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Detecting and Labelling Wireless Community Network Structures from Eigen-spectra

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Abstract

Wireless and *ad hoc* networks often have local regions of highly intra-connected modules or communities of nodes against a backdrop of sparser longer-range inter-community connectivities. We report on empirical observations and a prototype algorithm for correctly counting and labelling the number of dense community structures or modules within a fully connected network. We describe the eigen-spectral method applied to the Laplacian characteristic matrix. We illustrate the effect of different network characteristics on the eigen-spectrum and on properties of the eigenvectors corresponding to the N_c smallest eigenvalues in networks with N_c such dense community modules.

Keywords: wireless networks; ad hoc networks; community structure; modules; eigen-spectra; smallest eigen-values.

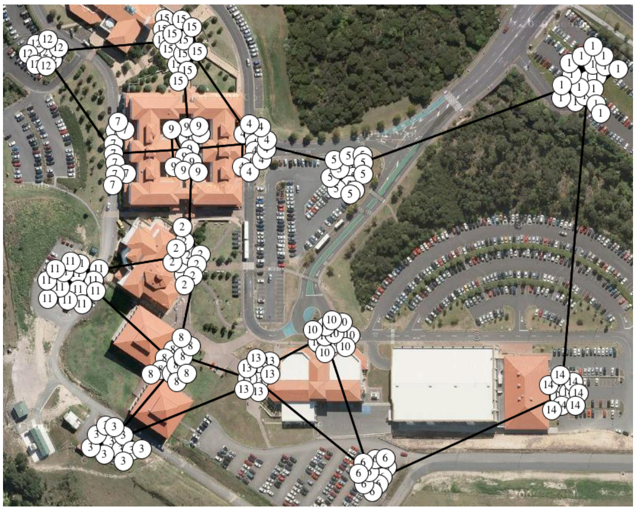

Figure 1: Spatial deployment pattern of connected devices on Massey's Albany campus, divided into 15 modular communities.

1 Introduction

Wireless networks are often highly complex systems [1,2] that are difficult to analyse for optimal design. Important considerations for the design of mobile, wireless networks [3] include: maintaining connectivity when nodes move; deploying resources only where required [4]; and managing power consumption [5]. Many *ad hoc* and mobile wireless device deployments give rise to highly structured connectivity patterns based upon localised regions or groups of units that are highly intra-connected amongst nodes of the same group, but where the groups themselves may be only sparsely interconnected. Such networks are fully connected in the sense that there are path-ways from any node to any other node, but have variations in connectivity strength such that nodes form tightly connected communities or modules with sparser "trunk routes" connecting communities.

This sort of community or modular structure within a fully-connected graph [6] is common in a variety of application contexts including: the Internet and World-Wide-Web; social and battle deployment situations; and also in gene-sequenced and neuronal biological networks. Mobile and wireless networks are particularly interesting however as we can easily visualise a spatial pattern [7] amongst the nodes of the network that is separate from the actual links or edges connecting nodes together. The actual bandwidth or number of packet hops can therefore be embedded in a spatial coordinate system but is completely separate from it in terms of its connectivity properties. Figure 1 shows a possible spatial deployment pattern of wireless-equipped “units” on Massey University’s Albany campus. Units are tightly clustered and have a high local connectivity but only the group leaders of each group are directly connected. The graph is therefore fully connected but highly modular.

Automatically detecting community or modular structure in networks is a non-trivial problem. In the case of networks or graphs that are **not** fully connected, this can be detected using graph colouring or node labelling algorithms and the answers are unambiguous. In the case of fully connected graphs but with a modular structure the identification of the modules or communities can be a matter of judgement and is not necessarily completely unambiguous. Generally community detection algorithms either follow a partitioning approach to divide the graph into predefined sizes so as to minimize the number of links between divisions, or alternatively they employ a hierarchical clustering or spectral approach [8] to iteratively cut up the graph into a number of “cut-sets.”

