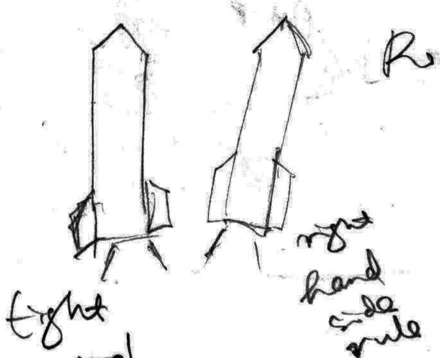
LOEB B149 2:35-3:55Pm TR

SYSC 4505 Automatic Control I

Introduction Control systems.

└ Temperature Control

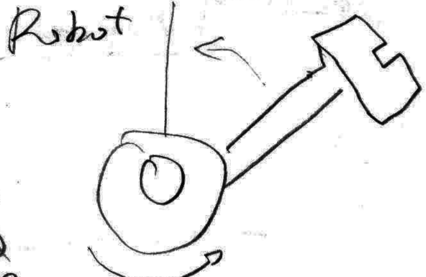
└ Home thermostat.



tight control

High Bandwidth

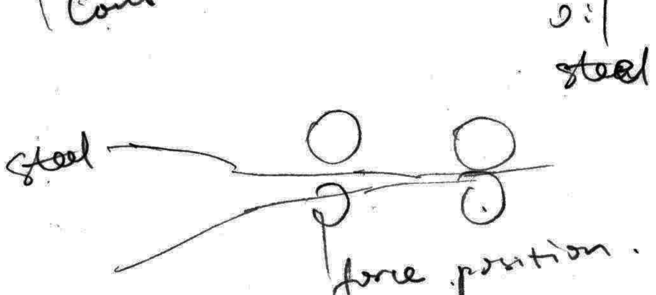
Control loop.



ABS - sensing more than control.

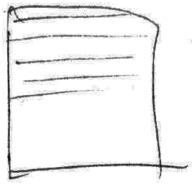
1800. loops in paper mill.

$500/minute.



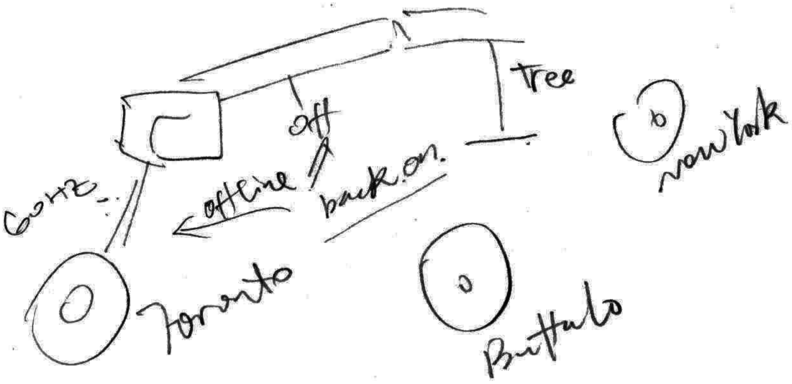
process control

Allen Bradley



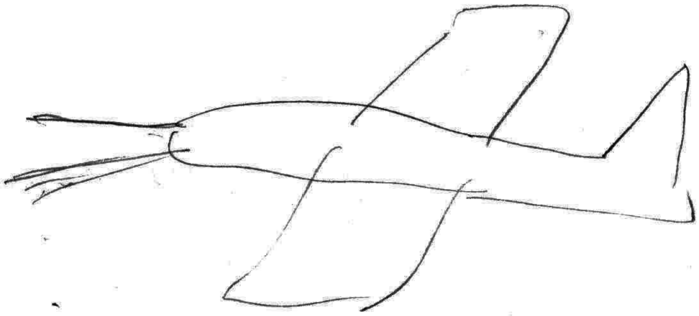
PLC

Programable Logic Controllers.

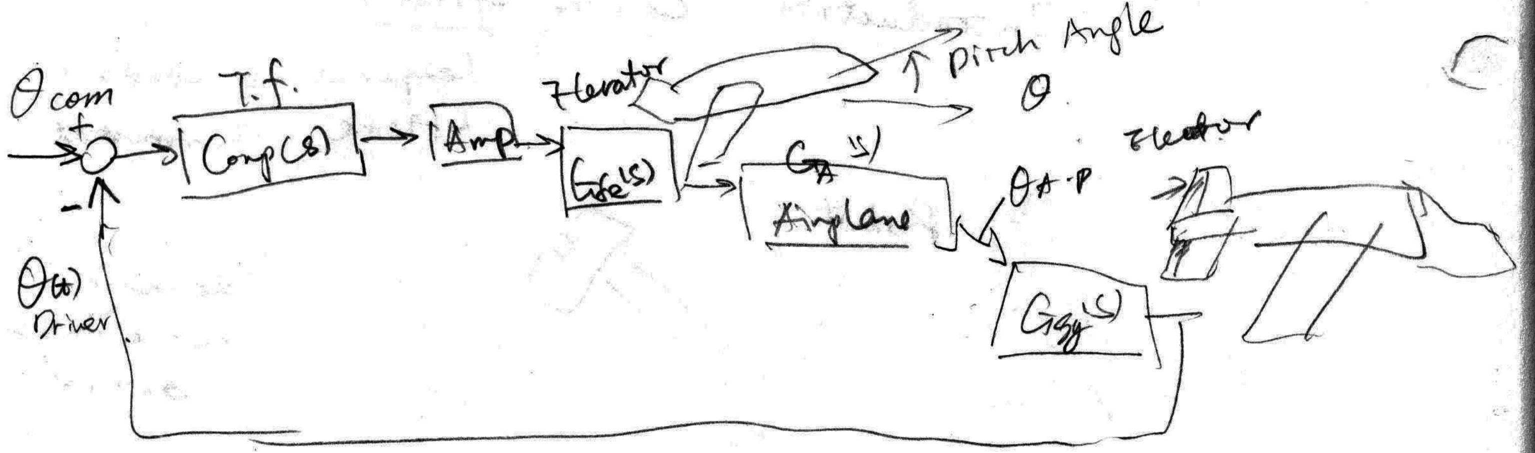


Kuestion?

why tight control have high bandwidth?

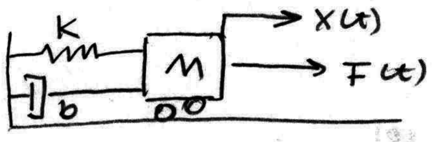


Example of Direction

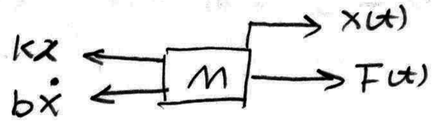


Review of Linear Systems.

One can model systems using D.E.



Draw FBD



Newton's 2nd law

$$M\ddot{x} = -kx - b\dot{x} + F(t)$$

$$M\ddot{x} + kx + b\dot{x} = F(t)$$

$$\ddot{x}(t) + \frac{b}{M}\dot{x}(t) + \frac{k}{M}x(t) = \frac{1}{M}F(t)$$

A physical system can be modelled as a D.E. of some order.

$$x^{n}(t) + a_{n-1}x^{n-1}(t) + a_{n-2}x^{n-2}(t) + \dots + a_0 x(t) =$$

$$b_m u^{m}(t) + b_{m-1}u^{m-1}(t) + \dots + b_0 u(t)$$

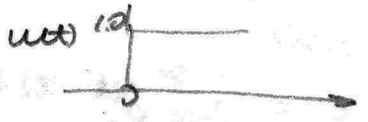
This is a linear time invariant (LTI) D.E.

We will use Laplace Transforms to solve & understand the dynamics of LTI systems.

Laplace Transforms

$$\mathcal{L}\{x(t)\} = \int_0^\infty e^{-st}x(t)\,dt$$

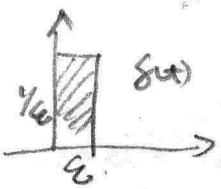
The unit step:



$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{L}\{u(t)\} = \int_0^\infty e^{-st}\overset{1.0}{u(t)}\,dt = -\frac{1}{s}e^{-st}\Big]_0^\infty = -0 + \frac{1}{s} = \frac{1}{s}$$

The unit Impulse



$$\mathcal{L}\{\delta(t)\} = \int_0^\infty \overset{1.0}{\cancel{e^{-st}}}\delta(t)\,dt = 1$$

Exponential $x(t) = e^{-at}$

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st}e^{-at}\,dt = \int_0^\infty e^{-(s+a)t}\,dt$$

$$= -\frac{1}{s+a}e^{-(s+a)t}\Big]_0^\infty = \frac{1}{s+a}$$

Derivative

$\mathcal{L}\{x(t)\} = X(s)$

$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0)$

$\mathcal{L}\{\overset{n}{x}(t)\} = s^n X(s) - s^{n-1}x(0) - s^{n-2}\dot{x}(0) - s^{n-3}\ddot{x}(0) - \cdots - x^{n-1}(0)$

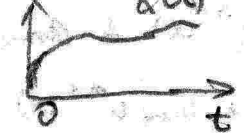
Integral

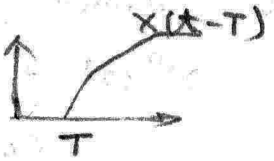
$\mathcal{L}\{\int x(t)dt\} = \frac{X(s)}{s}$

$\mathcal{L}\{\int\int\int \cdots \int x(t)dt\} = \frac{1}{s^n} X(s)$

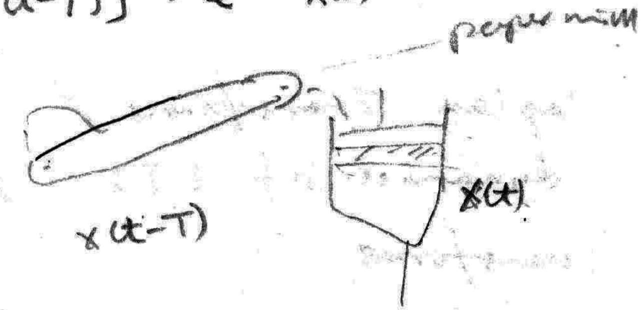
Time shift (chapter 2 Cover 2.7)

$X(t-T)$





$\mathcal{L}\{x(t-T)\} = e^{-sT}X(s)$



Shower temperature

Convolution

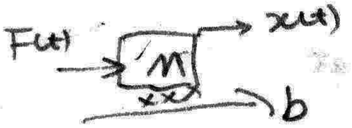
$g(t)$, $h(t)$

$g(t) * h(t) = \int_{\infty}^{\infty} g(t-x)h(x)dx$

$g(t) * h(t) = \int_{-\infty}^{\infty} g(t-\tau)h(\tau)d\tau$ the forced response/or the particular solution

$\mathcal{L}\{g(t) * h(t)\} = G(s)H(s)$

First Order Dynamic Systems



$m\dot{v}(t) = F(t) - bv(t)$

$m\dot{v}(t) + bv(t) = F(t)$

$\dot{v}(t) + \frac{b}{m}v(t) = \frac{F(t)}{m}$

Let's say F(t) is the unit step

take L.T of both sides

All I.C. are zero

$$sV(s) + \frac{b}{m}V(s) = \frac{1}{m}F(s)$$

$$V(s)\left(s+\frac{b}{m}\right) = \frac{1}{m}F(s)$$

$$V(s) = \frac{1/m}{s+b/m}F(s) \rightarrow \frac{V(s)}{F(s)} = \frac{1/m}{s+b/m}$$ → Transfer function

$F(s) = \frac{1}{s}$ because it is the unit step

$$V(s) = \frac{1/m}{s(s+b/m)},$$ we want to find v(t) t>0.

$$= \frac{A}{s} + \frac{B}{s+b/m}$$

$A = sV(s)\big|_{s=0} = \frac{1}{b}$

$B = \left(s+\frac{b}{m}\right)V(s)\big|_{s=-\frac{b}{m}} = \frac{1/m}{-b/m} = -\frac{1}{b}$

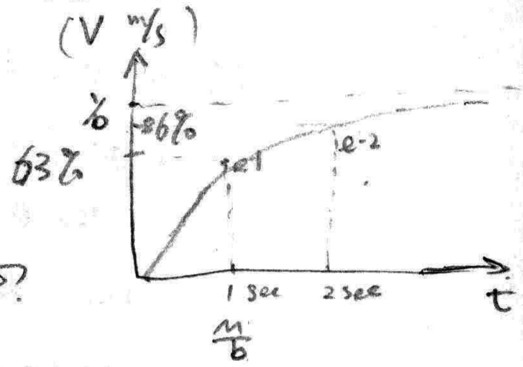
$$V(s) = \frac{1/b}{s} - \frac{1/b}{s+b/m}$$

Take Inverse L.T. we get:

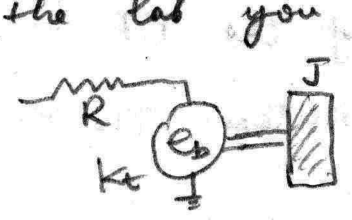
$$v(t) = \frac{1}{b}\left(1-e^{-b/m\,t}\right)$$

what do we call b/m & m/b ? $\frac{m}{b}$ is time constant

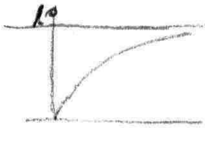
what does it look like?



