Proceedings of the 36th Chinese Control Conference
July 26-28, 2017, Dalian, China

Robustness of the complex networks by statistical physics method

Hai Lin[1], Jincheng Wang[1,2], Xiaocheng Li[1], Hongyuan Wang[1], Bin Ma[3],

1. Department of Automation, Shanghai Jiao Tong University and Key Laboratory of System Control and Information Processing, Ministry of Education of China, Shanghai 200240 P. R. China
E-mail: linhaicool@msn.com

2. Autonomous Systems and Intelligent Control nternational Joint Research Center, Xian Technological University, Xian, Shaanxi, China, 710021
E-mail: jcwang@sjtu.edu.cn

3. China Research Institute JIU YI, Shanghai P. R. China, 200135

Abstract: In this study, we present a novel analytical approach for robustness on Erdos-Renyi (ER) networks by introducing statistical physics method. We establish an exact mapping relation between ER networks and Ising model. Based on the mapping relation, we obtain the partition function of the ER networks and use it to determine the size of the giant component and the value of the critical edge present probability in ER networks. We extend the approach to investigate the size of remaining giant component and the critical fraction of the nodes removed in the ER networks whose nodes under the randomly attack.

Key Words: Percolation, Erdos-Renyi (ER) networks, Ising model, mapping relation

1 Introduction

In the past ten years, complex networks has been widely reaserched with the development of network and information technology. Power grid, mobile phone networks, and protein networks are examples analyzed by using complex networks conception[1-4]. In this conception, nodes (vertices) and edges (links) are introduced to describe the systems . The conspicuous diferent between traditional graph theory and complex networks conception motivate us to seek new methods to reaserch the complex networks. This is similar with the development of the statistical physics develops out from classical thermodynamics. Networks phase transition and percolation theory, mean field theory and generating function were proposed for treating complex networks[5-7].

In complex networks reaserch field, the study of the percolation has been popular in the last decades [8-12]. Percolation is the dynamical behaviors of the complex networks. It is called an mesoscopic level research in contrast with the microcosmic and macroscopic levels research. Mesoscopic behaviors shows the evolutionary process, interior structure and couipling of complex networks. Currently, generating function formalism[13-14]and percolation theory[15-19] are main methods to solve the mesoscopic level research of complex networks.

In random graphs works, generating function is widely-used which is a combinatorial mathematics tool. A framework for studying percolation of the network of the networks (NON) based on the generating function formalism is proposed by Gao [20]. Niu [21] Percolation of networks with directed dependency links] presented an analytical formalism for studying random networks with both connectivity links and directed dependency links under random node failures. Wang proposed a new statistical model with an additional cluster weight in the partition function with respect to the traditional site percolation model and designed a color-assigned cluster updating Monte Carlo simulation algorithm to simulate the percolation of the new statistical model [22].One constructive model, the Random Cluster Model (RCM) proposed by Fortuin and Kasteleyn [23] in the 1960s, combines the measure theory and percolation theory to lattice statistical models. RCM provides a unified description of several classical statistical model including the Potts [24], Ashkin-Teller [25], and the percolation models.

Methods mentioned above have proved to be valid and complete.Baesd on these methods Researchers accomplished a number of mature and systemic achievements. However,there are several significant challenges deserve us to solve.When using generating function, we often only concern the degree distribution of the nodes. This is only the microcosmic level information of the complex networks, meanwhile, there are plenty of additional information is abandoned. Furthermore, percolation theory limits the system in a lattice, which can't be completely used in the reaserch f complex networks.

We propose a statistical physics model to analyze the percolation problem on Erdos-Renyi (ER) networks. Since the conspicuous similarity between the complex networks and Ising model, include the structure and dynamic evolution process, it is rational to introduce the statistical physics theories into the research of complex networks. We establish a mapping relation between ER networks and Ising model and obtain the partition function of the ER networks and find exact analytical laws for percolation and collapse process of ER networks.

