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Automatica 40 (2004) 5–19
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Analysis of the Kalman filter based estimation algorithm: an orthogonal decomposition approach☆

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Received 22 January 2002; received in revised form 13 February 2003; accepted 10 July 2003

Abstract

In this paper we shall provide new analysis on some fundamental properties of the Kalman filter based parameter estimation algorithms using an orthogonal decomposition approach based on the excited subspace. A theoretical analytical framework is established based on the decomposition of the covariance matrix, which appears to be very useful and effective in the analysis of a parameter estimation algorithm with the existence of an unexcited subspace. The sufficient and necessary condition for the boundedness of the covariance matrix in the Kalman filter is established. The idea of directional tracking is proposed to develop a new class of algorithms to overcome the windup problem. Based on the orthogonal decomposition approach two kinds of directional tracking algorithms are proposed. These algorithms utilize a time-varying covariance matrix and can keep stable even in the case of unsufficient and/or unbounded excitation.

Keywords: Kalman filter; Recursive parameter estimation; Least squares algorithm; Windup; Directional tracking

1. Introduction

Tracking time-varying parameters of a dynamic system is an important issue in adaptive system design. In the world of adaptive control, tracking ability is usually provided by two kinds of recursive estimation algorithms: the exponentially weighted least squares (EWLS) algorithm and the Kalman filter based (KFB) algorithms. Although the EWLS algorithm in its form can be viewed as a special case of the Kalman filter (Ljung & Gunnarsson, 1990), these two kinds of algorithms track time-varying parameters based on different mechanisms. In principle, it can be said that the EWLS algorithm obtains its tracking ability by performing a forgetting operation on the information matrix; while in the KFB algorithms tracking ability is rendered by adding a nonnegative definite matrix to the covariance matrix. It is expected that this operational difference may lead to some significant differences in their behavior. One of the objectives of this paper is to clarify a main difference in terms of their behavior under the condition of unsufficient excitation.

This paper focuses on the KFB algorithms. The standard Kalman filter recursive algorithm is usually associated with a random walk parameter variation model and a linear regression equation described by

$$\theta_t = \theta_{t-1} + w_t, \tag{1}$$

$$y_t = \varphi_t^{\mathrm{T}} \theta_t + v_t. \tag{2}$$

In (1) θ_t represents the n-dimensional unknown system parameter vector, and w_t is a sequence of random vectors that drives the parameter's change. In (2), y_t is the scalar system's output, φ_t is the regression vector (also called regressor), and v_t is the measurement noise. Furthermore, it is usually assumed that both w_t and v_t are Gaussian process with zero mean value and the variances given by $Ew_t w_t^{\mathrm{T}} = Q$, $Ev_t^2 = r$, where Q is an $n \times n$ nonnegative definite matrix, and $r > 0$ is a scalar.

The standard Kalman filter for estimating θ_t in (1) is given by

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t(y_t - \varphi_t^{\mathrm{T}} \hat{\theta}_{t-1}), \tag{3}$$

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by the Associate Editor Antonio Vicino under the direction of Editor Torsten Söderström.

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0005-1098/$ - see front matter
doi:10.1016/j.automatica.2003.07.011

$$K_t = \frac{P_{t-1}\varphi_t}{r_t + \varphi_t^{\mathrm{T}} P_{t-1}\varphi_t}, \tag{4}$$

$$P_t = P_{t-1} - \frac{P_{t-1}\varphi_t\varphi_t^{\mathrm{T}} P_{t-1}}{r_t + \varphi_t^{\mathrm{T}} P_{t-1}\varphi_t} + Q_t. \tag{5}$$

As shown in Ljung and Gunnarsson (1990), if we exactly know the variances Q and r and let $Q_t = Q$, $r_t = r$ in the above equations, then the Kalman filter provides the best estimation of θ_t. However, in real applications we can never exactly know Q and r. Fortunately, the unknown variances Q and r do not restrict the applicability of the Kalman filter. On one hand, as shown in Isaksson (1987) and the references there, the unknown Q and r themselves can be estimated recursively to obtain the asymptotically optimal estimation on θ_t if the changes in θ_t are sufficiently slow. On the other hand, even when we cannot estimate Q and r reasonably well or the actual parameter change is quite different from the random walk model, the standard Kalman filter can still work very well. This is because the Kalman filter does not require an accurate model of the parameter variation (Niedźwiecki, 2000), and it can track time-varying parameters reasonably well as long as its gain vector K_t keeps away from zero. A nonzero gain vector K_t is guaranteed if the covariance matrix satisfies the matrix inequality

$$P_t \geqslant \alpha I, \quad \forall t, \tag{6}$$

where α is a positive scalar.

Therefore, we can design a Kalman filter in terms of its tracking performance by choosing a suitable matrix sequence $Q_t \geqslant 0$ to guarantee (6) for some appropriate positive number α. In this context, the choice of Q_t is exactly the same as choosing a suitable forgetting factor in the EWLS algorithm, and almost does not need the knowledge of how the parameters are changing. Although this design strategy is different from the original context of the Kalman filter, we still call the corresponding algorithm the Kalman filter based algorithm. For this kind of design, the important issue is to keep a reasonable tracking performance for all time. Of course, by adding the knowledge of the parameter variation to the choice of Q_t, we can expect to obtain more satisfactory tracking performance.

Besides a reasonable tracking performance it is also important to ensure the stability of a recursive algorithm. The stability of the Kalman filter is generally guaranteed if the covariance matrix satisfies the matrix inequality

$$P_t \leqslant \beta I, \quad \forall t \tag{7}$$

for some scalar $\beta > 0$.

In fact, as shown in Salgado, Goodwin, and Middleton (1988) and Parkum, Poulsen, and Holst (1992) the inequalities (6) and (7) represent two of the most desirable and most important properties for all of the tracking algorithms. They not only guarantee the tracking ability and stability of an algorithm, but also ensure some basic convergence properties (Parkum et al., 1992), which are needed in the analysis of adaptive control algorithms (Salgado et al., 1988). Therefore, a fundamental requirement for any tracking algorithm is to satisfy (6) and (7).

There are quite large number of publications available on the topic of the Kalman filter based algorithm. Here we mention the survey paper of Ljung and Gunnarsson (1990), the paper of Guo (1990), and the papers of Guo and Ljung (1995a, b) where the stability and tracking performance of the KFB algorithm are intensively analyzed based on stochastic excitation condition. This paper focuses on the behavior of the KFB algorithm in the case where the regressor φ_t is not persistently exciting and particularly there is a subspace in R^n which is almost not excited by φ_t. One of the main objectives of this paper is to inspect if the KFB algorithms satisfy (6) and (7) particularly in the case of unsufficient excitation based on *an orthogonal decomposition approach*. The orthogonal decomposition for positive semidefinite matrices is originally proposed in Cao and Schwartz (2001a) and has been successfully used in developing the directional forgetting algorithm of Cao and Schwartz (2000) and analyzing the windup phenomenon of the KFB algorithm (Cao & Schwartz, 2001b). In the present paper, we will continue to develop this useful analytical tool and make it more suitable to the analysis of the behavior of parameter estimation algorithms with unsufficient excitation. The objectives of this paper are three-fold.

1. To complement the orthogonal decomposition method proposed in Cao and Schwartz (2001a). In particular, the concepts of the *general* orthogonal decomposition and the *unique* orthogonal decomposition are established. New results (Theorem 2.2, and Lemma 2.1) are developed, which are directly applicable to the analysis of the KFB algorithm with unsufficient excitation.
2. Using the orthogonal decomposition method and the associated results to analyze the behavior of the KFB algorithms. By decomposing the covariance matrix into two parts based on the excited subspace, we can analyze the boundedness of the covariance matrix in an elegant way, so we easily characterize the property of the windup phenomenon in the KFB algorithm and establish the conditions for the boundedness of the covariance matrix in terms of the excitation condition.
3. To propose the new idea of *directional tracking* which leads to a new class of the KFB algorithms. The key idea of directional tracking is to restrict tracking directions of an algorithm to the excited subspace to avoid windup. Directional tracking appears as a parallel concept to directional forgetting and is particularly useful in modifying the KFB algorithm to improve its performance. Based on the orthogonal decomposition method and the associated results, two directional tracking algorithms are proposed and their main properties are established.

One of the main contributions of this paper is the introduction of the useful and effective analytical method based on the orthogonal decomposition approach. By applying this analytical method to the KFB algorithm, it is shown that this method is very useful in establishing the key property of an algorithm in the case of unsufficient excitation. Considering that there is not yet an effective analytical method available in the literature to handle the case of unsufficient excitation, this paper complements the existing theories and methods for analyzing the behavior of a parameter estimation algorithm.

In the authors previous work (Cao & Schwartz, 2001b), it has been proven that in the case of Q_t being a positive definite constant matrix and unsufficient excitation, some of the eigenvalues of P_t become unbounded as $t \to \infty$. That is, the so-called estimator windup does exist in some of the KFB algorithms. In the present paper, we will extend this analysis to the more general case of time-varying Q_t. In the case of constant Q_t and unsufficient excitation, it is shown that some of eigenvalues of P_t tend to infinity linearly with time. In addition, due to the new result developed in this paper (Theorem 2.2), the proof becomes more concise than that in Cao and Schwartz (2001b). Furthermore, the necessary and sufficient conditions on Q_t for P_t satisfying (7) is established in terms of the exciting condition.

This paper is organized as follows. In Section 2 we summarize the basic results concerning the orthogonal decomposition of a positive semidefinite matrix based on a subspace, and then establish some new results which complement the existing theories developed in Cao and Schwartz (2001a) and provide more direct and easy to use tools to analyze the KFB algorithm. In Section 3 we analyze the boundedness of the covariance matrix in terms of its two decomposed parts based on the theorems and lemmas established in Section 2. We clarify the property of the windup phenomenon in the standard Kalman filter and give sufficient and necessary conditions to avoid windup. In Section 4 we propose the idea of directional tracking for the KFB algorithms, which is a parallel concept to directional forgetting used in the modified EWLS algorithm. Based on the results established in Section 3, two new directional tracking algorithms are proposed, which have a bounded covariance matrix even in the case of unsufficient and/or unbounded excitation. Finally in Section 5, the conclusions are given.