A number of authors have proposed algorithms based on spectral detection [9] using various network properties and based on communities; anti-communities [10] and other modular properties [11,12]. Spectral method s make use of numerical properties of some characteristic matrix that describes the network in question. Usually this entails computing the eigen-values and eigenvectors of this characteristic matrix and analysing the resulting spectrum or density of values obtained [13]. This general approach gives rise to several useful results concerning network connectivity and bulk properties [14] and potential critical phenomena concerning connectivity break-up in complex networks [15].

In this paper we report some empirical observations on using spectral analysis of highly modular networks of a type that might arise from deployment of wireless devices or troops in a spatial pattern, especially when the network connectivity is evolving [16] in time. Our observations can be used to formulate a modular detection and labelling algorithm that is quite robust against different network details and which can form the basis for analysing modular structure in simulated wireless networks [17–19] as well in real network patterns [20].

In this paper we focus solely on undirected graphs, where the nodes are connected with bidirectional edges, rather than directed arcs [21]. This focus gives rise to characteristic matrices that are symmetric and simplifies the analysis considerably. Some work has been done using the complex eigenvalues that arise from asymmetric characteristic matrices [22] and this may be useful for some application network contexts in biological and chemical systems particularly where oscillatory [23] and network synchronisation phenomena [24] are involved.

We consider the Laplacian matrix of networks [25] and investigate it using dense linear algebra methods which give high precision for the relevant eigenvalues but which are computationally intensive and this feasible only for small- to medium-sized network systems. Sparse matrix techniques are required to scale-up to investigate very large networks [26].

We describe spectral methods for community structure detection and labelling in Section 2 and give specifics on: the Laplacian characteristic matrix in Section 2.1; the detection and labelling algorithm in Section 2.2 and an illustrative example in Section 2.3. Some other test networks with various properties are investigated in Section 3 and a discussion and some general observations given in Section 4 with a summary and tentative conclusions and areas for further work offered in Section 5.

2 Eigen Spectral Methods

We want to find an appropriate cut-set or set of cut-sets to partition the network up according to its communities. This requires a partition measure R_M based on some characteristic matrix M that defines the network of N nodes, and their connectivities. Following Newman and others [9,10], we define an index vector $\mathbf{s}$ so that $\mathbf{s}_i$ is ± 1 depending upon whether node i of the graph is in group 1 or group 2.

The partitioning measure on specification matrix M can then be written in the form:

$$R_M = s^T M s \tag{1}$$

using the usual transpose s^T notation to denote a row-vector in contra-distinction to a column-vector s. We need to maximise the partitioning measure subject to the constraint on $\mathbf{s}$ so that $\mathbf{s}^T\mathbf{s} = N$ and can therefore employ the Lagrange multiplier method [27]. This can be formulated as:

$$\Lambda = \mathbf{s}^T M \mathbf{s} - \lambda \mathbf{s}^T \mathbf{s} \tag{2}$$

with $\frac{d\Lambda}{d\mathbf{s}^T} = 0$, from which we obtain:

$$(M - \lambda)\mathbf{s} = 0 \tag{3}$$

and implying thats should be an eigenvector of M with corresponding eigenvalue λ

The eigenvalues λ_i and corresponding eigenvectors v_i for a matrix M are defined in the usual way by:

$$M.\mathbf{v}_i = \lambda_i \mathbf{v}_i \tag{4}$$

For the real symmetric matrices we discuss, the eigenvalues and vectors are all real. The main property we exploit in analysing a matrix that is associated with a graph or network is that of orthogonality. If we find the eigenvalues and their corresponding eigenvectors, we hope to use these to indicate separate (orthogonal) properties of different parts of the graph or network.

Generally the normalisation is such that the eigenvectors are not necessarily equal to ±1 and work reported in the literature suggests taking $\mathbf{s}$ in terms of the actual eigenvectors as the signum function of the actual eigenvectors $\mathbf{s}^*$.