2 Mapping relation between Random Cluster Model and Potts model.

Consider a finite graph $G=(V,E)$ without loops or multiple edges. An edge connects x and y is written as $e = \langle x,y \rangle \in E$. $p \in [0,1]$ is the independent present probability of each edge. Take set $\Omega = \{0,1\}^E$ as state space of edge. Members of Ω are 0/1 vectors $\omega = (\omega(e) : e \in E)$.We say the edge is open (presents) if $\omega(e) = 1$ and closed (not presents). The open edges set $F = \eta(\omega) =$

This work was supported by National Natural Science Foundation of China (No. 61533013, 61233004, 61433002,61633019), National 973 Program of Ch ina (No. 2013CB035406).Fund of National Engineering and Research Center for Commercial Aircraft Manufacturing(No.SAMC14-JS-15-050). Pudong New Area science and Technology Development Fundtechnology project(No.PKJ2015-Z06).

$\{e \in E : \omega(e) = 1\}$ is a subset of . Let $k(\omega)$ be the number of connected components, including isolated nodes. A random-cluster measure on G is defined by

$$\mu(\omega) = \frac{1}{Z_{RC}} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}, \omega \in \Omega \tag{1}$$

where normalizing constant Z_{RC} is given by

$$Z_{RC}(p,q) = \sum_{\omega \in \Omega} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}. \tag{2}$$

Remark 2.1 (1) is a parametric family of probability measures which provides a quantitative criteria to measure the number of edges and the size of component simultaneously. p measures the edge density while q measures cluster size.

Consider a finite graph $G = (V, E)$, in which the edges constructed in the same manner as the graph described in 2.1 but with the new properties of the nodes. σ takes values $\{0, 1, \ldots, q-1\}$ is assigned to each vertex x of G, and it represents the spin state of the vertex. Positive integer $q \in (0, +\infty)$ is the number of spin states. Take set $\Sigma = \{0, 1, \ldots, q-1\}^V$ as state space of vertex. Members of the Σ are vectors σ_x realizes one of q states on vertex x. To measure the intensity of open edges and the spin state of nodes simultaneously, we then define a probability mass function $\mu(\sigma, \omega)$ on product sample space $\Sigma \times \Omega$, with an appropriate normalizing constant Z,

$$\mu(\sigma, \omega) = \frac{1}{Z} \prod_{e = \langle x,y \rangle \in E} \{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\}, \tag{3}$$

where $\delta_e(\sigma) = \delta_{\sigma_x,\sigma_y}$ and δ is the Kronecker delta.

Remark 2.2 The configuration space of the model (3) is the set of all subsets of the Cartesian product $V \times E$. In this (3), p and q respectively represent the density of open edges and spin states of nodes. We regard this model as a vertex-edge model. When $q > 1$, we obatain a percolation model. When $q = 1$, it is an Ising or Potts model with a present probability p of each edge.

The measure $\mu(\sigma, \omega)$ has the two marginal measures. For the Potts measure, choose $\beta > 0$, such that $p = 1 - e^{-\beta}$

$$\begin{aligned} \mu_1(\sigma) &= \sum_{\omega \in \Omega} \mu(\sigma, \omega) \\ &= \tfrac{1}{Z_P} \exp\left\{ \beta \sum_{e \in E} \delta_e(\sigma) \right\} \end{aligned} , \tag{4}$$

for the Random cluster measure,

$$\begin{aligned} \mu_2(\omega) &= \sum_{\sigma \in \Sigma} \mu(\sigma, \omega) \\ &= \tfrac{1}{Z_{RC}} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)} \end{aligned} \tag{5}$$

For $|E|$the total number of edges, the following relationship holds between partition functions of the RCM and the Potts Model:

$$Z_{RC} = e^{-\beta|E|} Z_P \tag{6}$$

$\mu(\sigma, \omega)$ combines the Potts measure $\mu_1(\sigma)$ and RCM $\mu_2(\omega)$together. $p = 1 - e^{-\beta}$ is a bridge connects RCM and Potts model.Based on (6), we can obtain RCM partition function Z_{RC} fro0 Potts model partition.