In the lab you will have



Let's say $\frac{b}{m} = 1$,



$$H(s) = \frac{1}{s+1}.$$

Final Value Theorem. (FVT)

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0} sV(s) = 1$$

$$V(s) = \frac{1}{s(s+1)}.$$

Take arbitrary system

$$x^{n}(t) + a_{n-1} x^{n-1}(t) + \cdots + a_0 x(t) = b_m u^{m}(t) + b_{m-1} u^{m-1}(t) + \cdots + b_0 u(t)$$

Assuming zero initial conditions.

$$s^{n} x(s) + a_{n-1} s^{n-1} x(s) + \cdots + a_0 x(s) = b_m s^{m} u(s) + b_{m-1} s^{m-1} u(s) + \cdots + b_0 u(s)$$

And the T.F. becomes

$$\frac{X(s)}{U(s)} = \frac{b_m s^{m} + b_{m-1} s^{m-1} + \cdots + b_0}{s^{n} + a_{n-1} s^{n-1} + \cdots + a_0}$$

All roots of Denominator Polynomials with real coefficients will either be real or complex conjugate.

If all roots real, find the unit impulse resp $(U(s) = 1)$

Take inverse of L.T. do partial fraction Expansion

$$X(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \cdots + \frac{c_n}{s-p_n} = \frac{(s-z_1)(s-z_2)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$c_i = (s-p_i)\, X(s) \big|_{s=p_i}$$

$$x(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} + \cdots + c_n e^{p_n t}$$

The p_i's are the poles of the T.F. & they define the dynamics of the system. The z_i's are called zeros.

$$x(t) = c_1 e^{-0.1t} + c_2 e^{-10t} + \cdots + c_n e^{-n \times 10 t}$$

Chapter 2 - Laplace Transforms

P222 1^{st} order systems

P229 2^{nd} order systems

Systems with real roots

$$\frac{Y(s)}{X(s)} = H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0} = \frac{b_m (s-z_1) \cdots (s-z_m)}{(s-P_1) \cdots (s-P_n)}$$

$$= \frac{A_1}{s-P_1} + \frac{A_2}{s-P_2} + \cdots + \frac{A_n}{s-P_n} \quad , A_i = (s-P_i) H(s) \Big|_{s=P_i}$$

$$h(t) = A_1 e^{P_1 t} + A_2 e^{P_2 t} + \cdots + A_n e^{P_n t}$$

Systems with complex roots

2^{nd} order systems (underdamped)

If stable, P_i are -ve

$$H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

If $0 \le \xi < 1$, under damped

$\xi = 1$, critical damped

If $\xi > 1$, we have two real roots.

If $0 \le \xi < 1$, If we factor $s^2 + 2\xi\omega_n s + \omega_n^2$ into the roots

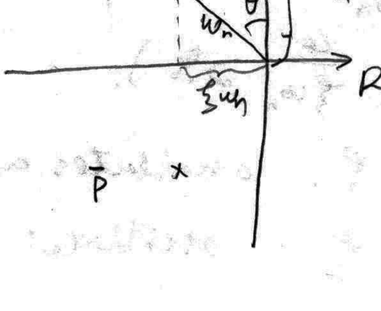
$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s-p)(s-\bar{p})$$

The roots $p, \bar{p}$ are complex

$$p, \bar{p} = -\xi\omega_n \pm \underbrace{\omega_n \sqrt{1-\xi^2}}_{\omega_d}$$

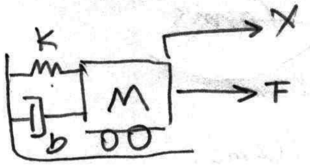
Assume stability

$Re\{p\} < 0$



$\sin\theta = \xi$

Example



model the system

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}F$$

T.F. is $\frac{X(s)}{F(s)} = \frac{1/m}{s^2 + b/m\, s + k/m}$

Let's say F(t) is a unit step.

$K = 36$ N/m , $b = 2$ N/m/s $M = 1$

$\frac{X(s)}{F(s)} = \frac{1}{s^2+2s+36}$ ($F(s) = \frac{1}{s}$ because it is unit step)

$X(s) = \frac{1}{s(s^2+2s+36)}$ Let's factor

$= \frac{1}{s(s-p)(s-\bar{p})}$

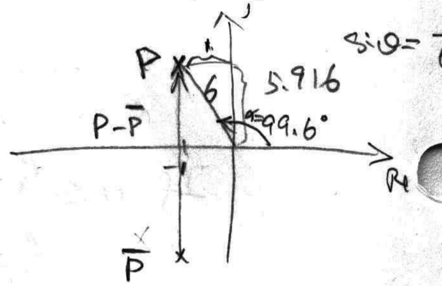
($\xi\omega_n = 1$, $\omega_n = 6$.

$p = -\xi\omega_n \pm \omega_n\sqrt{1-\xi^2}\,j$

$= -1 \pm 5.916j$)

$X(s) = \frac{A}{s} + \frac{B}{s-p} + \frac{\bar{B}}{s-\bar{p}}$

$x(t) = A + 2|B|e^{-\xi\omega_n t}\cos(\omega_d t + \angle\check{B})$



$\sin\theta = \frac{1}{6}.$

$A = s \cdot X(s)\big|_{s=0} = \frac{1}{(s-p)(s-\bar{p})} = \frac{1}{p\bar{p}}$

$= \frac{1}{36.} = 0.0278$

$B = (s-p)X(s)\big|_{s=p} = \frac{1}{s(s-\bar{p})} = \frac{1}{p(p-\bar{p})} = \frac{1}{6\angle 99.6 \cdot 2\times 5.916\times\angle 90^\circ}$

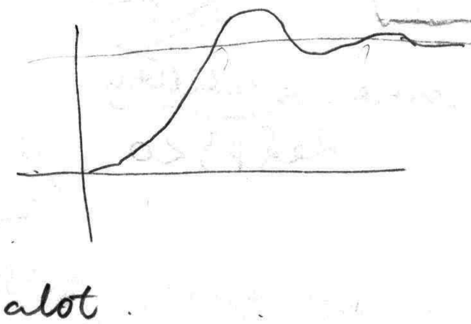
$= 0.0141\angle -189.6^\circ$

$x(t) = 1/36 + 0.028e^{-t}\cos(5.916t - 189.6^\circ)$

Settling time

$t_s \approx \frac{3}{\xi\omega_n}$ 5%.

(or $\frac{4}{\xi\omega_n}$ 2%).



Small ξ, oscillates alot.

Large ξ, oscillates a little.

$$H(s) = \frac{b_m s^m + \cdots b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0} = \frac{b_m(s - z_1) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)} =$$

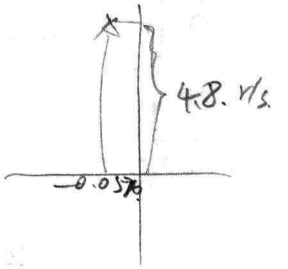
$$\frac{c_1}{s - p_1} + \overset{\curvearrowright \text{conjugate}}{\frac{c_2}{s - p_2} + \frac{c_3}{s - p_3}} + \cdots \quad , c_i = (s - p_i) H(s)\Big|_{s = p_i}$$

$$h(t) = c_1 e^{p_1 t} + 2|c_2| e^{-\xi \omega_n t} \cos(\omega_d t + \angle c_2) + \cdots$$

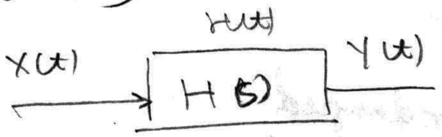
$$\left(T = \frac{4}{3} \right., \ 1.33 \text{ sec}. \quad f = \frac{3}{4} = 0.75 \text{ Hertz} \qquad 0.75 \times 2\pi = 4.8 \, r/s \left.\right)$$

$$17.5 = \frac{3}{\xi \times 4.8} \Rightarrow \xi = \frac{3}{17.5 \times 4.8} = 0.012$$

$$H(s) = \frac{\cancel{B}\ 23}{s^2 + 0.15 s + 23}$$



Frequency Response



$x(t) = A \sin(\omega t)$

$y_{ss}(t) = A\,|H(j\omega)| \sin(\omega t + \angle H(j\omega))$

We want to understand the behavior of systems for different freq. inputs.

We will plot the Bode Diagrams.

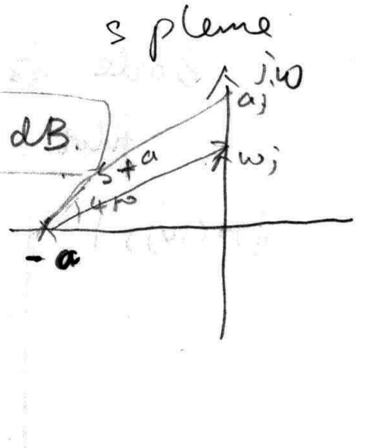
Recall Bode diagram for a first order system.

1st order system.