Newman and Zarei *et al.* both suggest using the Laplacian matrix L for M and looking at the eigenvector corresponding to the smallest eigenvalue. The argument is that this eigenvector will be most parallel to the true partition vector s. The general approach is to then use the sign of the individual values to determine whether node i is in group 1 or group 2 of the graph cut-sets. Various authors report some success with this method but do point out that it is is susceptible to error if the graph partition is not fully determinable and also that it requires knowledge that there e are in fact two and only two groups - of roughly the same size. In principle if a group is known to contain further r possible partitions, then it can be repeatedly applied to diminishing sub networks. It is not obvious how to terminate the repetition however without *a priori* knowledge of the number of groups resent. Other more complicated matrix choices have been reported in the literature to attempt to identify multiple groupings.

2.1 Laplacian Matrix of a Graph

Eigen spectral analysis of networks consists of studying properties of a characteristic matrix M that captures the essential properties of the graph or network. Common choices include the adjacency matrix A, the Laplacian $L = D - A$ or the sign-less Laplacian $|L| = D + A$ based upon the diagonal degree matrix D.

The Laplacian or admittance matrix is defined for undirected networks as:

$$L_{i,j} = k_i \delta_{i,j} - A_{i,j} \tag{5}$$

where k_i is the (edge) connectivity and $A_{i,j}$ the (symmetric) adjacency matrix.

In this work we use the Laplacian matrix for undirected (edge based) networks so that L is wholly real and is symmetric.

Figure 2 shows a small exemplar undirected network of 24 vertices arranged in 3 closely coupled communities. The corresponding Laplacian matrix is constructed from the diagonal degree matrix D and the adjacency matrix A and is shown in Figure 3

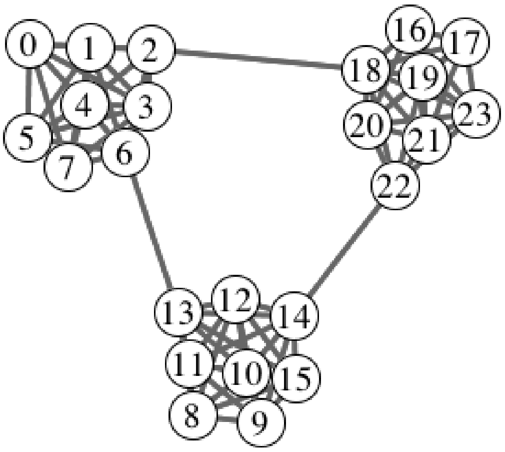

Figure 2: Undirected Graph of 24 Vertices in 3 closely-coupled communities.

```
  6 -1 -1 -1 -1 -1  0 -1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0      0
 -1  7 -1 -1 -1 -1 -1 -1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0      0
 -1 -1  6 -1 -1  0 -1  0  0  0  0  0  0  0  0  0  0  0 -1  0  0  0  0  0      0
 -1 -1 -1  7 -1 -1 -1 -1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0      0
 -1 -1 -1 -1  7 -1 -1 -1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0      0
 -1 -1  0 -1 -1  6 -1 -1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0      0
  0 -1 -1 -1 -1 -1  7 -1  0  0  0  0  0 -1  0  0  0  0  0  0  0  0  0  0      0
 -1 -1  0 -1 -1 -1 -1  6  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0      0
  0  0  0  0  0  0  0  0  6 -1 -1 -1 -1 -1  0 -1  0  0  0  0  0  0  0  0      0
  0  0  0  0  0  0  0  0 -1  6 -1 -1 -1  0 -1 -1  0  0  0  0  0  0  0  0      0
  0  0  0  0  0  0  0  0 -1 -1  7 -1 -1 -1 -1 -1  0  0  0  0  0  0  0  0      0
  0  0  0  0  0  0  0  0 -1 -1 -1  7 -1 -1 -1 -1  0  0  0  0  0  0  0  0      0
  0  0  0  0  0  0  0  0 -1 -1 -1 -1  7 -1 -1 -1  0  0  0  0  0  0  0  0      0
  0  0  0  0  0  0 -1  0 -1  0 -1 -1 -1  7 -1 -1  0  0  0  0  0  0  0  0      0
  0  0  0  0  0  0  0  0  0 -1 -1 -1 -1 -1  7 -1  0  0  0  0  0  0 -1  0      0
  0  0  0  0  0  0  0  0 -1 -1 -1 -1 -1 -1 -1  7  0  0  0  0  0  0  0  0      0
  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  6 -1 -1 -1 -1 -1  0 -1      0
  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 -1  6 -1 -1 -1 -1  0 -1      0
  0  0 -1  0  0  0  0  0  0  0  0  0  0  0  0  0 -1 -1  8 -1 -1 -1 -1 -1      0
  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 -1 -1 -1  7 -1 -1 -1 -1      0
  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 -1 -1 -1 -1  7 -1 -1 -1      0
  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 -1 -1 -1 -1 -1  7 -1 -1      0
  0  0  0  0  0  0  0  0  0  0  0  0  0  0 -1  0  0  0 -1 -1 -1 -1  6 -1      0
  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 -1 -1 -1 -1 -1 -1 -1  7      0

  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
```