3 Results on ER networks

In this section we uncover the mapping relation between ER networks and Potts model. Note that statistical physics systems and complex networks are analogical.Both th of them are multibody systems and the interior structures are complex. Therefore, we seek to apply the statistical physics theory to study the dynamical process and phase transaction of complex networks. We choose Ising model to be a conjugate model of the ER networks, since Ising model is a basically representative of Potts model. Other variants of ER networks, which include various nodes and edge presenting rules, can be studied along the same lines.

Sicne it is a lattice system, the neighbours of each particles in Ising model $z = 4$, while the nodes in ER networks have more neighbors than 4. Nodes in ER networks may be connected together randomly, we regard all other nodes as neighbors of a appointed node. Thus the neighbors of one node in ER networks are $N - 1$. The ER networks concerned in this paper are sparse networks, hence the parameter $p \ll 0.5$ and (5) is a monotonic decreasing function with respect to p. This is consistent with the equation (4),is also a monotonic decreasing function with respect to the number of spin pairs. Naturely, we regard the edge in ER networks as the spin pairs in Ising model.That is, an open edge in ER networks equivalents to a spinon in Ising model. Equation $p = 1 - e^{-\beta}$ joints the edge present probability with the temperature. Since $\beta = 1/kT$ is negatively correlated to T. The increasing of p equivalents to the decreasing of T, the changes respectively lead to the increasing of edges and spinons in RCM and Ising model. This corresponding relation verifies the proposition in equation (1). Therefore, we map the edge present probability p to the temperature T.

Table 1: Mapping relation

ER networks	Ising model
Edge	Spin pair
Node (with edges)	One spins of a spin pair
Node (without edge)	Single spin (not in pair)
Edge present probability	temperature
Giant component	Macroscopical magnetic moment
Edge loss	Spin pair vanished
Giant component emergence	Thase transition

Next, we study the emerging mechanism of the giant component in ER networks by applying method introduced from the Ising model. According to the equivalence relation of (6), the equation of the ER networks partition functions is

$$Z_{RC} = \sum_{\sigma \in \Sigma} \prod_{e \in E} \exp[\beta(\delta_e(\sigma) - 1)]. \tag{7}$$

As we know, if there are only two spin states spin-up and spin-down in Potts model. It becomes the famous Ising Ferromagnetic model. The spin-values are chosen randomly according to a certain probability measure, which is governed by interactions between neighboring nodes. In this paper, the $\delta_e(\sigma)$=$\sigma_x\sigma_y$ generally takes the formula

$$\delta_e(\sigma) = \sigma_x \sigma_y, \tag{8}$$

where $\delta_e(\sigma) = \{-1, 1\}$. As we know, the Ising model phase transformation curve is symmetric because of the two directions of spin. Naturally, the ER networks percolation curve only need to correspond either of the Ising model phase transformation curve on the two sides of the direction axis. Substituting (8) into (6), we can write the Ising model partition functions as

$$Z_{RC} = \sum_{\sigma\in\Sigma}\prod_{e\in E}\exp[\beta(\sigma_x\sigma_y - 1)]. \qquad (9)$$

If we regard (9) as a certain variant of the Ising model, then the Hamiltion of the variant is