2. Preliminaries

In this section we will summarize some basic results regarding decomposing a positive semidefinite (definite) matrix into the sum of two specific positive semidefinite matrices based on a given subspace (Cao & Schwartz, 2001a). We will also present some new results on this decomposition. All of these results play a central role in the analysis of the behavior of the KFB algorithm with unsufficient excitation.

Given an $n \times n$ positive semidefinite matrix A and a m-dimensional subspace $S \subset R^n$, $m \leqslant n$, we consider the problem of decomposing A into the form

$$A = B + C \tag{8}$$

where B and C are required to be positive semidefinite, and furthermore C is required to satisfy the "orthogonal" condition

$$CV = 0 \tag{9}$$

or equivalently B satisfies

$$BV = AV \tag{10}$$

where V is an $n \times m$ matrix whose columns constitute a basis of S. Since any vector $x \in S$ can be expressed as $x = Va$ where $a \in R^m$, we have $Cx = CVa = 0$, which implies $S \subseteq \operatorname{Ker} C$.[1] In the following, we call the decomposition (8) satisfying (9) an *orthogonal decomposition along the subspace* S. We assume that the rank of A is not less than m, the dimension of the given subspace S. Furthermore, we also assume that $\operatorname{Ker} A \cap S = 0$.[2] The reason for this assumption will become obvious later.

The fundamental problems for the orthogonal decomposition of (8) along the subspace S are whether the decomposition exists, and if it exists, whether it is unique. In Cao and Schwartz (2001a) it has been shown that if A satisfies $S \cap \operatorname{Ker} A = 0$, then the orthogonal decomposition exists; and furthermore, if the rank of B is required to be m, then the orthogonal decomposition is unique. These results are summarized in Theorem 2.1.

Theorem 2.1. *Given an $n \times n$ positive semidefinite matrix A with rank l, and a m-dimensional subspace S in R^n such that $S \cap \operatorname{Ker} A = 0$. Let V be an $n \times m$ matrix whose columns constitute a basis of S. Then there exists a unique pair of positive semidefinite matrices B_0 and C_0 such that $A = B_0 + C_0$, where B_0 satisfies $B_0 V = AV$ (or $C_0 V = 0$) and has rank m. Furthermore, B_0 and C_0 are given by*

$$B_0 = AV(V^{\mathrm{T}}AV)^{-1}V^{\mathrm{T}}A, \tag{11}$$

$$C_0 = A - B_0, \tag{12}$$

and the rank of C_0 is $l - m$.

Proof. Refer to Cao and Schwartz (2001a). □

The condition $S \cap \operatorname{Ker} A = 0$ ensures that the matrix $V^{\mathrm{T}}AV$ is invertible and $l \geqslant m$ (Cao & Schwartz, 2001a). We call $S \cap \operatorname{Ker} A = 0$ the *decomposable condition*. If the rank of A is equal to m, then $B_0 = A$, $C_0 = 0$. From $C_0 V = 0$ we have $S \subset \operatorname{Ker} C_0$. In the case where A is positive definite, we have rank $C_0 = n - m$ and hence $\operatorname{Ker} C_0 = S$.

[1] $\operatorname{Ker} C$ denotes the kernel space of C.

[2] As shown in Cao and Schwartz (2001a), $\operatorname{Ker} A \cap S = 0$ implies $\operatorname{rank}(A) \geqslant m$. Therefore, we often only state $\operatorname{Ker} A \cap S = 0$ without explicitly saying $\operatorname{rank}(A) \geqslant m$.

Corresponding to the property $S \subset \operatorname{Ker} C_0$, for the kernel space of B_0 we have the following lemma.

Lemma 2.1. *Let B_0 be defined by* (11). *Let x be a vector in R^n. Then $x \in \operatorname{Ker} B_0$ if and only if $Ax \in S^{\perp}$. That is,*

$$\operatorname{Ker} B_0 = \{x \in R^n \mid Ax \in S^{\perp}\}.$$

Proof. See the appendix. □

It should be noted that the conditions for the unique decomposition are $BV = AV$ and $\operatorname{rank} B = m$. As indicated in Remark 2.8 of Cao and Schwartz (2001a), if instead of requiring $\operatorname{rank} B = m$, we require that $\operatorname{rank} C = l - m$, then there are many positive semidefinite pairs of B and C satisfying $BV = AV$ and $\operatorname{rank} B \geqslant m$. Here we establish the following relationship between the unique decomposition given in Theorem 2.1 and the other orthogonal decompositions.

Theorem 2.2. *For any positive semidefinite pair B and C that constitutes an orthogonal decomposition of A, they satisfy*

$$B \geqslant B_0, \quad C \leqslant C_0,$$

where B_0 and C_0 are the unique orthogonal decomposition given in Theorem 2.1. *Furthermore, the ranks of B and C satisfy*

$$\operatorname{rank}(B) \geqslant m \tag{13}$$

$$\operatorname{rank}(A) - \operatorname{rank}(B) \leqslant \operatorname{rank}(C) \leqslant \operatorname{rank}(A) - m. \tag{14}$$

Proof. See the appendix. □

Theorem 2.1 characterizes the unique orthogonal decomposition of a positive semidefinite matrix along the given subspace S, while Theorem 2.2 characterizes all of the orthogonal decompositions along S based on the unique positive semidefinite pair B_0 and C_0. These two theorems underlie the theoretical analysis on the KFB algorithms given in the next section. Theorems 2.1 and 2.2 state that among all the orthogonal decompositions along S, the unique positive semidefinite matrix B_0 is minimal and has the feasible minimal rank m; and the unique positive semidefinite matrix C_0 is maximal and has the feasible maximal rank.

When analyzing the KFB algorithm, we often need to decompose a positive definite matrix which is equal to the sum of two positive semidefinite matrices. Assume that the positive definite matrix A has the form: $A = A_1 + A_2$, where both A_1 and A_2 are positive semidefinite and satisfy the decomposable condition. Based on Theorem 2.1, A_1 and A_2 can be decomposed as

$$A_1 = B_1 + C_1, \tag{15}$$

$$A_2 = B_2 + C_2, \tag{16}$$

where $\operatorname{rank}(B_1) = \operatorname{rank}(B_2) = m$, $B_1 V = A_1 V$, and $B_2 V = A_2 V$. Define

$$B = B_1 + B_2, \tag{17}$$

$$C = C_1 + C_2. \tag{18}$$

Then we have $A = B + C$. We can also see that the pair B and C forms an orthogonal decomposition along S and therefore $\operatorname{rank}(B) \geqslant m$ and $\operatorname{rank}(C) \leqslant \operatorname{rank}(A) - m$. Let the pair B_0 and C_0 be the unique orthogonal decomposition of A along S. Then generally $B_0 \neq B_1 + B_2$ and $C_0 \neq C_1 + C_2$. That is, the unique orthogonal decomposition of the sum of two matrices A_1 and A_2 is not, in general, equal to the sum of the two corresponding decompositions of the two matrices A_1 and A_2. Based on Theorem 2.2 we have the following relationship between B_0 and B, and C_0 and C.

$$B_0 \leqslant B = B_1 + B_2,$$

$$C_0 \geqslant C = C_1 + C_2.$$

3. Boundedness of the Kalman filter based algorithm

The estimator windup phenomenon in the EWLS algorithm is well known and is characterized as the exponential growth of some elements in the covariance matrix if the regression vector sequence φ is not persistently exciting (Åström & Wittenmark, 1995). The similar phenomenon in the KFB parameter estimator seems not to have been analyzed sufficiently. As shown in Niedźwiecki (2000, p. 284), it is quite easy to show that this kind of phenomenon does exist in the standard Kalman filter algorithm when no excitation is provided ($\varphi_t = 0$ for all t), and the covariance matrix tends to infinity at a linear rate in this case. However, a strictly theoretical analysis on the windup phenomenon in the KFB algorithm with a constant matrix Q_t has only recently been derived in Cao and Schwartz (2001b) for a relatively general excitation condition. Here we will extend the analysis to the more general case where Q_t could be time-varying.

The concepts of persistency of excitation and the excited subspace as well as the unexcited subspace are key properties in the paper. Their definitions are given below.

Persistency of excitation. The n-dimensional regression vector sequence $\varphi_t \in R^n$ is called persistently exciting in s steps if there exist constant $0 < a < \infty$ and an integer $s > 0$ such that

$$\sum_{i=t+1}^{t+s} \varphi_i \varphi_i^{\mathrm{T}} \geqslant aI \tag{19}$$

for all t.

This definition states that the n-dimensional real number space R^n can be spanned by φ_t uniformly in s steps when φ_t is persistently exciting.

The unexcited subspace. The following set:

$$\phi_{\mathrm{u}} = \{x \in R^n \mid x^{\mathrm{T}} \varphi_t = 0,\ \forall t\}$$

is defined as the unexcited subspace.

The above definition of the unexcited subspace is basically the same as that in Sethares, Lawrence, Johnson, and Bitmead (1986). Under this definition, the unexcited subspace is the collection of the directions in R^n which are never excited.[3]

The excited subspace. The orthogonal complement of ϕ_{u}, denoted by ϕ_{e}, is defined as the excited subspace.

The excited subspace is actually spanned by the regression vector sequence φ_t. In Sethares et al. (1986), the excited subspace ϕ_{e} is further decomposed into three subspaces based on the excitation condition. In this paper, we will consider the case where ϕ_{e} can be decomposed into two orthogonal subspaces: the persistently excited subspace ϕ_{p} and the subspace of decreasing excitation ϕ_{d} (Sethares et al., 1986). In Bittanti, Bolzern, and Campi (1990a), a similar definition for ϕ_{d} is introduced, where ϕ_{d} is called the unexcitation subspace. The following definition of the subspace of decreasing excitation is based on Bittanti et al. (1990a).

The subspace of decreasing excitation. The following set

$$\phi_{\mathrm{d}} = \left\{ x \in \phi_{\mathrm{e}} \mid \exists L < \infty, x^{\mathrm{T}} \sum_{1}^{N} \varphi_t \varphi_t^{\mathrm{T}} x < L,\ \forall N > 0 \right\}$$

is defined as the subspace of decreasing excitation.