Figure 3: Laplacian matrix for the 24 vertex cluster shown in Figure 2. The diagonal elements are the vertices' degree values and the off-diagonal elements come from the negative of the adjacency matrix. This network is undirected so the matrix is symmetric. The block structure arising from the three communities or modules can be seen since vertices have been deliberately contiguously indexed.

The Laplacian matrix has zero column and row sums – which fact is emphasised in Figure 3. This particular example has been arranged so that the three communities of vertices present are arranged contiguously in the matrix index space. The partitioning algorithm works with arbitrary matrix indices however and thus obviously does not require this arrangement to work correctly.

2.2 A Practical Algorithm

We find empirically that in the case of graphs of the form discussed in 1 above with very strong local connectivity, we can in fact extend this idea to examine the n smallest eigenvalues of the Laplacian matrix $L = M$ for which all elements are definitely positive and which are mutually orthogonal. We find that the following algorithm determines the number of unambiguous cut-set groups and can specifically label nodes by their group. In the cases of a mix of strongly defined and weakly defined cut-set groups we can use this method to at least count and label those groups and nodes that **are** unambiguous and leave the remaining nodes aside. The conventional method reported in the literature can then still be applied to the $n + 1$'th smallest eigenvalue's eigenvector and may be able to roughly split the remaining graph nodes into two further groups.

This is pragmatically a very useful and powerful result and appears to provide a robust algorithm as shown from the results presented in section 3.

2.3 An Illustrative Example

We show a specific example for the graph shown in figure 2 and give the sorted eigenvectors of the graph's Laplacian matrix in figure 4. The orthogonality of the three groupings is clearly seen and can be identified using the algorithm described above.

18	19	20	21	22	23
-0.1258	-0.1449	0.0000	0.0000	0.3333	0.0000
-0.1125	-0.1296	0.0000	0.0000	0.2981	0.0000
-0.1609	-0.1853	0.0000	0.0000	0.4262	0.0000
-0.1694	-0.1951	0.0000	0.0000	0.4487	0.0000
-0.1147	-0.1321	0.0000	0.0000	0.3037	0.0000
-0.1187	0.9340	0.0000	0.0000	0.3143	0.0000
-0.1202	0.0000	0.0000	0.0000	0.3184	0.0000
0.9355	0.0000	0.0000	0.0000	0.3532	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.3376
0.0000	0.0000	0.0000	0.0000	0.0000	0.3376
0.0000	0.0000	0.0000	0.0000	0.0000	0.3898
0.0000	0.0000	0.0000	0.0000	0.0000	0.4552
0.0000	0.0000	0.0000	0.0000	0.0000	0.3066
0.0000	0.0000	0.0000	0.0000	0.0000	0.2991
0.0000	0.0000	0.0000	0.0000	0.0000	0.3186
0.0000	0.0000	0.0000	0.0000	0.0000	0.3576
0.0000	0.0000	-0.1469	0.3855	0.0000	0.0000
0.0000	0.0000	-0.1469	0.3855	0.0000	0.0000
0.0000	0.0000	-0.0845	0.2218	0.0000	0.0000
0.0000	0.0000	-0.1783	0.4678	0.0000	0.0000
0.0000	0.0000	-0.1184	0.3107	0.0000	0.0000
0.0000	0.0000	-0.1253	0.3287	0.0000	0.0000
0.0000	0.0000	0.9424	0.3114	0.0000	0.0000
0.0000	0.0000	0.0000	0.3651	0.0000	0.0000