$$H_{RC}=\sum_{e\in E}(\sigma_x\sigma_y - 1). \qquad (10)$$

Consider the external field, the Hamiltion can be rewritten as

$$\begin{aligned} H_{RC} &= \sum_{<x,y>}\left[\tfrac{1}{2}(1+\sigma_x\sigma_y) - 1\right] + \mu\mathrm{H}\sum_{x=1}^{N}\sigma_x \\ &= \sum_{<x,y>}\sigma_x\sigma_y - \sum_{<x,y>}1 + \mu\mathrm{H}\sum_{x=1}^{N}\sigma_x \\ &= \sum_{<x,y>}\sigma_x\sigma_y + \mu\mathrm{H}\sum_{x=1}^{N}\sigma_x - \tfrac{1}{2}pN(N-1) \\ &= \sum_{x=1}^{N}\mu\sigma_x\left(\mathrm{H} + \tfrac{1}{\mu}\sum_{y=1}^{N}\sigma_y\right) - \tfrac{1}{2}pN(N-1) \\ &= \sum_{x=1}^{N}\mu\sigma_x(\mathrm{H}+h_i) - \tfrac{1}{2}pN(N-1) \end{aligned} \quad , \qquad (11)$$

where μ is the Bohr magnetons, is the H external field.

According to the mean field theory, we let

$$\begin{aligned} \bar{h}_i &= \tfrac{1}{\mu}\sum_{y=1}^{N}\bar{\sigma}_y \\ &= \tfrac{z}{\mu}\bar{\sigma} = \bar{h} \end{aligned} \quad , \qquad (12)$$

then the Hamiltion can be simplified as

$$H_{RC}=\sum_{x=1}^{N}\mu\sigma_x\left(\mathrm{H}+\bar{h}\right) - \frac{1}{2}pN(N-1) \qquad (13)$$

where

$$\bar{h} = \frac{z}{\mu}\bar{\sigma} \qquad (14)$$

$\bar{\sigma}$ is mean value of spins. As described above, in the derivation of (12), we regard all other nodes as the neighbors of node x, thus $z = N - 1$ is the coordination number.

Remark 3.1 An approximate treatment $\sum_{y=1}^{N}\bar{\sigma}_y = z\bar{\sigma}$ is proposed in (12) based on mean field theory. It is salutary for us to understand the mapping relation. If we expand the first term of the right-hand side of (13), we get an expression $z\bar{\sigma}\sigma_x$. It implies that the effect from neighbors on the spin of node x is approximates to a certain internal field. Furthermore, as we know, an open edge in Ising model equivalents to a two spinon in Ising model. Therefore we can map the internal field generated by the neighbors of particle x in Ising model to the average degree of the node x in ER networks, $z\bar{\sigma}\sigma_x \sim Np$.

Substituting (13) into (7), the partition functions becomes

$$\begin{aligned} Z_{RC} &= \sum_{\sigma_1}\sum_{\sigma_2}\ldots\sum_{\sigma_N}\left\{\prod_x\left\{\exp\left[\mu\sigma_x\left(\mathrm{H}+\bar{h}\right)\beta\right]\right\}\right. \\ &\left.*\exp\left[-\tfrac{1}{2}pN(N-1)\beta\right]\right\} \\ &= \exp\left[-\tfrac{1}{2}pN(N-1)\beta\right] \\ &*\prod_x\left\{\sum_{\sigma_x}\exp\left[\mu\sigma_x\left(\mathrm{H}+\bar{h}\right)\right]\beta\right\} \end{aligned} \qquad (15)$$

For the multiplication term in (15) can be transformed as

$$\begin{aligned} &\sum_{\sigma_x=\pm1}\exp\left[\mu\sigma_x\left(\mathrm{H}+\bar{h}\right)\right]\beta \\ &= 2\cosh\left(\mu\beta\mathrm{H} + \mu\beta\tfrac{z}{2\mu}\bar{\sigma}\right) \\ &= 2\cosh\left(\mu\beta\mathrm{H} + \beta\tfrac{z}{2}\bar{\sigma}\right) \end{aligned} \qquad (16)$$

Hence we obtain a compact form of ER networks partition function

$$Z_{RC} = \left[\exp\left[-\frac{1}{2}pN(N-1)\beta\right] * 2\cosh\left(\mu\beta\mathrm{H} + \beta z\bar{\sigma}\right)\right]^N \qquad (17)$$