It can be shown that for any $x \in \phi_{\mathrm{d}}, x^{\mathrm{T}}\varphi_t \to 0$ as $t \to \infty$. Therefore, each direction in ϕ_{d} is decreasingly excited.

The persistently excited subspace. The orthogonal complement of ϕ_{d} in ϕ_{e}, denoted by ϕ_{p} is defined as the persistently excited subspace.

It can be shown that for any $x \neq 0$ in ϕ_{p}, there exist a positive number a and an integer $s > 0$ such that

$$x^{\mathrm{T}} \sum_{i=t+1}^{t+s} \varphi_i \varphi_i^{\mathrm{T}} x \geqslant a \tag{20}$$

for all t. Inequality (20) indicates that ϕ_{p} is persistently excited.

Based on the above definitions, we can decompose the regressor φ_t as $\varphi_t = \varphi_{t,\mathrm{p}} + \varphi_{t,\mathrm{d}}$, where $\varphi_{t,\mathrm{p}} \in \phi_{\mathrm{p}}$ is called the persistently exciting component of φ_t, and $\varphi_{t,\mathrm{d}} \in \phi_{\mathrm{d}}$ is called the decreasingly exciting component. One can see that $\varphi_{t,\mathrm{d}} \to 0$ as $t \to \infty$. This is called the *asymptotic* zero excitation property.

In the following, we will analyze the behavior of the covariance matrix P_t under the condition that there exists an unexcited subspace ϕ_{u}. Furthermore, we assume that the dimension of ϕ_{e} is m and hence the dimension of ϕ_{u} is $n-m$.

[3] This definition seems very unrealistic because ϕ_{u} may never exist in the real world applications. However, as long as for a sufficiently long period $x^{\mathrm{T}}\varphi_t = 0$ or $x^{\mathrm{T}}\varphi_t$ is sufficiently small, then the definition is applicable and useful, just as the definition of persistency of excitation.

Now consider the update equation (5). Assume that P_t starts at $P_0 > 0$ and $Q_t \geqslant 0$. Let S_φ be an $n \times m$ matrix whose columns constitute a basis of the excited subspace ϕ_{e}. We can decompose P_t in the following way according to Theorem 2.1:

$$P_t = P_{t,\mathrm{o}} + P_{t,\mathrm{p}}, \tag{21}$$

where $P_{t,\mathrm{o}}$ and $P_{t,\mathrm{p}}$ are positive semidefinite and given by

$$P_{t,\mathrm{p}} = P_t S_\varphi (S_\varphi^{\mathrm{T}} P_t S_\varphi)^{-1} S_\varphi^{\mathrm{T}} P_t, \tag{22}$$

$$P_{t,\mathrm{o}} = P_t - P_{t,\mathrm{p}}. \tag{23}$$

From Theorem 2.1 we have $P_{t,\mathrm{o}} S_\varphi = 0$ and the rank of $P_{t,\mathrm{o}}$ is $n-m$.

Similarly, we can decompose Q_t in the same way as follows:

$$Q_t = Q_{t,\mathrm{o}} + Q_{t,\mathrm{p}}, \tag{24}$$

where $Q_{t,\mathrm{o}}$ and $Q_{t,\mathrm{p}}$ are positive semidefinite and given by

$$Q_{t,\mathrm{p}} = Q_t S_\varphi (S_\varphi^{\mathrm{T}} Q_t S_\varphi)^{-1} S_\varphi^{\mathrm{T}} Q_t, \tag{25}$$

$$Q_{t,\mathrm{o}} = Q_t - Q_{t,\mathrm{p}}. \tag{26}$$

Based on Theorems 2.1 and 2.2, we can give a lower bound for P_t in the KFB algorithm in terms of the matrix sequence Q_t, which is stated in the following theorem.

Theorem 3.1. *Assume that the regression vector sequence $\varphi_t \in R^n$ only spans an m-dimensional subspace in R^n. Then $P_{t,\mathrm{o}}$, the orthogonal part of the covariance matrix P_t to the excited subspace given by* (5), *satisfies the following matrix inequality*:

$$P_{t,\mathrm{o}} \geqslant P_{0,\mathrm{o}} + \sum_{i=0}^{t} Q_{i,\mathrm{o}}. \tag{27}$$

Proof. Define the following matrix:

$$\bar{P}_{t-1} = P_{t-1} - \frac{P_{t-1}\varphi_t\varphi_t^{\mathrm{T}}P_{t-1}}{r + \varphi_t^{\mathrm{T}}P_{t-1}\varphi_t}. \tag{28}$$

Then from (5) we have

$$P_t = \bar{P}_{t-1} + Q_t. \tag{29}$$

According to Theorem 2.1, we can decompose $\bar{P}_{t-1}$ along the excited subspace ϕ_{e} as

$$\bar{P}_{t-1} = \bar{P}_{t-1,\mathrm{o}} + \bar{P}_{t-1,\mathrm{p}}, \tag{30}$$

where $\bar{P}_{t-1,\mathrm{o}}$ and $\bar{P}_{t-1,\mathrm{p}}$ are positive semidefinite and given by

$$\bar{P}_{t-1,\mathrm{p}} = \bar{P}_{t-1} S_\varphi (S_\varphi^{\mathrm{T}} \bar{P}_{t-1} S_\varphi)^{-1} S_\varphi^{\mathrm{T}} \bar{P}_{t-1}, \tag{31}$$

$$\bar{P}_{t-1,\mathrm{o}} = \bar{P}_{t-1} - \bar{P}_{t-1,\mathrm{p}}. \tag{32}$$

From Theorem 2.1 we have $\bar{P}_{t-1,\mathrm{o}} S_\varphi = 0$. Furthermore, we know that the rank of $\bar{P}_{t-1,\mathrm{o}}$ is $n-m$ and the rank of $\bar{P}_{t-1,\mathrm{p}}$ is m.

Similarly, we can decompose Q_t in the same way

$$Q_t = Q_{t,\mathrm{o}} + Q_{t,\mathrm{p}}, \tag{33}$$

$$Q_{t,\mathrm{p}} = Q_t S_\varphi (S_\varphi^{\mathrm{T}} Q_t S_\varphi)^{-1} S_\varphi^{\mathrm{T}} Q_t, \tag{34}$$

$$Q_{t,\mathrm{o}} = Q_t - Q_{t,\mathrm{p}}, \tag{35}$$

where $Q_{t,\mathrm{o}}$ and $Q_{t,\mathrm{p}}$ are positive semidefinite and satisfy the same conditions as $\bar{P}_{t-1,\mathrm{o}}$ and $\bar{P}_{t-1,\mathrm{p}}$.

Thus, Eq. (29) can be written as

$$P_t = (\bar{P}_{t-1,\mathrm{o}} + Q_{t,\mathrm{o}}) + (\bar{P}_{t-1,\mathrm{p}} + Q_{t,\mathrm{p}}). \tag{36}$$

Based on Theorem 2.2, we can get the following inequality:

$$P_{t,\mathrm{o}} \geqslant \bar{P}_{t-1,\mathrm{o}} + Q_{t,\mathrm{o}}. \tag{37}$$

From (28) we have

$$\begin{aligned}\bar{P}_{t-1} &= P_{t-1} - \frac{P_{t-1}\varphi_t\varphi_t^{\mathrm{T}}P_{t-1}}{r + \varphi_t^{\mathrm{T}}P_{t-1}\varphi_t} \\ &= P_{t-1,\mathrm{o}} + P_{t-1,\mathrm{p}} - \frac{P_{t-1,\mathrm{p}}\varphi_t\varphi_t^{\mathrm{T}}P_{t-1,\mathrm{p}}}{r + \varphi_t^{\mathrm{T}}P_{t-1,\mathrm{p}}\varphi_t} \\ &\quad (\text{because } P_{t-1,\mathrm{o}}\varphi_t = 0), \\ &= P_{t-1,\mathrm{o}} + \hat{P}_{t-1,\mathrm{p}},\end{aligned} \tag{38}$$

where $\hat{P}_{t-1,\mathrm{p}} \geqslant 0$ is defined by

$$\hat{P}_{t-1,\mathrm{p}} = P_{t-1,\mathrm{p}} - \frac{P_{t-1,\mathrm{p}}\varphi_t\varphi_t^{\mathrm{T}}P_{t-1,\mathrm{p}}}{r + \varphi_t^{\mathrm{T}}P_{t-1,\mathrm{p}}\varphi_t}. \tag{39}$$

From (38) we have $\hat{P}_{t-1,\mathrm{p}}S_\varphi = \bar{P}_{t-1}S_\varphi$. Based on Lemma 4.4 in Cao and Schwartz (2001a), we can see that

$$\mathrm{rank}(\hat{P}_{t-1,\mathrm{p}}) = \mathrm{rank}(P_{t-1,\mathrm{p}}) = m.$$

Then from (38) we see that the pair $P_{t-1,\mathrm{o}}$ and $\hat{P}_{t-1,\mathrm{p}}$ constitutes the unique orthogonal decomposition of $\bar{P}_{t-1}$. Thus, we have

$$\bar{P}_{t-1,\mathrm{o}} = P_{t-1,\mathrm{o}}, \tag{40}$$

$$\bar{P}_{t-1,\mathrm{p}} = \hat{P}_{t-1,\mathrm{p}}. \tag{41}$$

From (40) and (37) we get

$$P_{t,\mathrm{o}} \geqslant P_{t-1,\mathrm{o}} + Q_{t,\mathrm{o}}, \tag{42}$$

which leads to (27). □

Since $P_t \geqslant P_{t,\mathrm{o}}$, inequality (42) gives a lower bound on P_t. On the other hand, it is quite easy to get an upper bound on P_t as follows.

From (5) it is obvious that

$$P_t \leqslant P_{t-1} + Q_t. \tag{43}$$

Using the above inequality recursively, one can get

$$P_t \leqslant P_0 + \sum_{i=0}^{t} Q_i. \tag{44}$$

From Theorem 3.1 we can see that if the sum of the orthogonal component of Q_t to the excited subspace is unbounded, then estimator windup happens in the KFB algorithm. This includes the case where Q_t is a positive definite constant matrix. In the case of Q_t being a constant matrix $Q > 0$, we can get the following result based on Theorem 3.1.