Figure 4: The last 6 eigenvectors corresponding to the 6 smallest eigenvalues of the Laplacian matrix for the 24 vertex cluster shown above. Note the last three are mutually orthogonal, non-overlapping and exclusive.

Figure 5 shows the sorted eigenvalues of the 24 vertex graph. There is a very obvious gap between the values of the three smallest eigenvalues that correspond to the 3 communities found in the graph and the remaining eigenvalues. This gap can also be used to help identify the number of community structures to label which is then an *a priori* unknown.

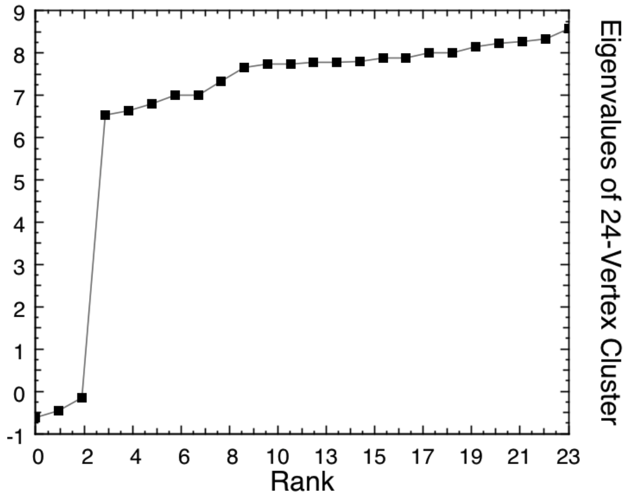

Figure 5: Ranked Eigenvalues for the 24 Vertex Network showing a gap before the last three smallest.

3 Results

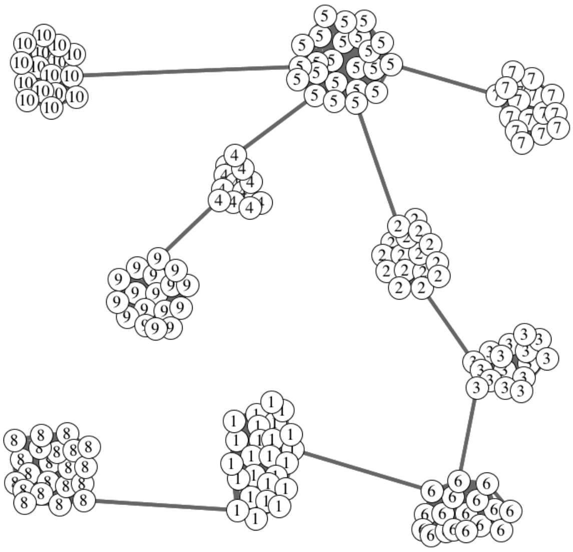

Figure 6: 179 nodes in 10 separate differently sized communities with trailing branches, correctly marked by the community partitioning algorithm.

A larger test data set with 179 vertices connected in a single cluster was used to investigate the scalability and robustness of the partitioning algorithm. In addition to deliberately muddling the located vertex nodes in the data structure, a range of different sized communities is used. A spatial structure was used to include trailing branches amongst communities. The algorithm correctly identifies ten communities of sizes: $25, 16, 16, 12, 23, 17, 16, 21, 17, 16$). A spatial all-all circular distance of 75 pixel units was used to generate this test set with long distance arcs added manually. The partitioning algorithm does not of course "see" the spatial distances, this is solely to aid the eye in presenting the test data set. Figure 6 shows the correctly marked ten communities of vertices.