As we know, partition function comprehends complete thermodynamic property of the system which is the primary challenge in thermodynamic calculation. Different with the generation function included only degree information and higher powers terms, thermodynamic calculation is more concise and complete. We can use it to deduce other thermodynamical functions of the ER networks. The free energy

$$\begin{aligned} F &= -kT\ln Z_{RC} \\ &= -NkT\ln\left[\exp\left[-\tfrac{1}{2}pN(N-1)\beta\right] * 2\cosh\left(\mu\beta\mathrm{H} + \beta z\bar{\sigma}\right)\right] \\ &= -NkT\left[-\tfrac{1}{2}pN(N-1)\beta + \ln 2\cosh\left(\mu\beta\mathrm{H} + \beta z\bar{\sigma}\right)\right] \end{aligned} \qquad (18)$$

The magnetization is given by

$$\bar{\mu} = N\mu\bar{\sigma} = -\frac{\partial F}{\partial \mathrm{H}} = N\mu\tanh\left(\mu\beta\mathrm{H} + \beta z\bar{\sigma}\right) \qquad (19)$$

The phase transition occurs at the time when the magnetization value cross the zero point. From (19) we know, it equates to the mean value of spins cross the zero point. Rearrange (19), we obtain the self-consistent equation with respect to $\bar{\sigma}$

$$\bar{\sigma} = \tanh\left(\mu\beta\mathrm{H} + \beta z\bar{\sigma}\right) \qquad (20)$$

Neglect the extern field

$$\bar{\sigma} = \tanh\left(z\beta\bar{\sigma}\right) \qquad (21)$$

As temperature β term changes, if the solution of (21) transforms from zero solution to nonzero, it implies that the system generates spontaneous magnetization and occurs the phase transition. To the ER networks, we regard the network includes a giant component. Thus, calculating the percolation point of ER networks equals to calculating the critical temperature point where (21) has a nontrivial solution. After obtain the value of the critical temperature point, we can use $p = 1 - e^{-\beta}$ to obtain the critical edges present probability p_c. We now take the graphical method to solve (21). $z\beta = 1$ is the critical case corresponds to the minimum value of the $\bar{\sigma}$, as shown in Fig. 2. Substituting $\beta = 1/z$, we obtain the critical value of p_c

$$p_c = 1 - e^{-\frac{1}{z}} \qquad (22)$$

The results of edges present probability p extraordinary approaches to the traditional method. In the traditional method, the size of the giant component is given by a self-consistent equation $S = 1 - e^{-k_i S}$, which is analogous to the formula of (22). k_i is the average degree of the nodes. The process of giant component size S transforms from zero solution to

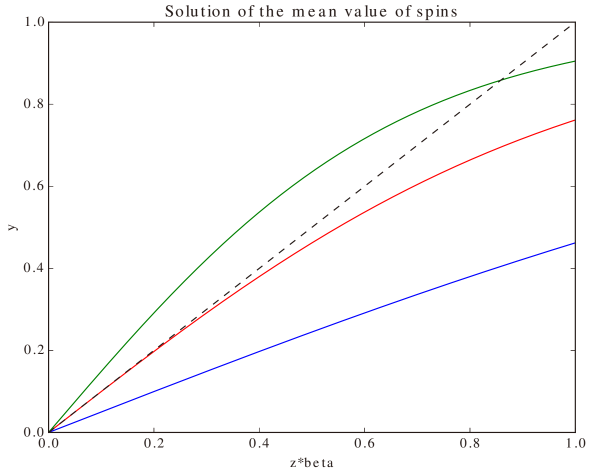

Fig. 1: Simulation results of relation between the value of $z\beta$ and the solution of $\bar{\sigma}$. For the ER network, $N = 5E{+}08$, $z\beta$ takes the value 0.8, 1 and1.8. The dash line is the curve of $y = \bar{\sigma}$ and the solid lines are the curves of $y = \tanh(z\beta\bar{\sigma})$ for the three values of $z\beta$.