Corollary 3.1. *Assume that the regression vector sequence $\varphi_t \in R^n$ only spans a m-dimensional subspace in R^n and Q_t is equal to a constant matrix $Q > 0$ in (5). Then there are $n - m$ eigenvalues of P_t given in (5) which will tend to infinity as $t \to \infty$.*

Proof. Based on Theorem 3.1, we have

$$P_t \geqslant P_{t,\mathrm{o}} \geqslant P_{0,\mathrm{o}} + tQ_{\mathrm{o}}. \tag{45}$$

Because the ranks of Q_{o} and $P_{0,\mathrm{o}}$ are $n - m$, therefore they have $n - m$ nonzero eigenvalues. Let the eigenvalues $\lambda_k(P_t), \lambda_k(P_{0,\mathrm{o}} + tQ_{\mathrm{o}})$ and $\lambda_k(Q_{\mathrm{o}})$ be arranged in increasing order. Then according to Corollary 7.7.4 and the Weyl Theorem in Horn and Johnson (1985), we have for $k = m+1, m+2, \ldots, n$

$$\begin{aligned}\lambda_k(P_t) &\geqslant \lambda_k(P_{t,\mathrm{o}}) \geqslant \lambda_k(P_{0,\mathrm{o}} + tQ_{\mathrm{o}}) \\ &\geqslant t\lambda_k(Q_{\mathrm{o}})\end{aligned} \tag{46}$$

which shows that there are $n - m$ eigenvalues of P_t that tend to infinity as $t \to \infty$. □

Inequality (46) shows that if there is an unexcited subspace and $Q_t = Q > 0$, then some of the eigenvalues of P_t will tend to infinity at least at a linear rate. We can also show that these eigenvalues will increase exactly at a linear rate as follows. Replacing Q_t with Q in (44) we get

$$P_t \leqslant P_0 + tQ. \tag{47}$$

Let the maximum eigenvalue of P_0 be $\lambda_{0,M}$. Then based on Corollary 7.7.4 and the Weyl Theorem in Horn and Johnson (1985), from (47) we can get for $k = m+1, m+2, \ldots, n$

$$\begin{aligned}\lambda_k(P_t) &\leqslant \lambda_k(P_0 + tQ) \\ &\leqslant \lambda_{0,M} + t\lambda_k(Q).\end{aligned} \tag{48}$$

Combining (46) and (48) in a compact form we get

$$t\lambda_k(Q_{\mathrm{o}}) \leqslant \lambda_k(P_t) \leqslant \lambda_{0,M} + t\lambda_k(Q) \tag{49}$$

for $k = m+1, m+2, \ldots, n$. Inequality (49) shows that there are $n - m$ eigenvalues of P_t that grow at a linear rate, which is determined by Q. This is quite different from the windup phenomenon in the EWLS algorithm where the covariance matrix grows exponentially. Generally, it can be expected that linear growth rate is much slower than an exponential growth rate, which means that the KFB algorithm with a constant matrix Q could be more robust to excitation failures than the EWLS algorithm (Niedźwiecki, 2000). Despite this significant difference, the KFB algorithm with a constant Q has the same tracking ability as that of the EWLS algorithm in the sense that both algorithms are exponentially convergent. These aspects suggest that the KFB algorithm could be a much better choice than the EWLS algorithm.

The possible presence of estimator windup in the KFB algorithms with unsufficient excitation indicates that the designer should be cautious in choosing Q_t when long-term unsufficient excitation is expected. In such a case, the conditions on Q_t for P_t being bounded from above is very helpful. From Theorem 3.1, one can see that a necessary condition for P_t being bounded from above is that the sum $\sum_i Q_{i,\mathrm{o}}$ is bounded. Fortunately, by using the orthogonal decomposition approach and the associated results in Section 2 we can analyze the boundedness of P_t based on its decomposed positive semidefinite parts $P_{t,\mathrm{o}}$ and $P_{t,\mathrm{p}}$, and develop some sufficient condition for P_t being bounded in the case where the unexcited subspace exists.

Define the following matrix:

$$M_t = \bar{P}_{t-1,\mathrm{p}} + Q_{t,\mathrm{p}}. \tag{50}$$

Then (36) becomes

$$P_t = M_t + \bar{P}_{t-1,\mathrm{o}} + Q_{t,\mathrm{o}}. \tag{51}$$

Decomposing M_t along the excited subspace based on Theorem 2.1, we get

$$M_t = M_{t,\mathrm{p}} + \Delta_t, \tag{52}$$

where $M_{t,\mathrm{p}}$ satisfies $M_{t,\mathrm{p}} S_\varphi = M_t S_\varphi$ and the rank of $M_{t,\mathrm{p}}$ is m, and $\Delta_t \geqslant 0$ is the orthogonal part of M_t to the excited subspace and the rank of Δ_t is given by

$$\mathrm{rank}(\Delta_t) = \mathrm{rank}(M_t) - m. \tag{53}$$

From (51) we have $M_{t,\mathrm{p}} S_\varphi = P_t S_\varphi$. Thus based on Theorem 2.1 it must be true that $M_{t,\mathrm{p}} = P_{t,\mathrm{p}}$. Therefore we get

$$M_t = P_{t,\mathrm{p}} + \Delta_t. \tag{54}$$

From (51), (54) and (40) we can have

$$P_t = P_{t,\mathrm{p}} + P_{t-1,\mathrm{o}} + Q_{t,\mathrm{o}} + \Delta_t. \tag{55}$$

From the above equation we get

$$P_{t,\mathrm{o}} = P_{t-1,\mathrm{o}} + Q_{t,\mathrm{o}} + \Delta_t. \tag{56}$$

which is the update equation for the orthogonal part of P_t to the excited subspace. Based on (54), (50) and (41) we can get the update equation for $P_{t,\mathrm{p}}$ as follows

$$\begin{aligned} P_{t,\mathrm{p}} &= M_{t,\mathrm{p}} = M_t - \Delta_t, \\ &= P_{t-1,\mathrm{p}} - \frac{P_{t-1,\mathrm{p}}\varphi_t\varphi_t^{\mathrm{T}} P_{t-1,\mathrm{p}}}{r_t + \varphi_t^{\mathrm{T}} P_{t-1,\mathrm{p}}\varphi_t} + Q_{t,\mathrm{p}} - \Delta_t. \end{aligned} \tag{57}$$

Since both $P_{t,\mathrm{p}}$ and $P_{t,\mathrm{o}}$ are positive semidefinite, $P_t = P_{t,\mathrm{p}} + P_{t,\mathrm{o}}$ is bounded if and only if both $P_{t,\mathrm{p}}$ and $P_{t,\mathrm{o}}$ are bounded. Thus, we can analyze the boundedness of P_t by separately inspecting the boundedness of $P_{t,\mathrm{p}}$ and $P_{t,\mathrm{o}}$.

In the following, we will show that the boundedness of $P_{t,\mathrm{p}}$ can be analyzed in terms of a reduced-order version of the Kalman filter for which the unexcited subspace does not exist.

Define the following matrix

$$W = [U \;\; V], \tag{58}$$

where U is an $n \times m$ matrix whose columns constitute an orthonormal basis of the excited subspace ϕ_e, and V is an $n \times (n-m)$ matrix whose columns constitute an orthonormal basis of ϕ_u. One can see that W is an orthogonal matrix and satisfies

$$WW^{\mathrm{T}} = UU^{\mathrm{T}} + VV^{\mathrm{T}} = I. \tag{59}$$

Also note that $U^{\mathrm{T}}U = I$, but $UU^{\mathrm{T}} \neq I$. The same is true for V.

From (5) one can get

$$U^{\mathrm{T}} P_t U = U^{\mathrm{T}} P_{t-1} U - \frac{U^{\mathrm{T}} P_{t-1}\varphi_t\varphi_t^{\mathrm{T}} P_{t-1} U}{r + \varphi^{\mathrm{T}} P_{t-1}\varphi_t} + U^{\mathrm{T}} Q_t U. \tag{60}$$

Noting that $V^{\mathrm{T}}\varphi_t = 0$ we have

$$\begin{aligned} U^{\mathrm{T}} P_{t-1}\varphi_t\varphi_t^{\mathrm{T}} P_{t-1} U &= U^{\mathrm{T}} P_{t-1}(UU^{\mathrm{T}} + VV^{\mathrm{T}}) \\ &\quad \times \varphi_t\varphi_t^{\mathrm{T}}(UU^{\mathrm{T}} + VV^{\mathrm{T}}) P_{t-1} U \\ &= U^{\mathrm{T}} P_{t-1} UU^{\mathrm{T}}\varphi_t\varphi_t^{\mathrm{T}} UU^{\mathrm{T}} P_{t-1} U \end{aligned}$$

and

$$\varphi_t^{\mathrm{T}} P_{t-1}\varphi_t = \varphi_t^{\mathrm{T}} UU^{\mathrm{T}} P_{t-1} UU^{\mathrm{T}}\varphi_t^{\mathrm{T}}.$$

Define the $m \times m$ matrix S_t as

$$S_t = U^{\mathrm{T}} P_t U. \tag{61}$$

Then we can get the following update equation for S_t based on (60)

$$S_t = S_{t-1} - \frac{S_{t-1}\psi_t\psi_t^{\mathrm{T}} S_{t-1}}{r + \psi_t^{\mathrm{T}} S_{t-1}\psi_t} + O_t, \tag{62}$$

where the m-dimensional vector ψ_t is defined by

$$\psi_t = U^{\mathrm{T}}\varphi_t \tag{63}$$

and the $m \times m$ matrix O_t is defined by

$$O_t = U^{\mathrm{T}} Q_t U = U^{\mathrm{T}} Q_{t,\mathrm{p}} U. \tag{64}$$

One can see that (62) has exactly the same form as the update equation (5). Therefore, (62) is the update equation of the covariance matrix for a mth-order Kalman filter. We have the following theorem on the relationship between S_t and $P_{t,\mathrm{p}}$.