$H(s) = \frac{a}{s + a}$ } unit gain

$|H(\omega j)| = \frac{a}{s + a}\Big|_{s = \omega j}$

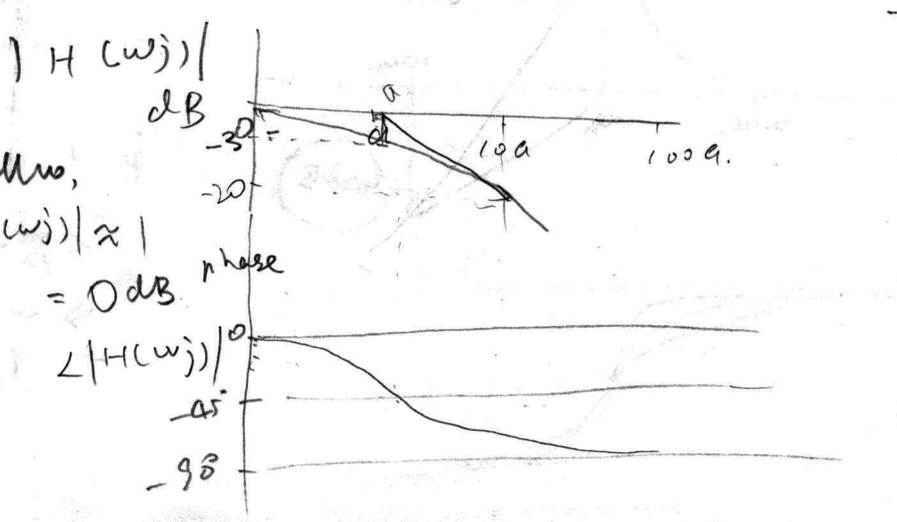
$\boxed{20 \log(K) = \text{dB.}}$



for small w,

$|H(\omega j)| \approx 1$

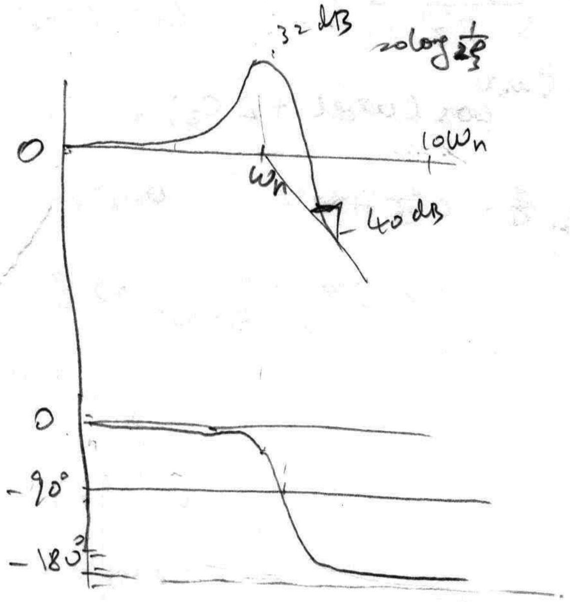
$= 0 \text{dB}$

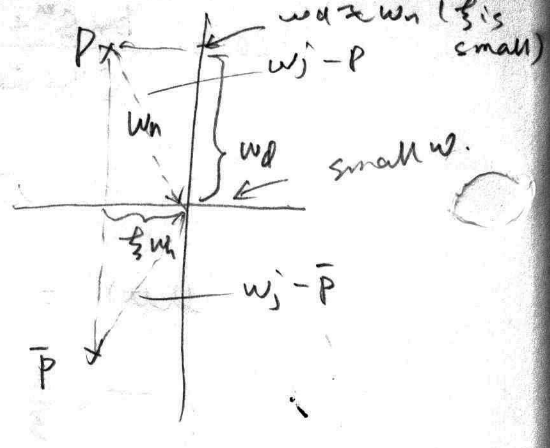


$$H(\omega j) = \frac{\omega_n^2}{(s-p)(s-\bar{p})}$$

↳ unit gain

$$= \frac{\omega_n^2}{\xi\omega_n 2\omega_n} \approx \frac{1}{2\xi}$$

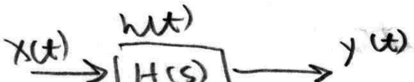




Bode Diagrams of underdamped

Review 2-nd order systems

$$H(s) = \frac{\omega^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$ } unit gain system use F.V.T.

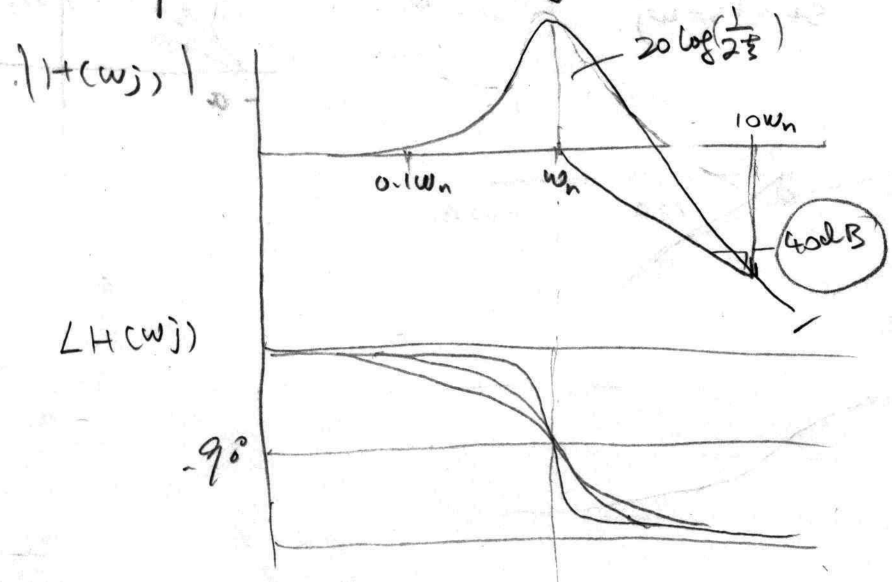


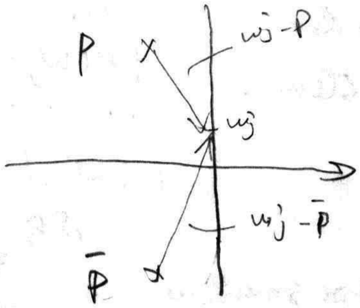
$X(t) = A\sin\omega t$

$Y_{ss}(t) = A|H(\omega j)| \sin(\omega t + \angle H(\omega j))$

Bode is log plot of ω. vs $|H(\omega j)|$

plot dB $= 20\log(|H(\omega j)|)$



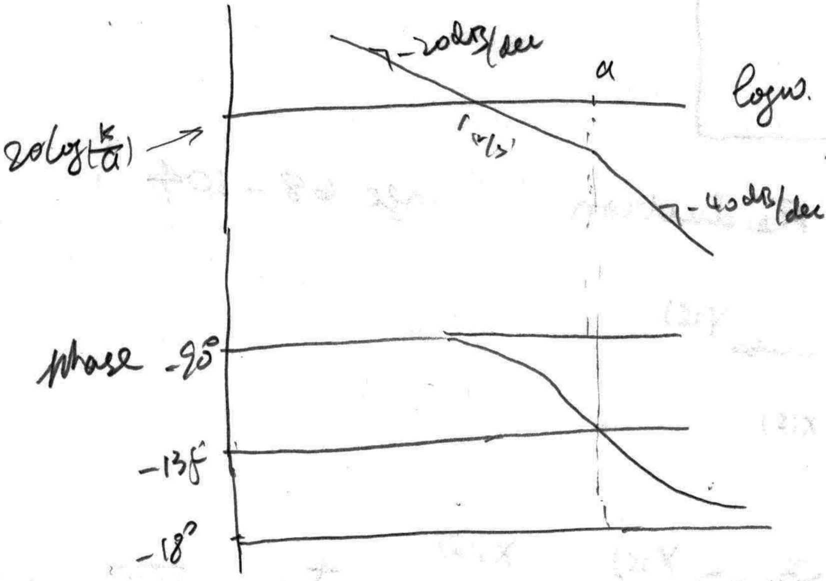


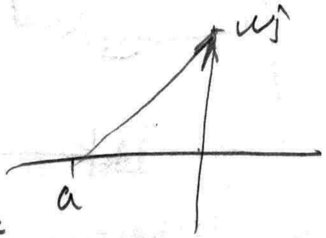
$\xi < \frac{1}{\sqrt{2}} < 0.707$

⇒ we get peak.

what about $H(s) = \frac{K}{s(s+a)}$

Find DC gain is K/a





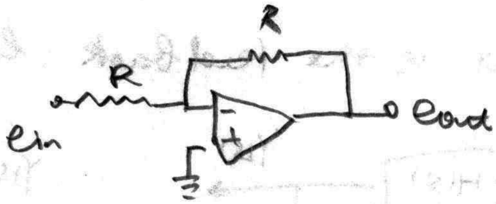
Simulation Diagram / Block Diagram

A simulation diagram depicts a D.E. using summer, multiplier & integrators.

$$\ddot{x} + a_1\dot{x} + a_0 x = b\,u(t) \Rightarrow \ddot{x} = -a_1\dot{x} - a_0 x + b\,u(t)$$

Op-Amps

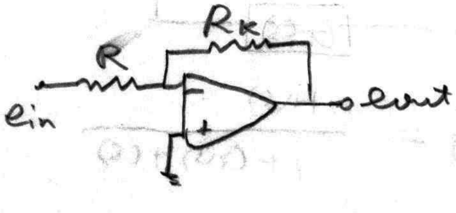
Inverters



$e_{out} = -\cancel{\frac{R}{R}} e_{in}$

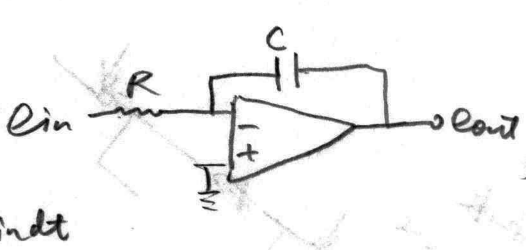
$e_{out} = -e_{in}$

multipliers



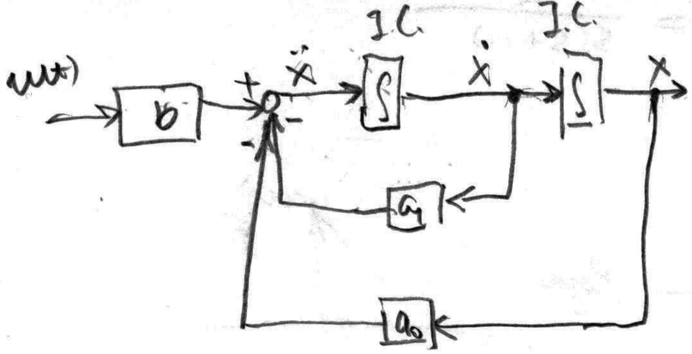
$e_{out} = -\frac{R_K}{R} e_{in}$

Integrators

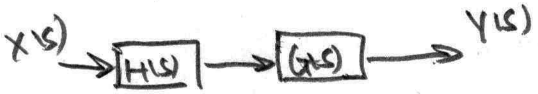


$e_{out} = -\frac{1}{RC}\int e_{in}dt$

you will use these in the lab.

