choice for computing a single scale factor for a line. It can be computed as

$$k_{\text{avg}} = \frac{k_1 + 4k_m + k_2}{6} \tag{F.59}$$

However, as with the elevation factor, this type of precision is seldom needed for the typical survey. Thus, the mean of the two endpoint scale factors is generally of sufficient accuracy for most surveys. In fact, it is not uncommon to use a single mean scale factor for an entire project.

The grid factor, GF, for the line is a product of the elevation factor, EF, and a scale factor, k_{avg}, and is computed as

$$\text{GF} = k_{\text{avg}} \times \text{EF} \tag{F.60}$$

Thus, a reduced grid distance, L_{grid}, is the product of the horizontal distance, L_m, and the grid factor, GF, and is computed as

$$L_{\text{grid}} = L_m \times \text{GF} \tag{F.61}$$

Example F.1 A distance of 536.07 ft is observed from station 1. The scale factors at observing, midpoint, and sighted stations are 0.9999587785, 0.9999587556, and 0.9999587328, respectively. The orthometric height at observing station is 1236.45 ft. Its geoidal height is −30.12 m and the radius in the azimuth is 6,366,977.077 m. Determine the length of the line on the mapping surface.

SOLUTION This solution will compare the grid factor computed using different radii in Equation (F.57) and different scale factors in Equation (F.60). Using the more precise methods, the grid factor is computed as follows. The orthometric height of the observing station in meters is

$$H = 1236.45 \text{ ft} \times \frac{12}{39.37} = 376.871 \text{ m}$$

Using the radius in the azimuth of the line and Equation (F.57), the elevation factor, EF, is

$$\text{EF} = \frac{6{,}366{,}977.077}{6{,}366{,}977.077 + 376.871 - 30.12} = 0.999945542$$

From Equation (F.59), the scale factor for the lines is

$$k_{\text{avg}} = \frac{0.9999587785 + 4(0.9999587556) + 0.9999587328}{6} = 0.999958756$$

From Equation (F.60), the grid factor for the line is

$$\text{GF} = 0.999945542 \times 0.999958756 = 0.99990430$$

Finally, the grid distance for this line is

$$L_{\text{grid}} = 0.99990430 \times 536.07 \text{ ft} = 536.02 \text{ ft}$$

Doing the problem again, this time with the mean radius of the Earth and the average of the two endpoint scale factors, yields

$$\text{EF} = \frac{6{,}371{,}000}{6{,}371{,}000 + 376.871 - 30.12} = 0.999945577$$

$$k_{\text{avg}} = \frac{0.9999587785 + 0.9999587328}{2} = 0.999958756$$

$$\text{GF} = 0.999945577 \times 0.999958756 = 0.99990433$$

$$L_{\text{grid}} = 0.99990433 \times 536.07 \text{ ft} = 536.02 \text{ ft}$$

Note that using the approximate radius of the Earth and the average scale factor for the endpoints of the line resulted in the same solution as the more precise computations. This is because the length of the distance observed has only five significant figures. The elevation factor computed using the mean radius of the Earth agreed with the radius in the azimuth to seven decimal places. This is also true of the scale factors, which agreed to nine significant figures. Thus, the grid factor was the same to seven decimal places and was well beyond the accuracy needed to convert a length with only five significant figures. This demonstrates why a common grid factor can often be used for an entire project that covers a small region.

F.4.2 Reduction of Geodetic Azimuths

Figure F.2 depicts the differences between geodetic azimuths, T, and grid azimuths, t. Since grid north (GN) at a point is parallel to the central meridian, the convergence angle, γ, is the largest correction between the two geodetic and grid azimuths. Additionally, there is a small correction to convert the arc on an ellipsoid to its equivalent chord on the mapping surface. This is known as the *arc-to-chord correction,* δ. The relationship between the geodetic azimuth and grid azimuth can be derived from Figure F.2 as

$$T = t + \gamma - \delta \tag{F.62}$$

As shown in Figure F.2, this equation works whether the line is east or west of the central meridian. For the Lambert Conformal Conic map projection, the arc-to-chord correction is computed as

$$\delta = 0.5(\sin \phi_3 - \sin \phi_0)(\lambda_2 - \lambda_1) \tag{F.63}$$

An analysis of Equation (F.63) shows that the worst cases for δ are for lines in the northern or southern extent of a map projection. Rearranging Equation (F.63) yields a change in longitude as

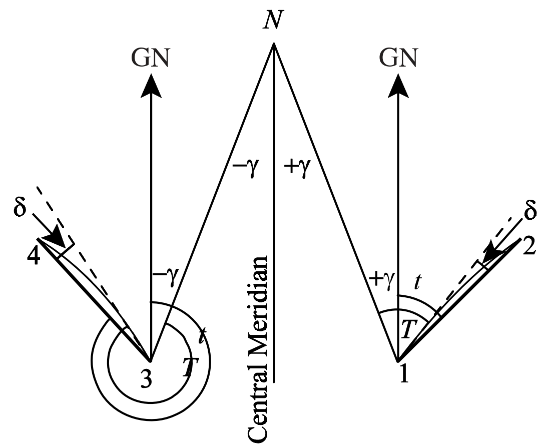

Figure F.2 Relationship of geodetic azimuth (T), grid azimuth (t), convergence angle (γ), and arc-to-chord correction (δ).

$$\lambda_2 - \lambda_1 = \Delta\lambda = \frac{2\delta}{\sin\phi_3 - \sin\phi_0} \tag{F.64}$$

As an example, assume that ϕ_3 is 42°30′. Further assume that the project is in the Pennsylvania North Zone, which has a sin ϕ_0 of 0.661539733812. If δ is to be kept below 0.5″, the maximum line in arc-seconds of longitude can be

$$\Delta\lambda = \frac{2(0.5'')}{\sin 42°30' - 0.661539733812} = 71.2''$$

At latitude 42°30′, this corresponds to a line of length of about 5334 ft, or 1.6 km. Few surveyors in northeastern Pennsylvania could find a line of this length to observe. Thus, the arc-to-chord correction is generally ignored in reductions, and Equation (F.62) can be simplified as

$$T = t + \gamma \tag{F.65}$$