Theorem 3.2. *$P_{t,\mathrm{p}}$ is bounded if and only if S_t is bounded.*

Proof. Once again based on Lemma 2.11 in Cao and Schwartz (2001a), we have

$$\mathrm{Ker}\, P_{t,\mathrm{p}} \oplus \mathrm{Ker}\, P_{t,\mathrm{o}} = R^n.$$

Since P_t is positive definite for all t, we have $\mathrm{Ker}\, P_{t,\mathrm{o}} = \phi_\mathrm{e}$. Thus, any vector x of unit length in R^n can be written as

$$x = y + z,$$

where $y \in \mathrm{Ker}\, P_{t,\mathrm{p}}$, $z \in \phi_\mathrm{e}$. Therefore,

$$P_{t,\mathrm{p}} x = P_{t,\mathrm{p}} z.$$

Since $z \in \phi_\mathrm{e}$, we have $z = Uz_1$ for some $z_1 \in R^m$, $z^\mathrm{T} z = z_1^\mathrm{T} U^\mathrm{T} U z_1 = z_1^\mathrm{T} z_1$. Therefore,

$$\begin{aligned} x^\mathrm{T} P_{t,\mathrm{p}} x &= z^\mathrm{T} P_{t,\mathrm{p}} z \\ &= z_1^\mathrm{T} U^\mathrm{T} P_{t,\mathrm{p}} U z_1 \\ &= z_1 U^\mathrm{T} P_t U z_1 = z_1 S_t z_1. \end{aligned} \tag{65}$$

Noting that both x and z_1 are bounded, from (65) we see that the bounded S_t leads to the bounded $P_{t,\mathrm{p}}$.

Next, assume that $P_{t,\mathrm{p}}$ is bounded. For any vector $u \in R^m$ with unit length, we have from (61)

$$\begin{aligned} u^\mathrm{T} S_t u &= u^\mathrm{T} U^\mathrm{T} P_t U u \\ &= v^\mathrm{T} P_t v, \end{aligned}$$

where $v = Uu \in \phi_\mathrm{e}$, $v^\mathrm{T} v = u^\mathrm{T} u = 1$. Since $P_{t,\mathrm{o}} v = 0$, we get $P_t v = (P_{t,\mathrm{p}} + P_{t,\mathrm{o}}) v = P_{t,\mathrm{p}} v$. Therefore,

$$u^\mathrm{T} S_t u = v^\mathrm{T} P_{t,\mathrm{p}} v. \tag{66}$$

From (66) we see that the bounded $P_{t,\mathrm{p}}$ leads to bounded S_t. □

Theorem 3.2 indicates that the boundedness of $P_{t,\mathrm{p}}$ is equivalent to the boundedness of S_t. For any nonzero vector $z \in R^m$, we have $z^\mathrm{T} \psi_t = (Uz)^\mathrm{T} \varphi_t$, which can be zero only at a finite number of t. Therefore, for the mth order Kalman filter defined by the update equation (62), there is no unexcited subspace in its parameter space R^m. Thus, based on the orthogonal decomposition method developed by the authors the behavior of a Kalman filter within the excited subspace can be analyzed without considering the existence of the unexcited subspace. In particular, if the subspace of decreasing excitation $\phi_\mathrm{d} = \{0\}$, then $\phi_\mathrm{e} = \phi_\mathrm{p}$ and the sequence $\{\psi_t\}$ is persistently exciting. Thus, the known results and methods under the condition of persistent excitation can be applied directly to determine the boundedness of $P_{t,\mathrm{p}}$.

In the case $\phi_\mathrm{d} \neq \{0\}$, we can use the asymptotic zero excitation property of decreasing excitation to analyze the behavior of the covariance matrix P_t as $t \to \infty$. We know that as $t \to \infty$ the decreasing exciting component $\varphi_{t,\mathrm{d}} \to 0$. Thus, for any small number $\varepsilon > 0$ we can find $0 < T < \infty$ such that for all $t \geqslant T$, $|\varphi_{t,\mathrm{d}}| < \varepsilon$, which means that for $t \geqslant T$, $\varphi_{t,\mathrm{d}}$ can be neglected compared with the persistently exciting component $\varphi_{t,\mathrm{p}}$. Therefore, there exists a sufficiently large and finite $T > 0$ such that for $t \geqslant T$ the subspace ϕ_d can be virtually viewed as a part of the unexcited subspace. Thus, for $t \geqslant T$ we can analyze the behavior of P_t based on Theorem 3.1 and Corollary 3.1. For example, in the case of $Q_t = Q > 0$ and $\phi_\mathrm{d} \neq \{0\}$, based on Corollary 3.1 we can see that some eigenvalues of P_t will tend to infinity as $t \to \infty$. Thus, the presence of decreasing excitation does not bring any special problems to our analysis as long as the ultimate behavior of the KFB algorithm, such as the boundedness of the covariance matrix P_t, is only concerned.

Now, we turn to the boundedness of $P_{t,\mathrm{o}}$. From (56) we can get

$$P_{t,\mathrm{o}} = P_{0,\mathrm{o}} + \sum_{i=1}^{t} Q_{i,\mathrm{o}} + \sum_{i=1}^{t} \Delta_i. \tag{67}$$

From the above equation we see that $P_{t,\mathrm{o}}$ is bounded if and only if both $\sum_{i=1}^{\infty} Q_{i,\mathrm{o}}$ and $\sum_{i=1}^{\infty} \Delta_i$ are bounded. Here, we should note that the bounded $\sum_{i=1}^{\infty} Q_{i,\mathrm{o}}$ alone does not guarantee the boundedness of $P_{t,\mathrm{o}}$, and the bounded $\sum_{i=1}^{\infty} \Delta_i$ is also necessary for $P_{t,\mathrm{o}}$ to be bounded. From the definition of Δ_t (refer to (50) and(52)), we see that Δ_t is dependent on both $Q_{t,\mathrm{p}}$ and $P_{t,\mathrm{p}}$, but we do not have an explicit expression for it. Therefore, it is very difficult to analyze whether the infinite sum $\sum_{i=1}^{\infty} \Delta_i$ is convergent. In the following, we will develop a sufficient condition for the boundedness of $P_{t,\mathrm{o}}$, which is independent of Δ_t and hence is easy to work with.

Define the following $(n-m) \times (n-m)$ matrix

$$L_t = V^\mathrm{T} P_t V, \tag{68}$$

where the $n \times (n-m)$ matrix V is the same as in (58). Then we have the following result.

Theorem 3.3. *$P_{t,\mathrm{o}}$ is bounded from above if L_t is bounded.*

Proof. Any vector $x \in R^n$ can be written as

$$x = y + z$$

where $y \in \phi_\mathrm{e}$, $z \in \phi_\mathrm{u}$. Therefore,

$$x^\mathrm{T} P_{t,\mathrm{o}} x = z^\mathrm{T} P_{t,\mathrm{o}} z. \tag{69}$$

The above equation indicates that $P_{t,\mathrm{o}}$ is bounded if $z^\mathrm{T} P_{t,\mathrm{o}} z$ is bounded for any vector $z \in \phi_\mathrm{u}$.

From $P_{t,\mathrm{o}} \leqslant P_t$ we get

$$z^\mathrm{T} P_{t,\mathrm{o}} z \leqslant z^\mathrm{T} P_t z.$$

The vector z can be written as

$$z = V z_1$$

for some vector $z_1 \in R^{n-m}$. Therefore, we have

$$z^\mathrm{T} P_{t,\mathrm{o}} z \leqslant z^\mathrm{T} P_t z = z_1^\mathrm{T} L_t z_1. \tag{70}$$

The conclusion follows from (69) and (70). □

Similar to Theorem 3.2, Theorem 3.3 connects the boundedness of $P_{t,\mathrm{o}}$ with that of the reduced-dimension matrix L_t.

Define the following matrix:

$$N_t = V^{\mathrm{T}} Q_t V, \tag{71}$$

where V is the same as in (58). Then we have

Corollary 3.2. *$P_{t,\mathrm{o}}$ is bounded from above if the sum $\sum_{i=1}^{\infty} N_i$ is convergent.*

Proof. We have

$$P_t \leqslant P_{t-1} + Q_t.$$

Therefore,

$$V^{\mathrm{T}} P_t V \leqslant V^{\mathrm{T}} P_{t-1} V + V^{\mathrm{T}} Q_t V$$

or

$$L_t \leqslant L_{t-1} + N_t. \tag{72}$$

From (72) we get

$$L_t \leqslant L_0 + \sum_{i=1}^{t} N_i. \tag{73}$$

From (73) and Theorem 3.3 the conclusion follows. □

The condition of the boundedness of $P_{t,\mathrm{o}}$ given in Corollary 3.2 is only dependent on Q_t and hence is much easier to check than the infinite sum of Δ_t (refer to Eq. (67)). From Corollary 3.2 we can further develop useful insight into the choice of Q_t, as will be shown soon.