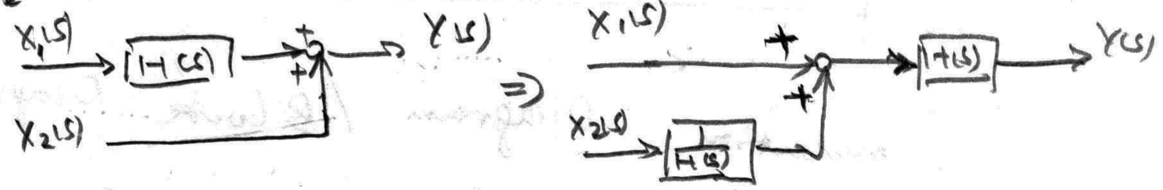


Block Diagram Reduction. (page 68 – 104)

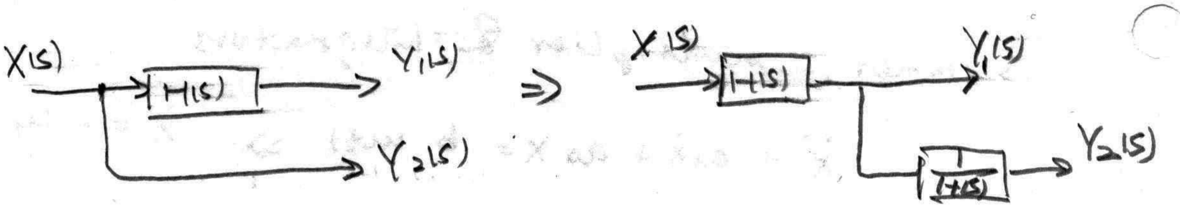


$Y(s) = H(s)\,G(s)\,X(s)$

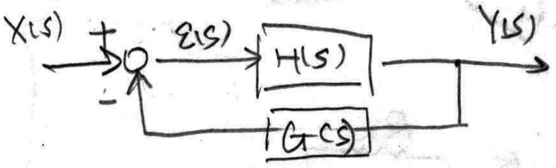
Move adders :



Pickoff points :



The key element is the feed back loop.



$$\frac{Y(s)}{X(s)} = \frac{H(s)}{1+G(s)H(s)}$$

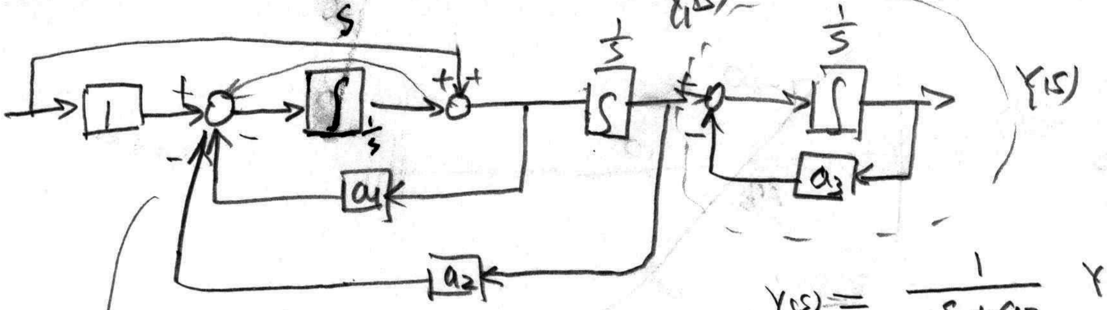
$Y(s) = H(s)\,E(s)$

$Y(s) = H(s)\,(X(s) - G(s)\,Y(s))$

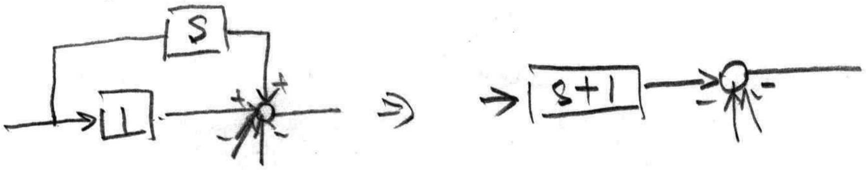
$= H(s)\,X(s) - H(s)\,G(s)\,Y(s)$

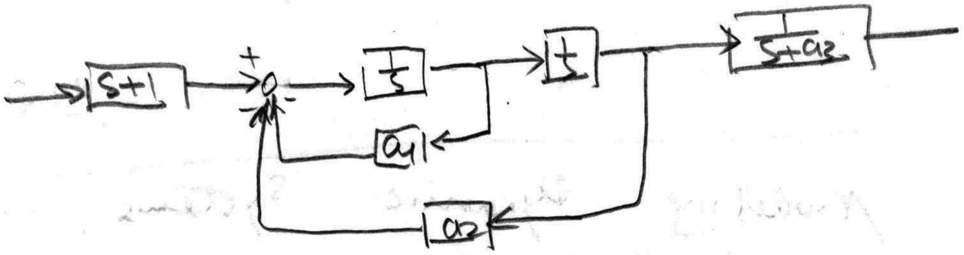
$(1 + H(s)\,G(s))\,Y(s) = H(s)\,X(s)$

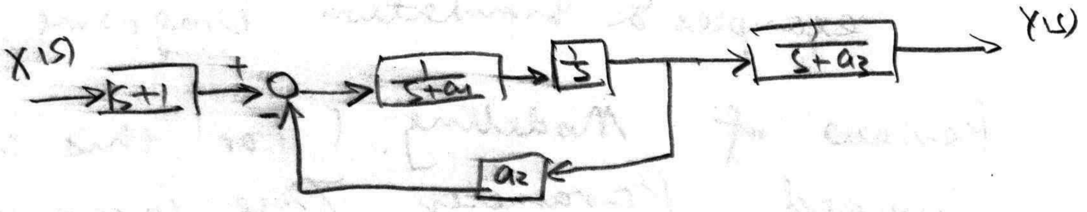
$$\frac{Y(s)}{Y(s)} = \frac{H(s)}{1+H(s)G(s)}$$



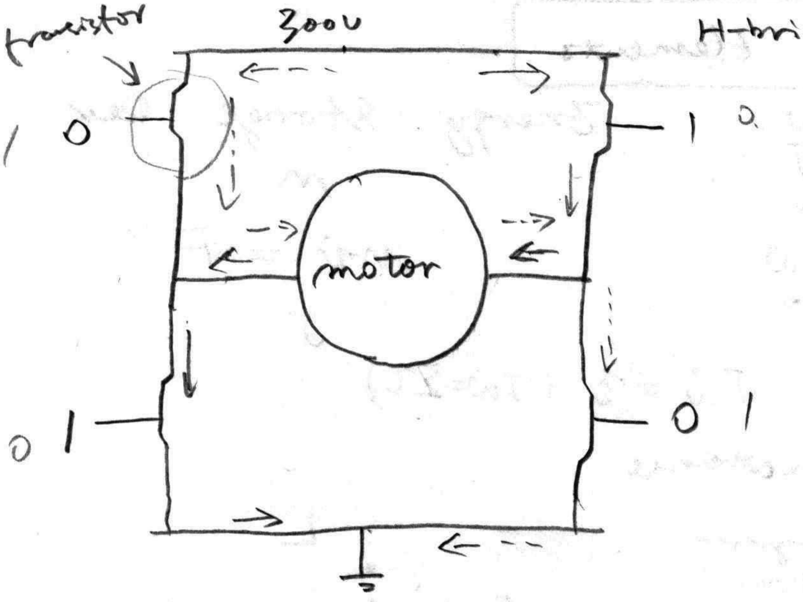
$$Y(s) = \frac{1}{s+a_3} Y_1(s)$$





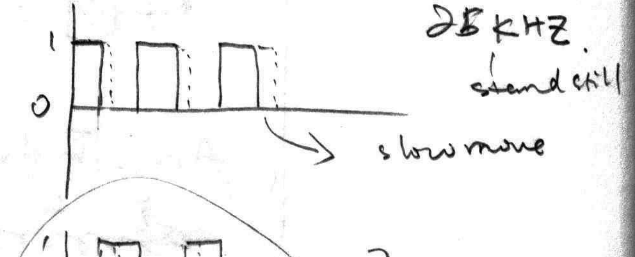


$$\frac{Y(s)}{X(s)} = \frac{s+1}{(s^2+a_1 s+a_2)(s+a_3)}$$



PWM

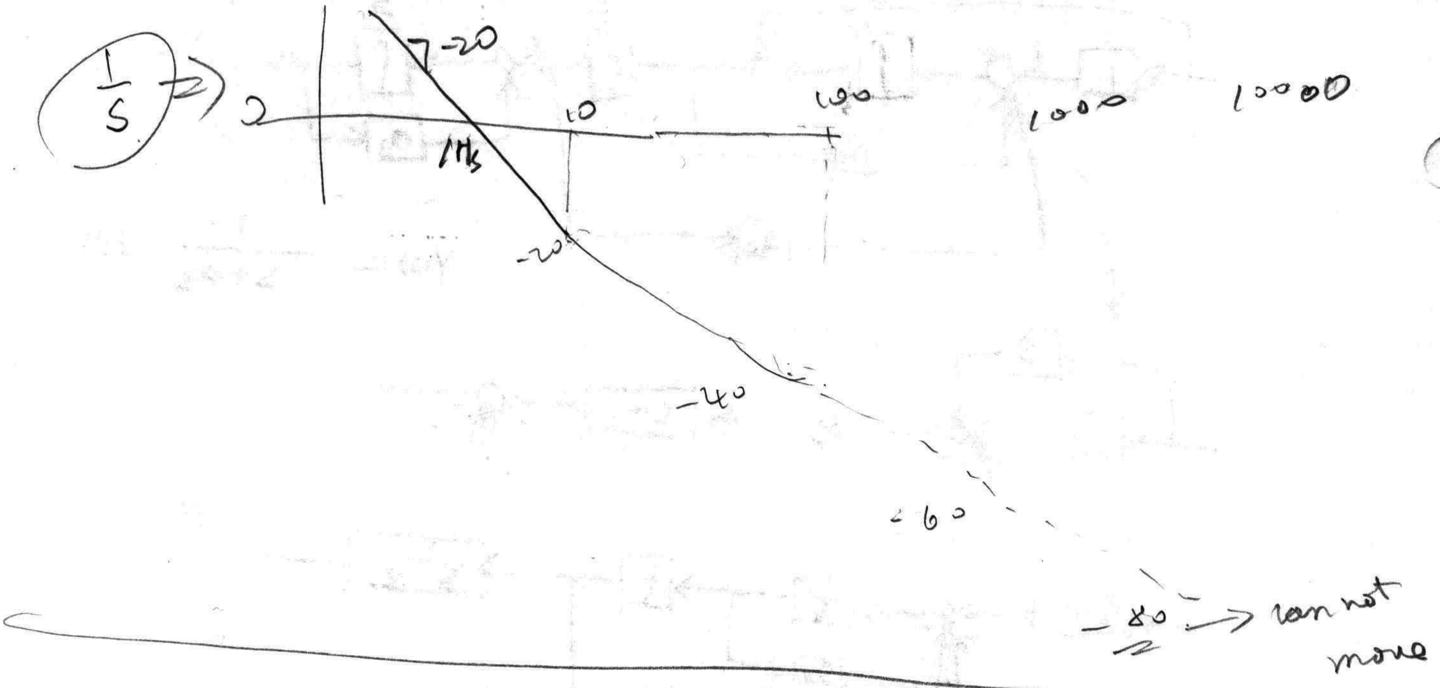
Pulse Width Modulation



$$J\dot{\omega} = \tau$$

$$\dot{\omega} = \frac{1}{J}\tau$$

$$s\omega = \frac{1}{J}\tau \Rightarrow \frac{\omega}{\tau} = \frac{1}{Js}$$



Modeling Dynamic Systems.

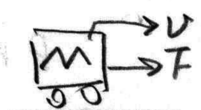
examples & Simulation Diagrams

Review of Modelling. (For this course)
Lumped Parameter Time Invariant
Linear systems.

Order of the system = # of energy storage devices

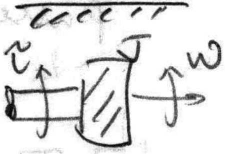
Inertial Elements

Energy Storage Dev. S.V.



M v

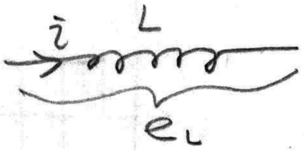
$m\dot{v} = F$.



J ω

$J\dot{\omega} = \tau \quad (J\dot{\omega}=\Sigma\tau)$

An Inductance

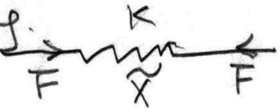


L i

$e_L = L\dot{i}$

3 Capacitive elements. K S.V. $\tilde{x}$

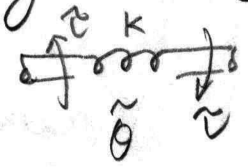
Spring



$F = K\tilde{x}$

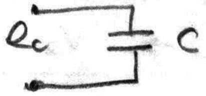
Rotary Spring. Sto. Rev. S.V.