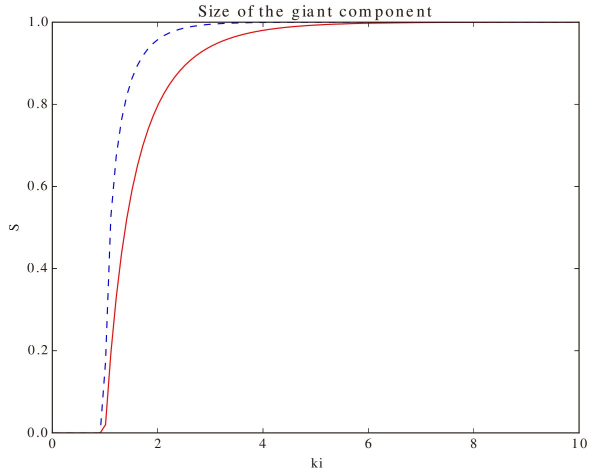

Fig. 2: Comparison of the traditional method and statistical physics method. For the ER network, $N = 5E{+}08$, we take continuously variable value of (convert to the average degree of the nodes k_i) to obtain the critical values of p_c in different cases of edges present probabilities. The dash line represents the curve of statistical physics method. The two methods take the same value at the phase transition point.

nonzero means the percolation process in ER networks. Figure. 2 shows the size of the giant component as a function of p. Note that the critical value p_c of in the two lines overlap at the same point and separate slightly after the critical points. Our method agrees well with the traditional method while the separation of the two lines is due to the approximate treatment in mean field theory.

4 Robustness of the ER networks

According to the map relation described above, randomly removing a fraction of $Q = 1 - P$ nodes (function failure) in ER networks equivalent with randomly removing a fraction of particles in Ising model. The size of Ising model reduces while the temperature remains the same. We seek to obtain the relation between Q and the fraction of P_∞ nodes which belongs to the giant component (Size of the the remaining giant component) after removing nodes. Another significant point is the critical value Q_c (or p_c)which triggers the percolation phase transition in ER networks, namely, the ER networks collapses under the randomly nodes attack. Once a fraction $Q = 1 - P$ of nodes is randomly removed from an ER networks, the edges present probability remains the same

$$p_2 = p \tag{23}$$

Coordination number

$$z_2 = NP - 1 \tag{24}$$

The critical value of p_{2c}

$$p_{2c} = 1 - e^{-\frac{1}{z_2}} = 1 - e^{-\frac{1}{NP-1}} \tag{25}$$

When $p_2 = p_{2c}$, the ER networks is at critical point of the percolation (collapses). Based on (24) and (25), we can obtain the critical value of P_c by the equation

$$p_2 = p = 1 - e^{-\frac{1}{NP_c-1}} = p_{2c} \tag{26}$$

Thus

$$P_c = \frac{\ln(1-p) - 1}{N\ln(1-p)} \tag{27}$$

The results of P_∞ extraordinary approaches to the traditional method (see Fig.5). The separation of the two lines is due to the approximate treatment in mean field theory. P_∞ directly characterizes the robustness of the network. Larger P_∞ means a more robust network against the attack. According to the map relation between RCM and Ising model and (19), the P_∞ is given by

$$P_\infty = \sigma_\infty = P\bar{\sigma} = P\tanh(z_2\beta\bar{\sigma}) = P\tanh(zP\beta\bar{\sigma}) \tag{28}$$

5 Conclusion

In summary, we have developed a statistical physics method to study the percolation process of the ER networks. We established a mapping relation between ER networks and Ising model. Based on the mapping relation, we obtained the partition function of ER networks. For studying percolation of ER networks, we derived an exact analytical law (21), and for robustness of the ER networks in the case of randomly nodes attack, we derived an exact analytical law (28). In particular, we found that, ER networks is precisely corresponds to the Ising model both on mechanism mathematics and dynamical evolution process. The equation $p = 1 - e^{-\beta}$ builds a bridge between the edges present probability in ER networks and the temperature in Ising model. These findings show that the using of statistical physics methods on study ER networks is brief, informative and accurate.