For any vector $x \in R^{n-m}$ we have

$$\begin{aligned} x^{\mathrm{T}} \sum_{i=1}^{\infty} N_i x &= x^{\mathrm{T}} \left(\sum_{i=1}^{\infty} V^{\mathrm{T}} Q_i V \right) x \\ &= (Vx)^{\mathrm{T}} \left(\sum_{i=1}^{\infty} Q_i \right) Vx \\ &= y^{\mathrm{T}} \left(\sum_{i=1}^{\infty} Q_i \right) y, \end{aligned} \tag{74}$$

where $y = Vx \in \phi_{\mathrm{u}}$. From (74) we see that the boundedness of $\sum_{i=1}^{\infty} N_i$ is equivalent to the condition

$$y^{\mathrm{T}} \left(\sum_{i=1}^{\infty} Q_i \right) y < \infty, \quad \forall y \in \phi_{\mathrm{u}}. \tag{75}$$

Thus, the condition given in (75) is also a sufficient condition for the boundedness of $P_{t,\mathrm{o}}$. Now consider the orthogonal decomposition of Q_t along the *unexcited* subspace ϕ_{u}.[4] Assume that Q_t satisfies the decomposable condition defined in Theorem 2.1, then along ϕ_{u}, Q_t can be decomposed as

$$Q_t = Q_{t,\mathrm{o}}^{\mathrm{u}} + Q_{t,\mathrm{p}}^{\mathrm{u}} \tag{76}$$

where $Q_{t,\mathrm{o}}^{\mathrm{u}}$ is the orthogonal part to the unexcited subspace, that is, $Q_{t,\mathrm{o}}^{\mathrm{u}} x = 0$ for any nonzero vector $x \in \phi_{\mathrm{u}}$. The rank of $Q_{t,\mathrm{o}}^{\mathrm{u}}$ is m, and the rank of $Q_{t,\mathrm{p}}^{\mathrm{u}}$ is $n-m$. The matrix N_t defined in (71) can be written as

$$N_t = V^{\mathrm{T}} Q_{t,\mathrm{p}}^{\mathrm{u}} V. \tag{77}$$

Thus (75) becomes

$$y^{\mathrm{T}} \left(\sum_{i=1}^{\infty} Q_{i,\mathrm{p}}^{\mathrm{u}} \right) y < \infty, \quad \forall y \in \phi_{\mathrm{u}}. \tag{78}$$

Therefore, if the matrix sum $\sum_{i=1}^{\infty} Q_{i,\mathrm{p}}^{\mathrm{u}}$ is bounded, so is $\sum_{i=1}^{\infty} N_i$. Then based on Corollary 3.2 one can see that the boundedness of $\sum_{i=1}^{\infty} Q_{i,\mathrm{p}}^{\mathrm{u}}$ is a *sufficient* condition for the boundedness of $P_{t,\mathrm{o}}$, while the boundedness of $\sum_{i=1}^{\infty} Q_{i,\mathrm{o}}$ is a *necessary* condition for the boundedness of $P_{t,\mathrm{o}}$ (refer to (67)).

We have

$$\begin{aligned} x^{\mathrm{T}} \sum_{i=1}^{\infty} N_i x &= y^{\mathrm{T}} \sum_{i=1}^{\infty} Q_i y \\ &= y^{\mathrm{T}} Q_1 y + \cdots + y^{\mathrm{T}} Q_t y + \cdots, \end{aligned}$$

where $y = Vx \in \phi_{\mathrm{u}}$. If $\sum_{i=1}^{\infty} N_i$ is bounded, it must be true that

$$y^{\mathrm{T}} Q_t y \to 0 \quad \text{as } t \to \infty. \tag{79}$$

Since $Q_t \geqslant 0$, then (79) means (refer to Horn & Johnson, 1985, p. 400)

$$Q_t y \to 0 \quad \text{as } t \to \infty. \tag{80}$$

Eq. (80) is the *necessary* condition for the boundedness of $\sum_{i=1}^{\infty} N_i$. Noting that (80) should be true for any vector in ϕ_{u}, (80) means that in order to keep $\sum_{i=1}^{\infty} N_i$ bounded the unexcited subspace ϕ_{u} should asymptotically become the kernel space of Q_t as $t \to \infty$. In other words, as $t \to \infty$, Q_t should asymptotically become singular and its $n-m$ eigenvalues should tend to zero with the associated eigenvectors belonged to the unexcited subspace. These observations may be helpful in the choice of Q_t.

4. Directional tracking algorithms based on the Kalman filter

4.1. Directional forgetting and directional tracking

In the previous section, it has been shown that estimator windup does exist in the standard Kalman filter based algorithm when the regressor is not persistently exciting, and it is characterized as linear growth of the covariance matrix. Compared with exponential estimator windup in the exponentially weighted least squares (EWLS) algorithm, linear estimator windup may not cause severe consequences due to the fact that the covariance matrix grows linearly rather than exponentially. However, windup can never be positive in any estimation algorithms, because unbounded growth of the covariance matrix means that the algorithm may become

[4] Up to now, the orthogonal decompositions we have used are conducted based on the excited subspace ϕ_{e}.

extremely sensitive to noise and disturbances. Windup is a potential threat to the stability and performance of an algorithm. In addition, as indicated in Salgado et al. (1988) and Parkum et al. (1992) the boundedness of P_t (expressed in (6) and (7)) is of fundamental importance for an estimation algorithm, as it is the key property in connection with the performance analysis of adaptive systems. Therefore, it is desirable and significant to develop parameter estimation algorithms that can overcome the windup drawback. Theorems 3.2–3.5 as well as Corollaries 3.2 and 3.3 established in the previous section can provide us useful directions as to develop such algorithms.

To overcome the windup problem in the EWLS algorithm, many modified EWLS algorithms have been proposed during the last two decades. These algorithms can be characterized either as nonuniform time forgetting (time-varying forgetting) or nonuniform space forgetting (directional forgetting), or a combination of these two (selective forgetting in Parkum et al., 1992). For the windup problem in the Kalman filter based algorithm, relatively few research results have been reported in the literature. Among them are the fading memory Kalman filter algorithm (Niedźwiecki, 2000) and the modified KFB algorithm (Cao & Schwartz, 2001b) derived based on the directional forgetting method of Cao and Schwartz (2000).

In this section, we will develop some modified Kalman filter based algorithms by choosing an appropriate matrix series $\{Q_t\}$. We call these algorithms the directional tracking Kalman filter based (DTKFB) algorithm. To explain why these algorithms are characterized as *directional tracking*, we take a look at the following update equation for the information matrix in the EWLS algorithm

$$R_t = \mu R_{t-1} + \varphi_t \varphi_t^{\mathrm{T}}, \tag{81}$$

where $\mu < 1$ is the forgetting factor. Obviously, at each update the old information contained in R_{t-1} is discounted uniformly in all directions and thus windup takes place. The directional forgetting strategy is to modify the above update equation for R_t so that the old information is only discounted in certain directions at each update.

On the other hand, the Kalman filter is described by the updated equation for the covariance matrix P_t (refer to (5)), and generally no update equation is explicitly formed for the information matrix. The tracking ability of the algorithm is obtained by ensuring $Q_t \geqslant 0$. Since $P_t > Q_t$, it can be seen that if $Q > 0$ then the algorithm can track the time-varying parameters in any direction. As shown in Ljung and Gunnarsson (1990), the EWLS algorithm can be viewed as a special case of the Kalman filter with a specific Q_t which is not singular. This example shows that there is a direct connection between the forgetting directions and tracking directions. If Q_t is singular for some t, then tracking can only happen in certain directions at these time instants. Therefore, by adjusting Q_t we can control the tracking directions. In a general sense, any algorithm that uses a singular Q_t during some period has the directional tracking property. Here we will focus on the algorithms that track the time-varying parameters only in the excited subspace.

Directional tracking and directional forgetting are dual concepts in the estimation methods, and therefore, they are of equal significance. Directional forgetting is based on the update equation of the information matrix, which determines how the old information is discounted when new information is available. The concept of directional tracking is applied to the update equation of the covariance matrix, which determines the algorithm's gain vector and hence its tracking direction. Generally speaking, unlike the covariance matrix the information matrix is not involved with the implementation of a recursive algorithm. The information matrix is mainly used in deriving an algorithm and analyzing its performance. When an algorithm is derived in terms of the information matrix, the inverse of the information matrix, which appears as the covariance matrix, must be given in a recursive form in order to avoid matrix inversion operation at each update. Therefore, designing a recursive algorithm directly based on the covariance matrix is implementation orientated and may be more computationally efficient than the algorithm designed based on the information matrix. The idea of directional tracking is useful in developing such kinds of computationally efficient algorithms.

4.2. Directional tracking algorithms

In this section, we will propose two kinds of directional tracking algorithms which have the property of tracking time-varying parameters only in the excited subspace. The idea is based on the fundamental principle of parameter estimation: tracking can happen in some direction only if there is an excitation in the same direction. Based on this principle, an estimation algorithm should track time-varying parameters *only within the excited subspace*. This requires that the rank of the matrix Q_t should asymptotically coincide with the dimension of the excited subspace. The attempt to track in unexcited directions is *useless* or even dangerous.

To evaluate the proposed algorithms, we will analyze the boundedness of the covariance matrix P_t in two situations: (1) φ_t is persistently exciting; (2) φ_t is not persistently exciting and there exists an unexcited subspace, but the subspace of decreasing excitation does not exist. [5] For the case of nonpersistent excitation, we will only consider the boundedness of $P_{t,\mathrm{o}}$, the orthogonal part of P_t to the excited subspace. As has been shown in the previous section, the boundedness of $P_{t,\mathrm{p}}$ can be analyzed based on the condition of persistent excitation when the subspace of decreasing excitation does not exist. Therefore, for the case of nonpersistent excitation we will not discuss the boundedness of $P_{t,\mathrm{p}}$, since it is completely the same to the boundedness of P_t with persistent excitation.

[5] As has been stated in Section 3, the boundedness of P_t as $t \to \infty$ in the case of decreasing excitation can be treated as the case where an unexcited subspace exists.

4.2.1. Directional tracking algorithm with rank one Q_t matrix

In this kind of directional tracking algorithm, the rank of Q_t is required to be one for all t. Therefore, Q_t can be written as

$$Q_t = \gamma\psi_t\psi_t^{\mathrm{T}}, \tag{82}$$

where $\gamma > 0$ is a scalar and ψ_t is a vector that belongs to the excited subspace ϕ_{e}. The update equation for P_t becomes

$$P_t = P_{t-1} - \frac{P_{t-1}\varphi_t\varphi_t^{\mathrm{T}}P_{t-1}}{r + \varphi_t^{\mathrm{T}}P_{t-1}\varphi_{t-1}} + \gamma\psi_t\psi_t^{\mathrm{T}}. \tag{83}$$

With the Q_t defined in (82), it is easy to see that its orthogonal part $Q_{t,\mathrm{o}}$ to the excited subspace is a zero matrix. To specify ψ_t, we require that ψ_t is persistently exciting whenever φ_t is. In addition, if φ_t only excites a subspace in R^n, then ψ_t should excite the same subspace. By choosing ψ_t in such a way, we can obtain a symmetric update equation for the covariance matrix and information matrix as shown in the following.