K $\tilde{\theta}$



$\tau = K\tilde{\theta}$

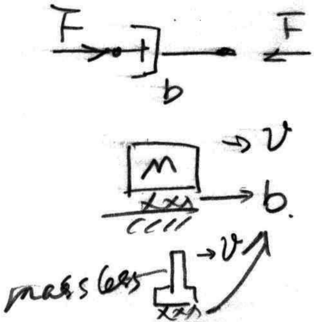
Capacitance C q.



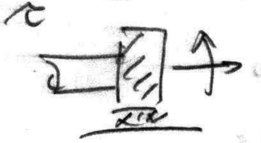
$e_c = \frac{1}{c} q$

We have dissipative elements.

Dampers. $F = b \cdot v$

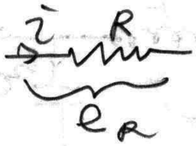


Similarly for rotary inertia



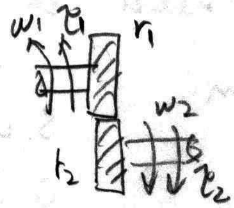
$\tau = b\omega$

Resistants



$\Rightarrow e_R = Ri$

Gears.



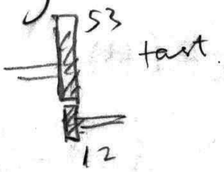
Does not store energy

$\tau_2 = \frac{r_2}{r_1} \tau_1$ power in = power out

$\omega_2 = \frac{r_1}{r_2} w_1$

32-40, 53 teeth

Bicycle big gear infront



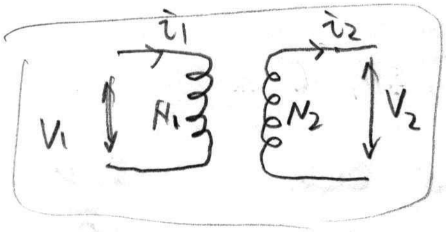
32

back 12 - 25

25 armin space. 2000 : 1

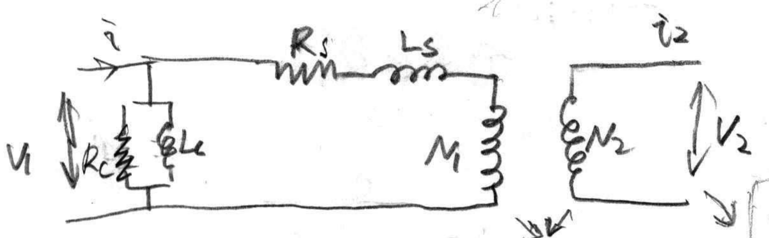
Electrical Transformers

ideal



$$V_2 = \frac{N_2}{N_1} V_1 ,$$

$$i_2 = \frac{N_1}{N_2} i_1$$

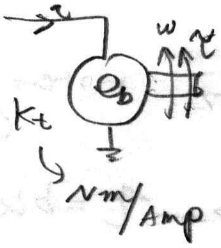


Rate

short → get i_1
open → V_2

99.5%
99.7%
power

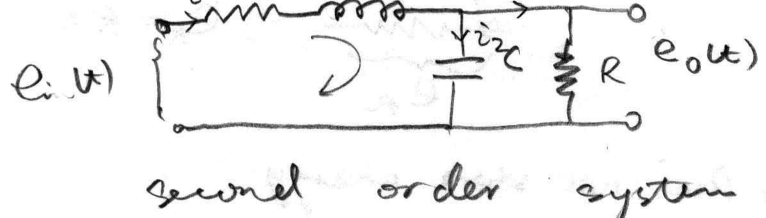
Motors



$$\tau = K_t i$$

$$e_b = K_t w$$

Example :



second order system, we need 2 S.V.

Energy state variable form

$C \rightarrow q$

$L \rightarrow i$

$$e_i(t) = Ri + L\dot{i} + \frac{1}{C}q$$

$$\dot{i} = -\frac{R}{L}i - \frac{1}{LC}q + \frac{1}{L}e_i(t)$$

$$\left.\begin{array}{l}\dot{q} = i - i_3 = i_2 \\ \frac{1}{C}q = i_3 R \Rightarrow i_3 = \frac{1}{CR}q\end{array}\right\} \Rightarrow \dot{q} = i - \frac{1}{CR}q$$

S.V. forms

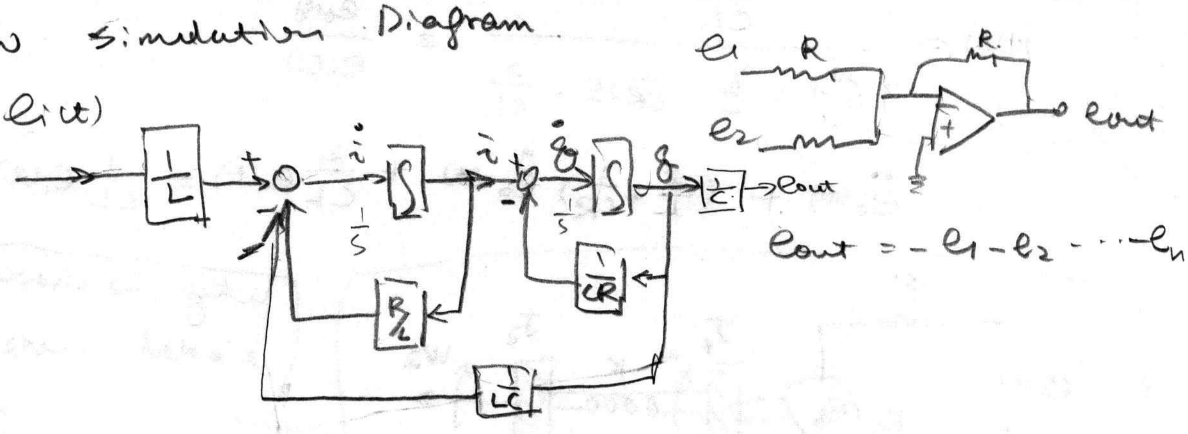
$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots$ + inputs

$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots$ + inputs

$\vdots$

$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots$ + inputs

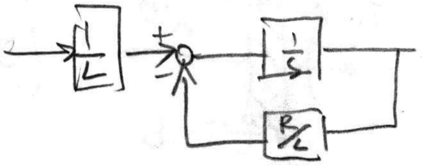
Draw simulation Diagram



$e_{out} = -e_1 - e_2 \cdots - e_n$

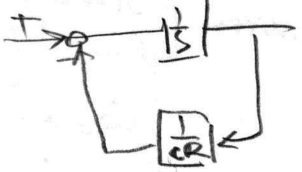
$$\frac{e_o(s)}{R_i(s)} = H(s)\ ?$$

Do block diagram reduction

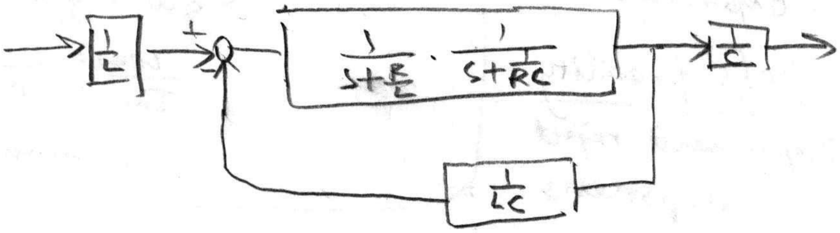


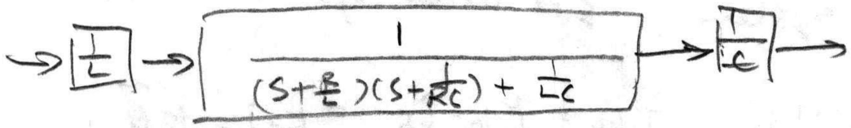
$$\rightarrow \frac{H(s)}{1+H(s)G(s)} = \frac{\frac{1}{s}}{1+\frac{1}{s}\cdot\frac{R}{L}} = \frac{1}{s+\frac{R}{L}}$$

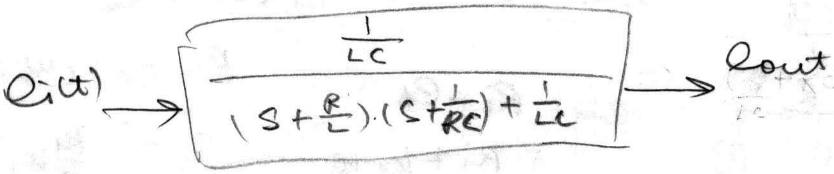
similarly

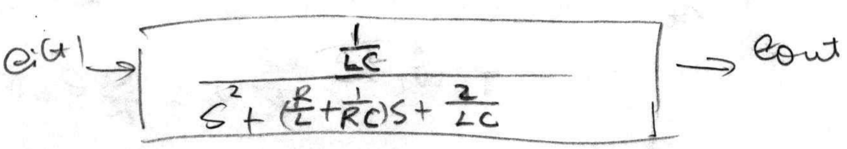


$$\rightarrow \frac{1}{s+\frac{1}{RC}}$$



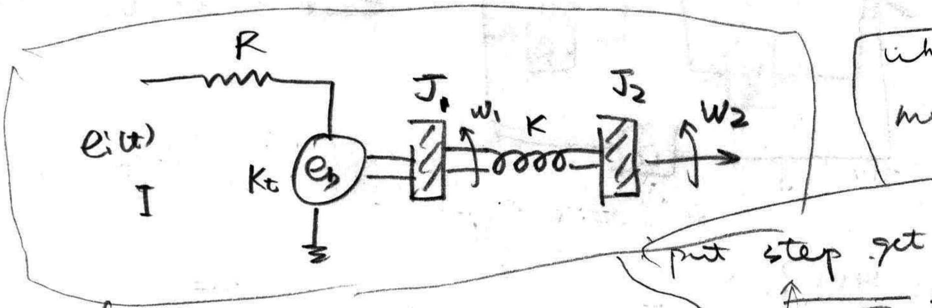




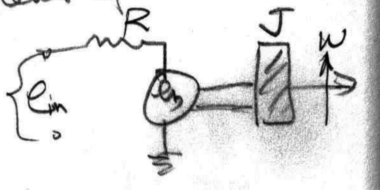


$$H(s) = \frac{\frac{1}{CL}}{s^2 + (\frac{R}{L} + \frac{1}{CR})s + \frac{2}{CL}} = \frac{e_o(s)}{e_i(s)}$$

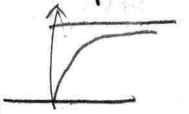
$$\ddot{e}_o(t) + (\frac{R}{L} + \frac{1}{CR})\dot{e}_o(t) + \frac{2}{CL}e_o(t) = \frac{1}{CL}e_i(t)$$



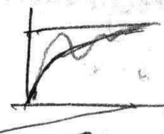
why we choose this model instead of



put step get results,

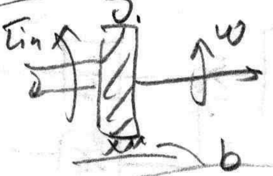


that's good enough.



may be reject, (disturbance?)

→ Let's model this system & draw simulation diagram, reduce simulation diagram & get T.F.



$$\boxed{J_1\dot{\omega} = \tau_{in} - b\omega}$$ ✓

$$\tau_{in} = J_1\dot{\omega} - b\omega$$ X

Initial Condition

$$\boxed{J_1\dot{\omega} = b\omega + \cancel{\tau_{in}}}\ \ 0$$

1, 1, 6 r/s

$\dot{\omega} = 6\ r/s^2$ → unstable.

$$\dot{\omega} - \frac{b}{J_1}\omega = \tau_{in}$$

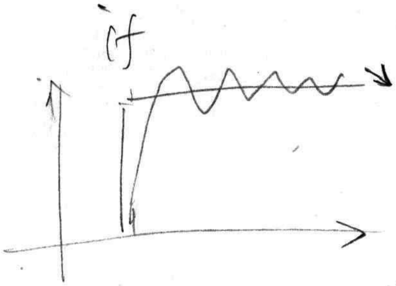
$$\Downarrow\ s\omega - \frac{b}{J_1}\omega = \tau_{in}$$

$$\frac{\omega(s)}{\tau_{in}} = \frac{1}{s - \frac{b}{J_1}}$$

unstable.

pole: $\frac{b}{J_1}$

you may be get above system not right one.