References

[1] Clusella, P.,Grassberger, P.,Perez-Reche, F. J.andPoliti, A. (2016). Immunization and Targeted Destruction of Networks

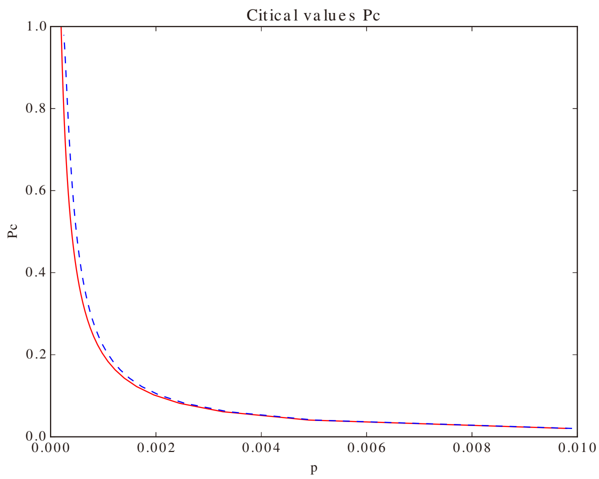

Fig. 3: Comparison of the traditional method and statistical physics method. For the ER network, $N = 5E{+}03$, we take continuously variable value of p to obtain the critical values of P_c in different cases of edges present probabilities. The solid line is the curve of the traditional method and the dash line is the curve statistical physics method. That two lines coincide extremely shows the validity of the statistical physics method.

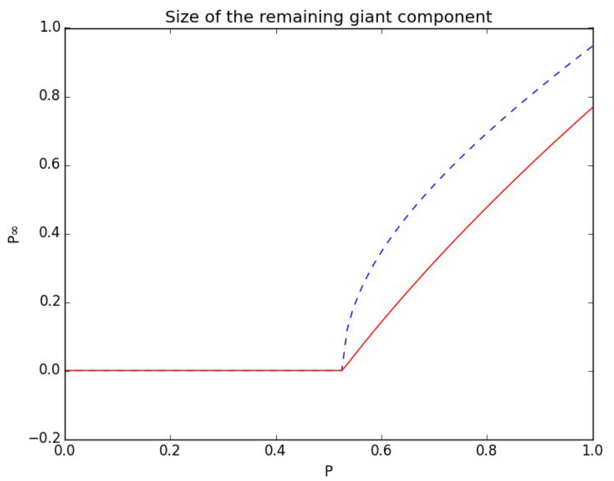

Fig. 4: Comparison of the traditional method and statistical physics method. For the ER network, $N = 5E{+}08$, we take continuously variable value of P to obtain the size of the the remaining component P_c in different fractions of attack. The solid line is the curve of the traditional method and the dash line is the curve statistical physics method. The two curves take the same value at the phase transition point.

using Explosive Percolation. Physical Review Letters, volume (117), 208301.

[2] Hantsch, M.and Huchzermeier, A. (2016). Transparency of risk for global and complex network decisions in the automotive industry. International Journal of Production Economics, volume(175), 81-95.

[3] Lacasa, L.and Gomez-Gardenes, J. (2013). Correlation Dimension of Complex Networks. Physical Review Letters, volume (110), 168703.

[4] Skardal, P. S.,Taylor, D.and Sun, J. (2014). Optimal Synchronization of Complex Networks. Physical Review Letters, volume (113), 144101.

[5] Pastor-Satorras, R.,Castellano, C.,Van Mieghem, P.and Vespignani, A. (2015). Epidemic processes in complex networks. Reviews of Modern Physics, volume (87), 925-979.

[6] Markovich, N. M. (2015). Extremes Control of Complex Systems With Applications to Social Networks. IFAC-PapersOnLine, volume (48), 1296-1301.