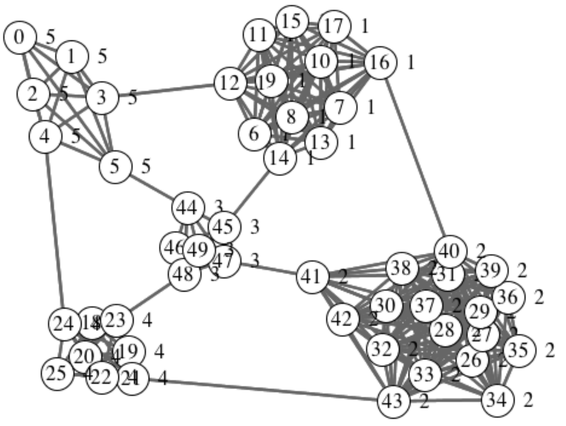

Figure 7: 50 nodes in a spoked-hub pattern of 5 separate communities.

In order to investigate how robust the algorithm was against multiple edges connecting communities an additional test set arranged in a hub and spoke pattern as shown in Figure 7. This data set was again correctly marked as having five communities of sizes $(12, 18, 6, 8, 6)$. This pattern is not unlike a command and control battlefield deployment pattern.

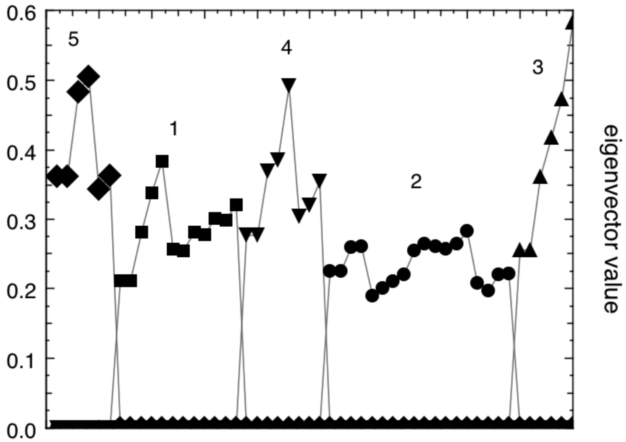

Figure 8: The smallest eigenvalue's eigenvectors, which are mutually orthogonal and label their member nodes.

It is interesting to look more closely how the eigenvectors of the network's Laplacian Matrix are partitioned. Figure 8 shows the five eigenvectors corresponding to the smallest eigenvalues in the spoked-hub network of fifty vertices. The nodes have been conveniently arranged in contiguous index order to emphasise the mutual orthogonally and lack of overlap amongst the eigenvectors. The plot symbols and annotations indicate how each of the five eigenvectors neatly span the space and this identify each community and its constituent vertices.

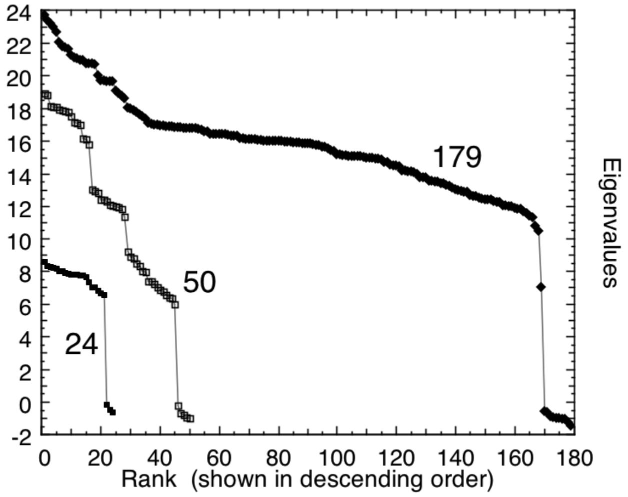

Figure 9: Eigenvalues for the three test networks, showing characteristic gap structure.