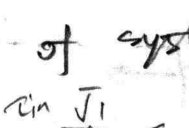
at least 2nd order. reject right 1st order.

Hypothesis
- Experiments

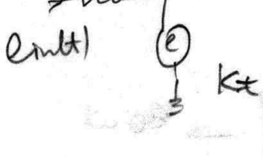
→ falsifiability.

Experiment reject Hypothesis.

Order of system 3rd order system J_1, K, J_2



$$J_1\omega_1 = \tau_{in} - \tau_s = K_t i - K\tilde{\theta}$$



$$e_{in} = Ri + e_b$$
$$= Ri + K_t\omega_1$$

$$Ri = -K_t\omega_1 + e_{in}(t)$$

$$\dot{i} = -\frac{K_t}{R}\omega_1 - \frac{1}{R}e_{in}(t)$$

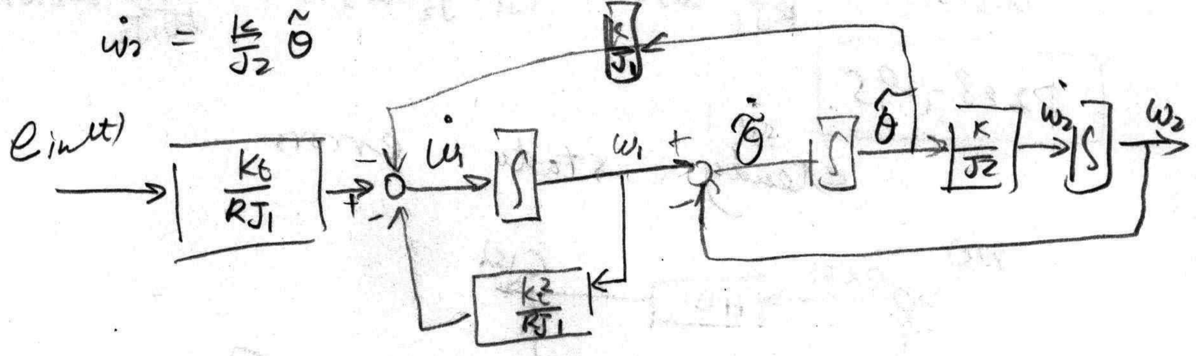
$$J_1\dot{\omega}_1 = -\frac{K_t^2}{R}\omega_1 + \frac{K_t}{R}e_{in}(t) - K\tilde{\theta}$$

$$\dot{\omega}_1 = -\frac{K_t^2}{RJ_1}\omega_1 - \frac{K}{J_1}\tilde{\theta} + \frac{K_t}{RJ_1} e_{in}(t)$$

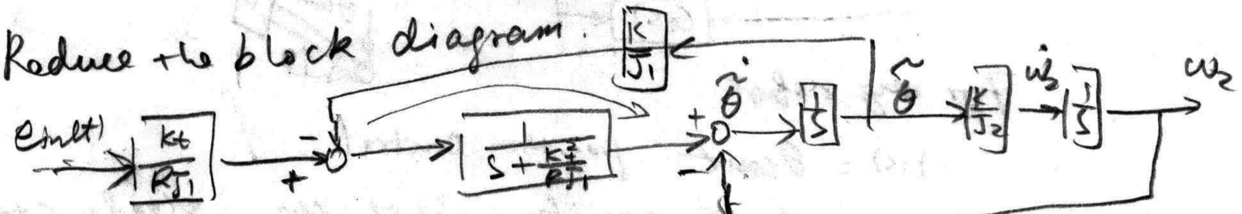
$$\dot{\tilde{\theta}} = \omega_1 - \omega_2$$

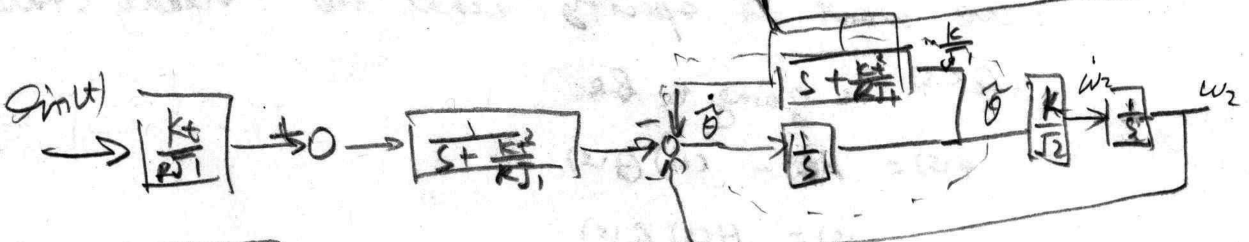
$$J_2 \dot{\omega}_2 = K\tilde{\theta}$$

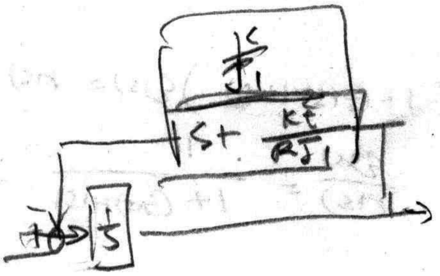
$$\dot{\omega}_2 = \frac{K}{J_2}\tilde{\theta}$$



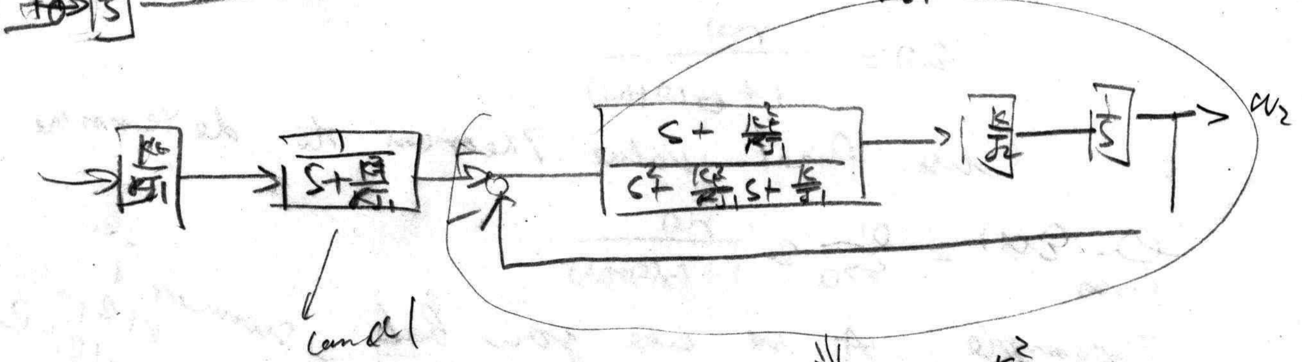
Reduce the block diagram.







$$\Rightarrow H(s) = \frac{\frac{1}{s}}{1+\frac{1}{s}\cdot\left(\frac{K/J_1}{s+\frac{K_t^2}{RJ_1}}\right)} = \frac{s+\frac{K_t^2}{RJ_1}}{s^2+\frac{K_t^2}{RJ_1}s+\frac{K}{J_1}}$$



$$\Downarrow \quad \frac{\dfrac{\left(s+\frac{K_t^2}{RJ_1}\right)\frac{K}{J_2}}{s\left(s^2+\frac{K_t^2}{RJ_1}s+\frac{K}{J_1}\right)}}{1+}$$

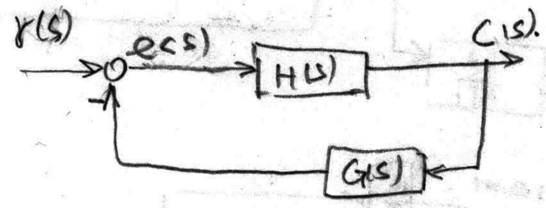
$$\frac{\frac{K}{J_2}\left(s+\frac{K_t^2}{RJ_1}\right)}{s^3+\frac{K_t^2}{RJ_1}s^2+\left(\frac{K}{J_1}+\frac{K}{J_2}\right)s+\frac{KK_t^2}{RJJ_2}} \Leftarrow$$

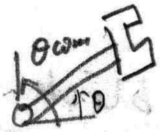
$$H(s) = \frac{\frac{kk_t}{RJ_1J_2}}{s^3 + \frac{k_t^2}{RJ_1}s^2 + (\frac{k}{J_1} + \frac{k}{J_2})s + \frac{kk_t^2}{RJ_1J_2}} = \frac{w_2(s)}{e_{in}(s)}$$

$$\dddot{\omega}_2(t) + \frac{k_t^2}{RJ_1}\ddot{\omega}_2(t) + (\frac{k}{J_1} + \frac{k}{J_2})\dot{\omega}_2(t) + \frac{kk_t^2}{RJ_1J_2}w_2(t) = \frac{kk_t}{RJ_1J_2}e_{in}(t)$$

$\boxed{\text{p288-295.}}$

steady state errors





For my robot.

$r(s) = \theta_{comt}$ position control

We want to specify what the steady state error $\varepsilon(s)$ is going to be.

$\varepsilon(s) = r(s) - c(s)G(s)$

$c(s) = H(s)\varepsilon(s)$

$\varepsilon(s) = r(s) - G(s)H(s)\varepsilon(s) \Rightarrow (1 + G(s)H(s))\varepsilon(s) = r(s)$

$\Rightarrow \frac{\varepsilon(s)}{r(s)} = \frac{1}{1 + G(s)H(s)}$ → T.F. to error.

$$\varepsilon(s) = \frac{r(s)}{1 + G(s)H(s)}.$$

use final value Theorem to determine S.S. error.