Define the information matrix R_t as

$$R_t = (P_t - \gamma\psi_t\psi_t^{\mathrm{T}})^{-1} = P_{t-1}^{-1} + r^{-1}\varphi_t\varphi_t^{\mathrm{T}}. \tag{84}$$

By using the matrix inversion lemma one can find the update equation for R_t is

$$\begin{aligned} R_t &= (R_{t-1}^{-1} + \gamma\psi_{t-1}\psi_{t-1}^{\mathrm{T}})^{-1} + r^{-1}\varphi_t\varphi_t^{\mathrm{T}} \\ &= R_{t-1} - \frac{R_{t-1}\psi_t\psi_t^{\mathrm{T}}R_{t-1}}{\gamma^{-1} + \psi_t^{\mathrm{T}}R_{t-1}\psi_{t-1}} + r^{-1}\varphi_t\varphi_t^{\mathrm{T}}. \end{aligned} \tag{85}$$

Comparing (85) with (83) one can see that they are completely symmetric. The role of ψ_t in (83) is the same as that of φ_t in (85). In particular, if ψ_t and φ_t have the same property, then so do R_t and P_t. This structural symmetry between (83) and (85) has two advantages: (1) it helps to choose the vector series $\{\psi_t\}$; (2) it can simplify the analysis of P_t or R_t. Symmetry between the information matrix and covariance matrix is also noticed in Gunnarsson (1994), where the design method of Q_t matrix to prevent P_t from tending to zero is called covariance modification. In Gunnarsson (1994) the use of regularization [6] to avoid windup, and the relationship between covariance modification and regularization is discussed. As will be shown below, we can avoid windup by appropriately choosing ψ_t and no regularization is needed.

Symmetry between (83) and (85) suggests that one possible choice for ψ_t is

$$\psi_t = \frac{\varphi_t}{\sqrt{\varepsilon + \varphi_t^{\mathrm{T}}\varphi_t}}, \tag{86}$$

where ε is a positive scalar, which ensures that ψ_t is well defined even with $\varphi_t = 0$. Eq. (86) means that ψ_t is the normalized regressor φ_t. Choosing ψ_t according to (86) ensures that: (1) ψ_t is persistently exciting whenever φ_t is; (2) ψ_t is bounded in spite of the boundedness of φ_t. As will be shown later, property (2) is important when the algorithm is used in an adaptive control system.

Now the proposed directional tracking algorithm can be described by the following equations:

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t(y_t - \varphi_t^{\mathrm{T}}\hat{\theta}_{t-1}),$$

$$K_t = \frac{P_{t-1}\varphi_t}{r + \varphi_t^{\mathrm{T}}P_{t-1}\varphi_t},$$

$$P_t = P_{t-1} - \frac{P_{t-1}\varphi_t\varphi_t^{\mathrm{T}}P_{t-1}}{r + \varphi_t^{\mathrm{T}}P_{t-1}\varphi_t} + \frac{\gamma}{\varepsilon + \varphi_t^{\mathrm{T}}\varphi_t}\varphi_t\varphi_t^{\mathrm{T}}.$$

To simplify the notation, we will call the above equations Algorithm I.

With Q_t chosen as (82) the unexcited subspace is the kernel space of Q_t. Based on Corollary 3.2 we see that $P_{t,\mathrm{o}}$ is bounded from above. Therefore, there is no windup for Algorithm I in the case of nonpersistent excitation.

In the following, we will establish the boundedness of P_t for the case of persistent excitation for Algorithm I. First, we will show that P_t is bounded from above. Then based on the symmetric property between P_t and R_t, we show that P_t is also bounded from below.

Lemma 4.1. *Assume that $\{\psi_t\}$ is a bounded persistently exciting sequence of s steps. Then any vector x of unit length can be represented by*

$$x = \sum_{i=t+1}^{t+s} \sigma_x(i,t)\psi_t,$$

where the scalar $\sigma_x(i,t)$ is uniformly bounded, that is, there is a positive number d such that $|\sigma_x(i,t)| \leqslant d$ for all t and x, $|x| = 1$.

Lemma 4.1 is proposed in Bittanti, Bolzern, and Campi (1990b) and is needed in the proof of the following theorem.

Theorem 4.1. *Assume that φ_t is persistently exciting in s steps. Then the covariance matrix P_t of Algorithm I is bounded from above for all t.*

Proof. Basically we follow the approach of Bittanti et al. (1990b).

Since $P_t > 0$, one can write P_t as $P_t = M_t^2$, where M_t is a positive definite matrix. Thus, $x^{\mathrm{T}}P_{t+s}x = |M_{t+s}x|^2$. From (86) we have

$$|\psi_t| = \frac{|\varphi_t|}{\sqrt{\varepsilon + |\varphi_t|^2}} \leqslant 1. \tag{87}$$

Then based on Lemma 4.1 we can get

$$\begin{aligned} x^{\mathrm{T}}P_{t+s}x &= |M_{t+s}x|^2 \\ &\leqslant \sum_{i=t+1}^{t+s} \sigma_x^2(i,t)|M_{t+s}\psi_i|^2. \end{aligned}$$

[6] Regularization is usually to add a constant positive definite matrix to the information matrix, and this method generally increases computational complexity.

That is

$$x^{\mathrm{T}} P_{t+s} x \leqslant \sum_{i=t+1}^{t+s} \sigma_x^2(i,t) \psi_i^{\mathrm{T}} P_{t+s} \psi_i. \tag{88}$$

From (83) we can get the following inequality

$$\begin{aligned} P_t &\leqslant P_{t-1} + \gamma \psi_t \psi_t^{\mathrm{T}} \\ &\leqslant P_{t-1} + \gamma I. \end{aligned} \tag{89}$$

Recursively applying inequality (89) to the right hand side of (88) for all of the terms in the form $\psi_i^{\mathrm{T}} P_k \psi_i$, where $i < k$, until all of them having the form: $\psi_i^{\mathrm{T}} P_i \psi_i$, one can get the following inequality:

$$x^{\mathrm{T}} P_{t+s} x \leqslant \sum_{i=t+1}^{t+s} \sigma_x^2(i,t) \psi_i^{\mathrm{T}} P_i \psi_i + \sum_{i=t+1}^{t+s-1} \gamma_x(i,t) |\psi_i|^2, \tag{90}$$

where $\gamma_x(i,t) \geqslant 0$ is a function of $\sigma_x^2(i,t)$, $i \in [t+1, t+s-1]$ and γ. The uniform boundedness of $\sigma_x(i,t)$ leads to the uniform boundedness of $\gamma_x(i,t)$.

Based on (86) we have $\varphi_t = a_t \psi_t$, where a_t satisfies

$$a_t = \sqrt{\varepsilon + |\varphi_t|^2} \geqslant \sqrt{\varepsilon}.$$

From (83) we can get

$$\begin{aligned} \psi_t^{\mathrm{T}} P_t \psi_t &= \frac{r \psi_t^{\mathrm{T}} P_{t-1} \psi_t}{r + \varphi_t^{\mathrm{T}} P_{t-1} \varphi_t} + \gamma (\psi_t^{\mathrm{T}} \psi_t)^2 \\ &= r \frac{\psi_t^{\mathrm{T}} P_{t-1} \psi_t}{r + a_t^2 \psi_t^{\mathrm{T}} P_{t-1} \psi_t} + \gamma |\psi_t|^4 \\ &\leqslant \frac{r}{a_t^2} \frac{a_t^2 \psi_t^{\mathrm{T}} P_{t-1} \psi_t}{r + a_t^2 \psi_t^{\mathrm{T}} P_{t-1} \psi_t} + \gamma |\psi_t|^4 \\ &\leqslant \frac{r}{a_t^2} + \gamma \\ &\leqslant \frac{r}{\varepsilon} + \gamma. \end{aligned} \tag{91}$$

Substituting (91) into (90) we have

$$x^{\mathrm{T}} P_{t+s} x \leqslant \left(\frac{r}{\varepsilon} + \gamma \right) \sum_{i=t+1}^{t+s} \sigma_x^2(i,t) + \sum_{i=t+1}^{t+s-1} \gamma_x(i,t). \tag{92}$$

Noting that s is a finite integer, then from (92) and the uniform boundedness of $\sigma_x(i,t)$ and $\gamma_x(i,t)$, we conclude that $x^{\mathrm{T}} P_t x$ is bounded from above for all x ($|x| = 1$) and t. □

Remark 4.1. In Cao and Schwartz (2001b), the vector ψ_t is chosen as $\psi_t = b_t \varphi_{t+1}$ for some positive scalar b_t, and it is shown that P_t is bounded from above for the case of bounded regressor φ_t. Here, by choosing ψ_t as (86), we have proven that P_t is bounded from above *without* the assumption that φ_t is bounded.

Theorem 4.2. *Assume that φ_t is bounded and persistently exciting. Then there is a scalar $\alpha > 0$ such that $P_t \geqslant \alpha I$.*

Proof. Since φ_t is bounded and persistently exciting, Eqs. (83) and (85) are completely symmetric and R_t must have the same behavior as that of P_t. Then from Theorem 4.1 we see that there is a positive number σ such that $R_t \leqslant \sigma I$. From (84) it follows that

$$\begin{aligned} P_t &= R_t^{-1} + \gamma \psi_t \psi_t \geqslant R_t^{-1} \\ &\geqslant \sigma^{-1} I. \qquad \square \end{aligned}$$

Remark 4.2. Unlike the upper bound of P_t established in Theorem 4.1, the lower bound established in Theorem 4.2 is dependent on the assumption that φ_t is bounded. As indicated in Salgado et al. (1988), $\alpha I \leqslant P_t \leqslant \beta I$ is the key property in establishing the basic error properties of a algorithm, which are very useful in a wide range of applications in adaptive filtering and control (see also Goodwin & Sin, 1984). It is also indicated that these properties should hold irrespective of the boundedness of the regressor. In this context, Algorithm I may have some weakness if it is used in an adaptive control system because it is not theoretically proven that $P_t \geqslant \alpha I$ in the case of unbounded regressor. Fortunately, as remarked in Parkum et al. (1992) the basic error properties can be established based on the weaker condition $P_t > 0$. Thus, Algorithm I can guarantee the basic error properties in the case of unbounded regressor.