[7] Ruiz-Martin, C.,Paredes, A. L.and Wainer, G. A. (2015). Applying Complex Network Theory to the Assessment of Organizational Resilience. IFAC-PapersOnLine, volume (48), 1224-1229.

[8] Baxter, G. J.,Dorogovtsev, S. N.,Goltsev, A. V.and Mendes, J. F. F. (2010). Bootstrap percolation on complex networks. Physical Review E, volume (82), 011103.

[9] Goltsev, A. V.,Dorogovtsev, S. N.and Mendes, J. F. F. (2006). k-core (bootstrap) percolation on complex networks: Critical phenomena and nonlocal effects. Physical Review E, volume (73), 056101.

[10] Kawamoto, H.,Takayasu, H.and Takayasu, M. (2015). Analysis of Network Robustness for a Japanese Business Relation Network by Percolation Simulation. Proceedings of the International Conference on Social Modeling and Simulation, plus Econophysics Colloquium 2014. Cham: Springer International Publishing.

[11] Shuai, S.,Xuqing, H.,Stanley, H. E.and Shlomo, H. (2015). Percolation of localized attack on complex networks. New Journal of Physics, volume (17), 023049.

[12] Morone, F.and Makse, H. A. (2015). Influence maximization in complex networks through optimal percolation. Nature, volume (524), 65-68.

[13] Gao, J.,Buldyrev, S. V.,Havlin, S.and Stanley, H. E. (2012). Robustness of a network formed by n interdependent networks with a one-to-one correspondence of dependent nodes. Physical Review E, volume (85), 066134.

[14] Dong, G.,Gao, J.,Du, R.,Tian, L.,Stanley, H. E.and Havlin, S. (2013). Robustness of network of networks under targeted attack. Physical Review E, volume (87), 052804

[15] Powell, M. J. (1980). Site percolation in random networks. Physical Review B, volume (21), 3725-3728.

[16] Zhou, D.,Gao, J.,Stanley, H. E.and Havlin, S. (2013). Percolation of partially interdependent scale-free networks. Physical Review E, volume (87), 052812.

[17] Li, D.,Zhang, Q.,Zio, E.,Havlin, S.and Kang, R. (2015). Network reliability analysis based on percolation theory. Reliability Engineering & System Safety, volume (142), 556-562.

[18] Hayasaka, S. (2016). Explosive percolation in thresholded networks. Physica A: Statistical Mechanics and its Applications, volume (451), 1-9.

[19] Jian, Y.,Liu, E.,Zhang, Z.,Qu, X.,Wang, R.,Zhao, S.andLiu, F. (2015). Percolation and Scale-Free Connectivity for Wireless Sensor Networks. IEEE Communications Letters, volume (19), 625-628.

[20] Gao, J.,Buldyrev, S. V.,Stanley, H. E.,Xu, X.and Havlin, S. (2013). Percolation of a general network of networks. Physical Review E, volume (88), 062816.

[21] Niu, D.,Yuan, X.,Du, M.,Stanley, H. E.and Hu, Y. (2016). Percolation of networks with directed dependency links. Physical Review E, volume (93), 042312.
[22] Wang, S.,Zhang, W.and Ding, C. (2015). Percolation of the site random-cluster model by Monte Carlo method. Physical Review E, volume (92), 022127.
[23] Fortuin, C. M.and Kasteleyn, P. W. (1972). On the random-cluster model. Physica, volume (57), 536-564.
[24] Wu, F. Y. (1982). The Potts model. Reviews of Modern Physics, volume (54), 235-268.
[25] Pfister, C. E.and Velenik, Y. (1997). Random-cluster representation of the ashkin-teller model. Journal of Statistical Physics, volume (88), 1295-1331.
[26] Edwards, R. G., and Sokal, A. D. (1988). Generalization of the fortuin-kasteleyn-swendsen-wang representation and monte carlo algorithm. Physical review D, volume 38(6), 2009-2012.