$$\lim_{t\to\infty} \varepsilon(t) = \lim_{s\to 0} s\frac{r(s)}{1 + G(s)H(s)}$$

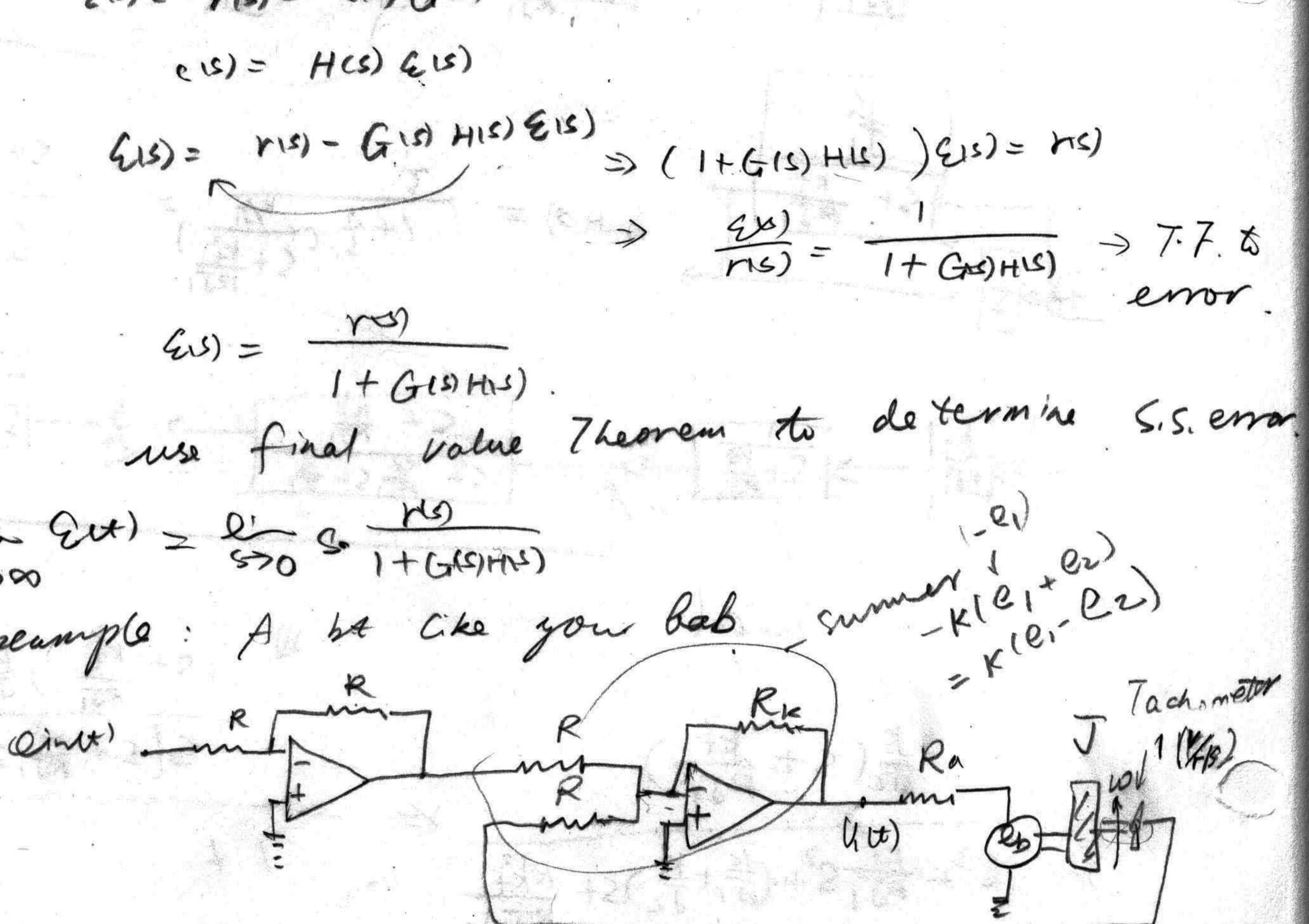
Example: A bit like your lab

summer

$(-e_1)$

$-K(e_1 + e_2)$

$= K(e_1 - e_2)$



20

$$J\dot{\omega} = K_t i$$

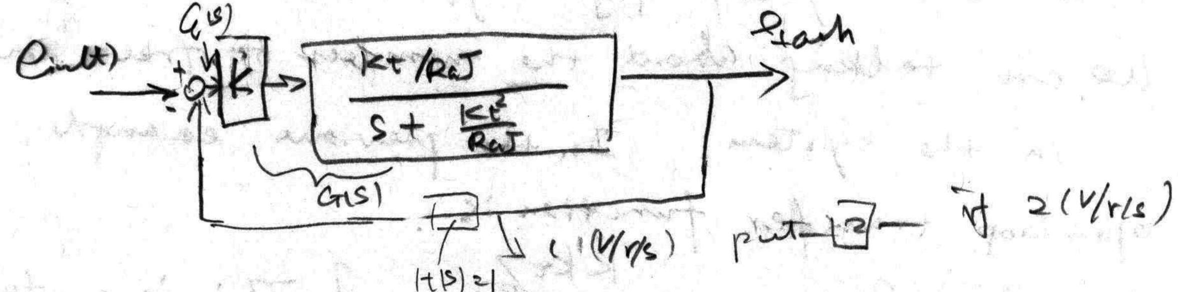
$$R_a i = V_1(t) - K_t\omega$$

$$i = -\frac{K_t}{R_a}\omega + \frac{1}{R_a}V_1(t)$$

$$J\dot{\omega} = -\frac{K_t^2}{R_a}\omega + \frac{K_t}{R_a}V_1(t)$$

$$\dot{\omega} = -\frac{K_t^2}{R_aJ}\omega + \frac{K_t}{R_aJ}V_1(t)$$

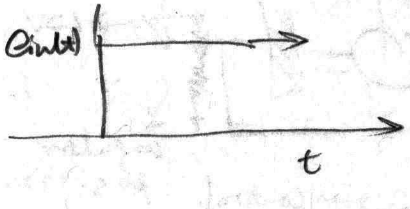
$$\frac{W(s)}{V_1(s)} = \frac{K_t/R_aJ}{s+\frac{K_t^2}{R_aJ}}$$, Draw the Block Diagram



The error is

$$e_{in}(t) - e_{tach} = e_{in}(t) - \omega(t) = \varepsilon(t)$$

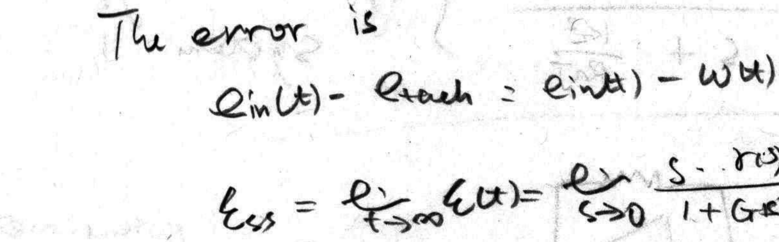
$$\varepsilon_{ss} = \lim_{t\to\infty}\varepsilon(t) = \lim_{s\to 0}\frac{s\cdot r(s)}{1+G(s)H(s)}$$ $r(s) \rightarrow e_{in}/s$ (because I put in a step input $e_{in}(t)$)



$$= \frac{e_{in}}{1+\lim_{s\to 0}G(s)H(s)}$$

$$= \lim_{s\to 0}\frac{e_{in}}{1+\frac{KK_t/R_aJ}{s+\frac{K_t^3}{R_aJ}}} = \frac{e_{in}}{1+K/K_t}$$

$K_p \Rightarrow$ The error coefficient for a step input.



ε_{ss} steady state error

$$\varepsilon_{ss} = \lim_{s\to 0}\frac{s\cdot\frac{R}{s}}{1+G(s)H(s)}$$ $R = \omega_{com}$

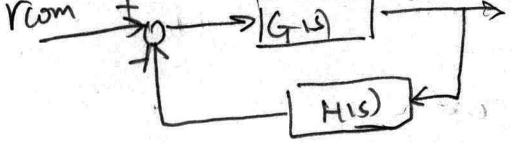
$$\boxed{\varepsilon_{ss} = \frac{R}{1+K_p}}$$

$$K_p = \lim_{s\to 0}G(s)H(s) = K/K_t$$ $\Rightarrow$ error coefficient for step input

Concept of System Type

If the open loop transfer function is :

$$S_{OL} = G(s)\,H(s)$$



$$S_{OL} = G(s)\,H(s)$$

whereas the closed loop T.F. is

$$\frac{r(s)}{r_{com}(s)} = \frac{G(s)}{1+G(s)\,H(s)}.$$

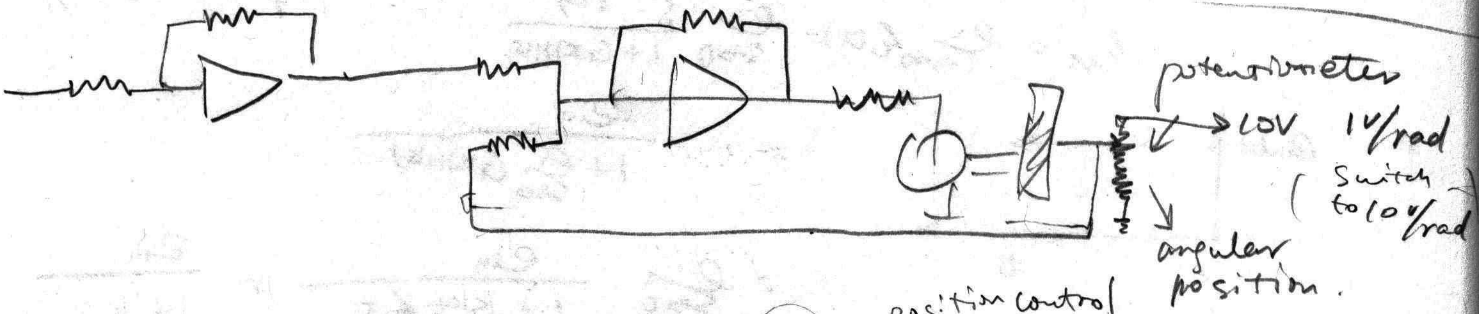
$$S_{OL} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^j(s-p_1)(s-p_2)(s-p_3)\cdots(s-p_n)} = \frac{b_m(s-z_1)(s-z_2)\cdots(s-z_m)}{s^j(s-p_1)(s-p_2)\cdots(s-p_n)}$$

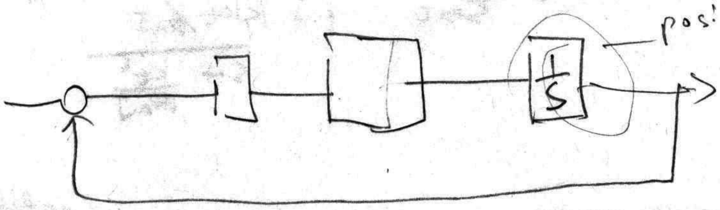
This is a system of jth type.

We are talking about the number of free integrators in the system. In the previous example, the open loop ~~transfer~~ function is :

$$S_{OL} = \frac{KK_t/R_aJ}{s + \frac{K_t^2}{R_aJ}} \Bigg\}$$ This is a type 0 system

---





$$\varepsilon_{ss} = \lim_{s\to0} \frac{s\cdot\theta_{com}/s}{1 + \underbrace{\frac{\frac{KK_t}{R_aJ}}{s(s+\frac{K_t^2}{R_aJ})}}} \xrightarrow{} \text{This now becomes a type 1 systems}$$

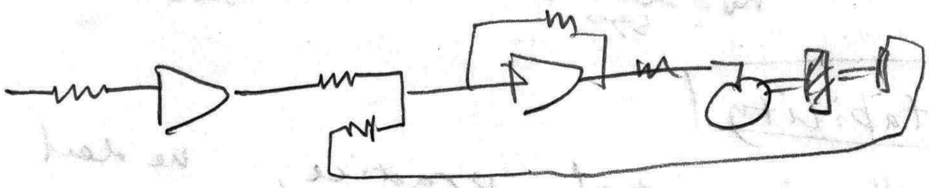
where $G(s)H(s) = \dfrac{KK_t/R_aJ}{s(s+K_t^2/R_aJ)}$

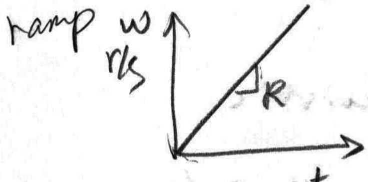
$$K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{K k_t/R_aJ}{s(s + k_t^2/R_aJ)} = \infty$$

$\Rightarrow \mathcal{E}_{ss} = 0$.

So, A type 1 system will have $k_p = \infty$ and steady state error due to a step input will be zero!