The data sets we have employed all share the common characteristic that they have densely-connected communities that are relatively sparsely interconnected. The network or graph is fully connected but the community or modular structure can still be unambiguously identified and different modules differentiated. The corresponding eigenvalue spectrum for each of the three test networks used is shown in Figure 9. Generally, in each case the number of communities N_c corresponds strongly to the smallest N_c eigenvalues. Furthermore there is a strong tendency for the eigen-spectrum to have sharply delineated gaps between these N_c eigenvalues and the rest of the spectrum.

4 Discussion

We have used a dense matrix formulation with double precision implementation of the Jacobi rotation method for obtaining the eigenvalues of a real symmetric matrix [28]. This can be coded in a number of ways, but we have used a C programming language implementation [29] alongwith a fast sorting method to identify the smallest eigenvalues. The Jacobi rotation method

Algorithm 1 Summary of modular network community detection and labelling algorithm.

```
 1: load adjacency matrix A for given network
 2: compute diagnonal degree matrix $D$
 3: compute Laplacian matrix $L = D - A$
 4: compute Eigenvalues $\lambda_i$ and Eigenvectors $\mathbf{s}_i$ of $L$
 5: sort $\lambda_i$ (and aassociated $\mathbf{s}_i$)
 6: starting with $i$ for the smallest $\lambda_i$ set $N_c = 0$
 7: while $s_{i,j} > 0, \forall j$ do
 8:     increment $N_c$
 9:     if $s_{i,j} > 0$ then
10:         label vertices $j$ by $N_c$
11:     end if
12: end while
```

satisfactorily computes the orthogonality of the lowest eigenvectors to sufficient precision for us to make simple comparison tests. Less precision would require use of a more sophisticated and expensive test criteria using dot product comparisons for orthogonality instead of direct comparisons tests against zero.

Algorithm 1 summarises the modular network community detection and labelling algorithm and is satisfactory providing the smallest eigenvalues of the Laplacian matrix can be computed to good precision and accuracy.

The Jacobi rotation method has a relatively high accuracy for the smallest eigenvalues but it is $\mathcal{O}(N^3)$ in computational complexity as well as being $\mathcal{O}(N^2)$ in memory usage. Other algorithms for computing eigenvalues of a sparse matrix such as the (iterated) Lanczos method [30]might be expected to give better computational performance as well as meomory utilisation, but may not yield sufficient accuracy for the small eigenvalues and individual elements of the corresponding eigenvectors. The test at line 7 of Algorithm 1 could be replaced by test based on a dot product to ensure continued orthogonality of vectors s_i as i is rolled up from the smallest eigenvalue indices.

5 Summary and Conclusions

We have described how an examination of the N_c smallest eigenvalues' corresponding eigenvectors that are entirely positive can be used to identify that there are N_c unambiguous cut-sets in a network and furthermore to identify and label the nodes belonging to each cut-set. We have shown that this algorithm is quite robust for graphs of the sort discussed. The algorithm works well for networks with highly dense community modules.

This algorithm can be applied to nodes in a fully connected *ad hoc* network in various contexts including those of wireless-sensor deployment, troop or other asset movements and other situations where it is strongly desirable or efficient resource management to designate local groupings or hubs of control or management.

This algorithm can likely be used in conjunction with others that have been reported in the literature to further analyse networks with a mix of both unambiguous (as reported here) and also more ambiguous groupings. The work reported here used a simple dense matrix method of determining the smallest eigenvalues. We are further investigating sparse methods and their relative accuracies to support use of this algorithm with very large networks and their Laplacian matrices. Our algorithm is not lengthy to implement and could likely be embedded in a mobile management device dynamically explore device deployments for small- to medium-sized mobile networks.

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