4.2.2. Directional tracking algorithm with varying rank Q_t

In this algorithm, Q_t has exactly the same form as the information matrix in the EWLS algorithm (refer to (81)),

$$Q_t = \mu Q_{t-1} + \gamma \psi_t \psi_t^{\mathrm{T}}. \tag{93}$$

In (93) ψ_t is chosen as (86). The algorithm is

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t (y_t - \varphi_t^{\mathrm{T}} \hat{\theta}_{t-1}),$$

$$K_t = \frac{P_{t-1} \varphi_t}{r + \varphi_t^{\mathrm{T}} P_{t-1} \varphi_t},$$

$$P_t = P_{t-1} - \frac{P_{t-1} \varphi_t \varphi_t^{\mathrm{T}} P_t}{r + \varphi_t^{\mathrm{T}} P_{t-1} \varphi_t} + Q_t,$$

$$Q_t = \mu Q_{t-1} + \frac{\gamma}{\varepsilon + \varphi_t^{\mathrm{T}} \varphi_t} \varphi_t \varphi_t^{\mathrm{T}}.$$

and is called Algorithm II.

We see that the tracking direction of Algorithm II depends not only on the current regression vector but also on the old regression vector, whose effects are discounted by the forgetting factor $\mu < 1$. We decompose Q_t into $Q_t = Q_{t,\mathrm{o}} + Q_{t,\mathrm{p}}$ based on Theorem 2.1. Then based on the well established properties for the information matrix in a EWLS algorithm, one can see that $Q_{t,\mathrm{o}}$ will tend to zero if there is an unexcited subspace. Therefore, tracking directions will be asymptotically limited to the excited subspace and Algorithm II has the ability to choose tracking directions. Obviously, the rank of Q_t is dependent on the excitation condition.

To show that Algorithm II is windup free, we need to prove that $P_{t,\mathrm{o}}$ is bounded from above when there exists an unexcited subspace. We can establish the following theorem based on Corollary 3.2.

Theorem 4.3. *Assume that there exists an l-dimensional unexcited subspace in R^n. Decompose P_t of Algorithm II as $P_t = P_{t,\mathrm{o}} + P_{t,\mathrm{p}}$ along the excited subspace. Then $P_{t,\mathrm{o}}$ is bounded from above.*

Proof. Define N_t as

$$N_t = V^{\mathrm{T}} Q_t V,$$

where V is an $n \times l$ matrix whose columns constitute a basis of the unexcited subspace. Then we have

$$N_t = \mu N_{t-1}. \tag{94}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{\infty} N_i &= N_1 + N_2 + N_3 + \cdots \\ &= (1 + \mu + \mu^2 + \cdots) N_1 \\ &= \frac{1}{1-\mu} N_1 < \infty.\end{aligned} \tag{95}$$

The conclusion follows from Corollary 3.2. □

Next, we consider the property of P_t for the case where the regressor φ_t is persistently exciting. Noting that ψ_t is bounded and also persistently exciting, from Johnstone, Johnson, Bitmead, and Anderson (1982) we have

$$q_1 I \leqslant Q_t \leqslant q_2 I, \tag{96}$$

where $q_2 > q_1 > 0$.

Based on the above inequality, it can be shown that

$$P_t > Q_t \geqslant q_1 I. \tag{97}$$

That is, P_t is bounded below away from zero. Furthermore, this does not depend on the bounded regressor assumption.

On the other hand, following the same procedure as in the proof of Theorem 4.1, we can establish the upper bound of P_t as stated in the following theorem.

Theorem 4.4. *Assume that φ_t is persistently exciting, then for Algorithm II the matrix P_t is bounded from above for all t.*

4.2.3. Comparison between Algorithms I and II

Algorithm II can provide more choices than Algorithm I. If μ is chosen very close to 1, then tracking directions do not change much at each update and the Algorithm II's behavior is expected to be similar to the standard Kalman filter in the case of persistent excitation. On the other hand, if μ is very small, then the old tracking directions are discounted quickly and Algorithm II's behavior is expected to be close to that of Algorithm I. Therefore, generally it can be said that Algorithm II is something between the standard Kalman filter and Algorithm I.

To illustrate the possible difference between Algorithms I and II, let us assume the system to be estimated is described by

$$y_t = \varphi_t^{\mathrm{T}} \theta_0,$$

where θ_0 is a constant vector. Define the parameter estimate error

$$\tilde{\theta}_t = \hat{\theta}_t - \theta_0. \tag{98}$$

Introduce the Lyapunov function

$$V_t = \tilde{\theta}_t P_t^{-1} \tilde{\theta}_t. \tag{99}$$

It can be proven that

$$V_t \leqslant \tilde{\theta}_{t-1}^{\mathrm{T}} (P_{t-1} + J_t^{\mathrm{T}} Q_t J_t)^{-1} \tilde{\theta}_{t-1}, \tag{100}$$

where J_t is defined by

$$J_t = I + r^{-1} \varphi_t \varphi_t^{\mathrm{T}} P_{t-1}.$$

For Algorithm I, since Q_t has rank one the matrix $J_t^{\mathrm{T}} Q_t J_t$ is positive semidefinite. From (100) we have

$$V_t \leqslant \tilde{\theta}_{t-1}^{\mathrm{T}} P_{t-1}^{-1} \tilde{\theta}_{t-1} = V_{t-1} \tag{101}$$

which shows that V_t is not increasing.

For Algorithm II, since Q_t is positive definite in the case of persistent excitation the matrix $J_t^{\mathrm{T}} Q_t J_t$ is also positive definite. Thus, from (100) we have

$$V_t < \tilde{\theta}_{t-1}^{\mathrm{T}} P_{t-1}^{-1} \tilde{\theta}_{t-1} = V_{t-1} \tag{102}$$

which shows that V_t is strictly monotonically decreasing.

Comparison between (101) and (102) shows that Algorithm II may have better convergence property than Algorithm I since its Lyapunov function is strictly monotonically decreasing irrespective of the direction of $\tilde{\theta}_t$.

Finally, the boundedness of the P_t matrix for Algorithm II is independent of the assumption of regressor boundedness; while for Algorithm I only the upper bound of P_t is established without the same assumption. This difference could be a theoretical advantage of Algorithm II over Algorithm I when they are involved with the stability analysis of an adaptive control system.

5. Conclusions

A theoretical framework for analyzing the behavior of parameter estimation algorithms has been developed based on an orthogonal decomposition approach. The application of this approach to the analysis of the Kalman filter based algorithm has shown that this framework is especially effective in the case where an unexcited subspace exists. This framework is not only suitable to the analysis of the Kalman filter based algorithms, but also applicable to the analysis of the other kinds of algorithms, such as the exponential weighted least squares algorithm and its variants. By the orthogonal decomposition approach, the behavior of the covariance matrix can be analyzed in terms of two decomposed parts whose boundedness are much easier to investigate than the overall covariance matrix. Sufficient and necessary conditions to avoid windup has been established for the Kalman filter based algorithm, which provide useful directions for deriving new algorithms free of windup.

The idea of directional tracking has been introduced for the Kalman filter based algorithm, which is similar to the concept of directional forgetting introduced for the exponential forgetting least squares algorithm. Two kinds of directional tracking algorithms have been proposed, which can overcome the windup problem in the standard Kalman filter. In addition, it has been shown that these algorithms have a bounded covariance matrix in the case of unsufficient and/or unbounded excitation. These algorithms will enrich the family of parameter estimation algorithms and provide more choices to the designer especially in the field of adaptive control.

Appendix.

Proof (Proof of Lemma 2.1). If $y=Ax\in S^{\perp}$, then $V^{\mathrm{T}}y=0$, $B_0x=AV(V^{\mathrm{T}}AV)^{-1}V^{\mathrm{T}}y=0$. Therefore, $x\in \operatorname{Ker} B_0$.

On the other hand, if $x\in \operatorname{Ker} B_0$, then $0=B_0x=AVz$, where $z=(V^{\mathrm{T}}AV)^{-1}V^{\mathrm{T}}Ax$. Noting that the columns of V are the basis of S, we have $Vz\neq 0$ unless $z=0$. Therefore, $AVz=0$ leads to $z=0$ or $Vz\in \operatorname{Ker} A$. First, assume $Vz\in \operatorname{Ker} A$. Since the columns of V constitute a basis of S, we also have $Vz\in S$. Therefore, $Vz\in \operatorname{Ker} A\cap S$. However, A satisfies the decomposition condition $\operatorname{Ker} A\cap S=0$. Therefore, we have $Vz=0$ and hence $z=0$. From $z=0$ we get $V^{\mathrm{T}}Ax=0$, which indicates the vector $y=Ax$ is orthogonal to the basis of S. Thus we conclude $y=Ax\in S^{\perp}$. □

Proof (Proof of Theorem 2.2). From $BV=AV$ we have $D=V^{\mathrm{T}}BV=V^{\mathrm{T}}AV$. Then from the condition $S\cap \operatorname{Ker} A=0$ and Lemma 2.1 in Cao and Schwartz (2001a) we can see that D is positive definite and $S\cap \operatorname{Ker} B=0$. Therefore, B satisfies the decomposable condition. Based on Theorem 2.1 we can decompose B as $B=B_1+C_1$, where $B_1V=BV=AV$ and $C_1\geqslant 0$, and furthermore, $\operatorname{rank} B_1=m$ and $\operatorname{rank} C_1=\operatorname{rank}(B)-m$. Thus, we have

$$A=B+C=B_1+(C_1+C).$$

Noting that the pair B_1 and C_1+C is the unique orthogonal decomposition of A, it must satisfies

$$B_0=B_1=B-C_1,$$

$$C_0=C+C_1.$$

Since $C_1\geqslant 0$, we conclude that $B_0\leqslant B$ and $C_0\geqslant C$.

From $B_0\leqslant B$, one immediately gets (13). Similarly, from $C_0\geqslant C$ one gets

$$\operatorname{rank}(C)\leqslant \operatorname{rank}(C_0)=\operatorname{rank}(A)-m. \tag{103}$$

On the other hand, from $A=B+C$ one can get

$$\operatorname{rank}(A)\leqslant \operatorname{rank}(B)+\operatorname{rank}(C). \tag{104}$$

Combining (103) and (104) we get (14). □

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