Let's take the case where the input is a ramp. and look at the type "0"





$W_{com}(t) = Rt$

$W_{com}(s) = \frac{R}{s^2}$

Then the steady state error becomes:

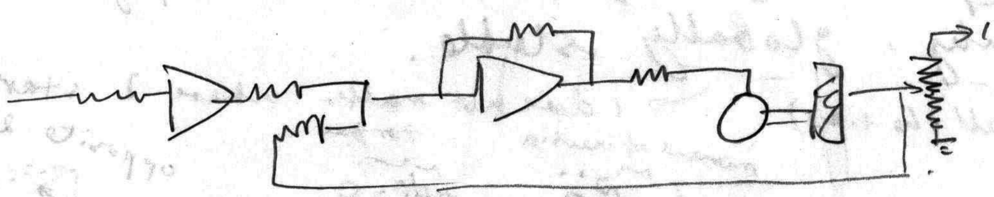
$$\mathcal{E}_{ss} = \lim_{s \to 0} s \cdot \frac{R/s^2}{1 + G(s)H(s)} = \lim_{s \to 0} \frac{R}{\cancel{s}^{\,0} + sG(s)H(s)}$$

$$= \frac{R}{k_v}$$ where $K_v \Rightarrow$ coefficient of error for a ramp or velocity input.

$$K_v = \lim_{s \to 0} sG(s)H(s) \Leftarrow$$

$$= \lim_{s \to 0} \frac{s\,Kk_t/R_aJ}{s + k_t^2/R_aJ} = 0$$

$\mathcal{E}_{ss} \Rightarrow \infty$. A type "0" system can not follow a ramp input.



Now let's see what happens with a type "1" system.

$$\mathcal{E}_{ss} = \lim_{s \to 0} \frac{s\,R/s^2}{1 + \frac{1}{s} \cdot \frac{K\,k_t/R_aJ}{s + \frac{k_t^2}{R_aJ}}} = \frac{R}{\cancel{s}^{\,0} + \frac{K \cdot k_t/R_aJ}{s + k_t^2/R_aJ}} = \frac{R}{K/k_t} = \frac{R}{K_V}$$

A type "1" system can now follow a ramp input.

System Type	Step	Ramp	Parabola
0	$E_{ss} = \frac{R}{1+K_p}$	∞	∞
1	$E_{ss} = 0$	$\frac{R}{K_v}$	∞
2	$E_{ss} = 0$	0	R/K_a

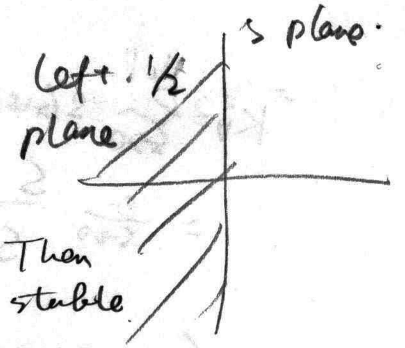
where $K_p = \lim_{s\to0} G(s)H(s)$

$K_v = \lim_{s\to0} s\,G(s)H(s)$, $K_a = \lim_{s\to0} s^2 G(s)H(s)$

Stability

Typically, in control practice, we deal with linear time invariant systems.

In this case, one looks at the characteristic equation (pole location) and determine if all poles have -ve real parts

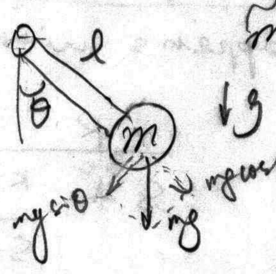


$$H(s) = \frac{b_m (s-z_1) \cdots (s-z_n)}{(s-p_1) \cdots (s-p_n)}$$

Such as system would be

Called exponentially (decay rate), asymptotically → (keep goes to zero, no ~~return~~ to 0) uniformly, globally stable.

(all the time) → (does not matter where I start)



$$\overbrace{ml^2}^{\text{moment of inertia}}\underline{\ddot{\theta}} = \overbrace{-mgl\sin\theta}^{\text{torque}}$$

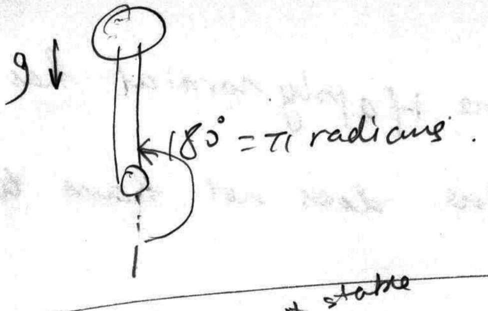
↓g $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$ → non linear.

$\theta = 0$ is equilibrium position

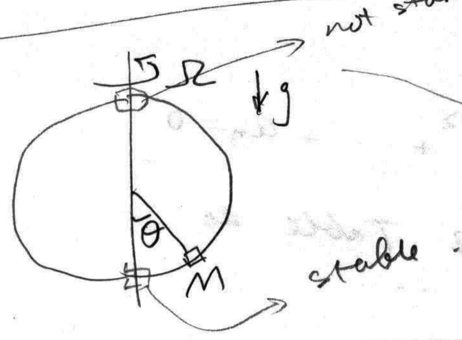
For small θ, $\sin\theta = \theta$

$\ddot{\theta} + \frac{g}{l}\theta = 0$ linearization about equilibrium $\theta = 0$

not exponentially
not asymptotically
but uniformly, globally

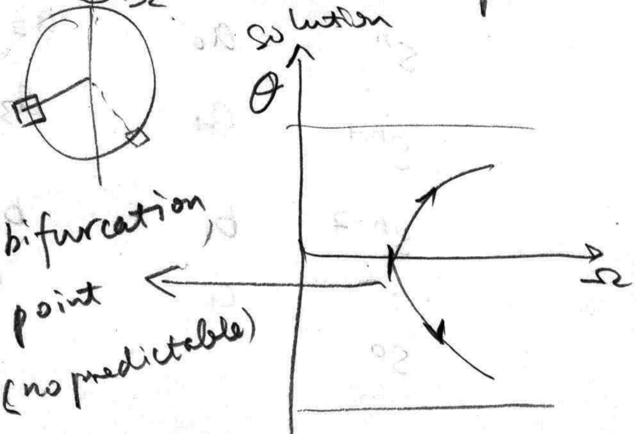


$S\theta = -\theta$
unstable



$\ddot{\theta} = f(\dot{\theta}, \Omega)$ fourize and linearize
found it is not stable

as Ω ↑ get different equilibrium position.



bifurcation point
(no predictable)

(Chaotic system)

Routh Horwitz method for stability. P.275-280

The idea is to determine stability of a characteristic equation without explicitly find the roots.
he will use this method to assist us in plotting root-locus.

Given the characteristic Equation.

$$p(s) = s^n + a_{n-1} s^{n-1} + \dots + a_0$$

it it is a stable polynomial. all roots have +ve real parts. If stable, it may be referred to as Hurwitz polynomial.

P275 - 280 Routh - Hurwitz

This is a method to determine if a polynormial has my roots in the right ½ s-plane. One does not have to find the actual roots.

Given a polynormial

$$p(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n = 0$$

One builds a Routh_Hurwitz Table as

$$\begin{array}{llll} s^n & a_0 & a_2 & a_4 \quad \cdots \\ s^{n-1} & a_1 & a_3 & a_5 \quad \cdots \\ s^{n-2} & b_1 & b_2 & b_3 \\ \vdots & c_1 & c_2 & c_3 \\ s^0 & & & \end{array}$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1},$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, \quad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

The number of roots of $p(s)$ in the right ½ s-plane equals the number of sign changes in the first column of the Routh-Hurwitz table. For $p(s)$ to be Hurwitz (stable) (all roots in open left ½ plane) the all coefficient a_i must be nonzero & +ve. This is necessary but not sufficient condition.

↳ (even nonzero & +ve, still have roots in right ½ plane.)

Example of a 3rd order system.

$$P(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

$a_i > 0$, what is the requirement for stability?

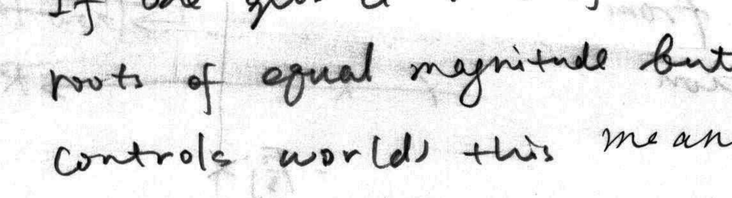
Let's build the Routh-Hurwitz table.

s^3	a_0	a_2	0
s^2	a_1	a_3	0
s	$\frac{a_1 a_2 - a_0 a_3}{a_1}$	0	0
s^0	a_3		

The condition for stability of a 3rd order system is $a_1 a_2 > a_0 a_3$

If one gets a row of zero's then this implies roots of equal magnitude but opposite sign (In the controls world) this means purely imaginary (conjugate) roots. We can find the location of the imaginery roots by solving the auxiliary equation which is the row above, in the 3rd case, $a_1 s^2 + a_3 = 0$

P280 Example.



how big can I make "k" for this to remain stable.

Find Characteristic Equation.

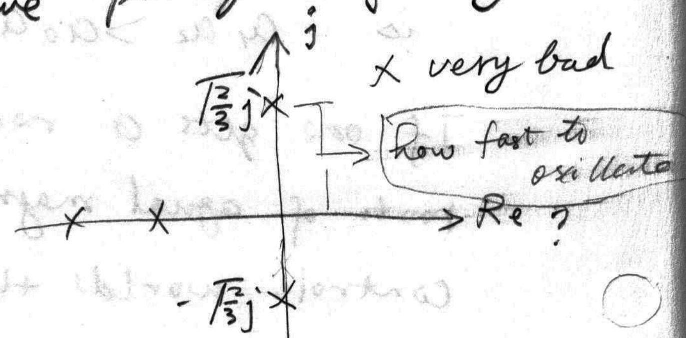
$$\frac{C(s)}{r(s)} = \frac{H(s)}{1 + H(s)G(s)} = \frac{\frac{K}{s(s^2+s+1)(s+2)}}{1 + \frac{K}{s(s^2+s+1)(s+2)}} = \frac{K}{s(s^2+s+1)(s+2)+K} \Rightarrow$$

$$\frac{K}{S^4+3S^3+3S^2+2S+K}$$

$$\begin{array}{l|llll}
S^4 & 1 & 3 & K \\
S^3 & 3 & 2 & 0 \\
S^2 & \frac{7}{3} & K & 0 \\
S & \frac{\frac{14}{3}-3K}{7/3} & 0 & \\
S^0 & K & &
\end{array}$$

from $\frac{14}{3}-3K>0 \Rightarrow K<\frac{14}{9}$

For this system to be stable, $K<\frac{14}{9}$. at $K=\frac{14}{9}$ then row s^1 is zero and we have purely imaginary roots.



× very bad

how fast to oscillate

We find the roots from the auxillary equation

Set $K=14/9$

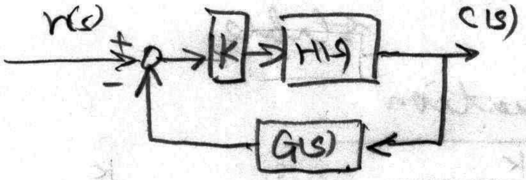
$$7/3\, s^2 + 14/9 = 0 \Rightarrow S^2+\frac{2}{3}=0 \Rightarrow (s+\sqrt{2/3}\,j)(s-\sqrt{2/3}\,j)$$

The imaginary roots are at $\omega j = \pm\sqrt{2/3}\,j$

The Root Locus Method.

Find the loci of possible closed loop pole locations from the open loop transfer function.

The standard closed loop structure is



The closed loop transfer function is:

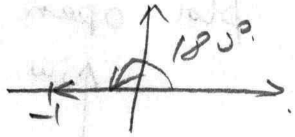
$$\frac{C(s)}{r(s)}=\frac{K\,H(s)}{1+K\,H(s)G(s)}$$

The open loop T.F. is

$$S_{OL} = KH(S)G(S)$$

The poles of the closed loop transfer function are given by

$$1 + KH(S)G(S) = 0.$$



This happens when $KH(S)G(S) = -1$

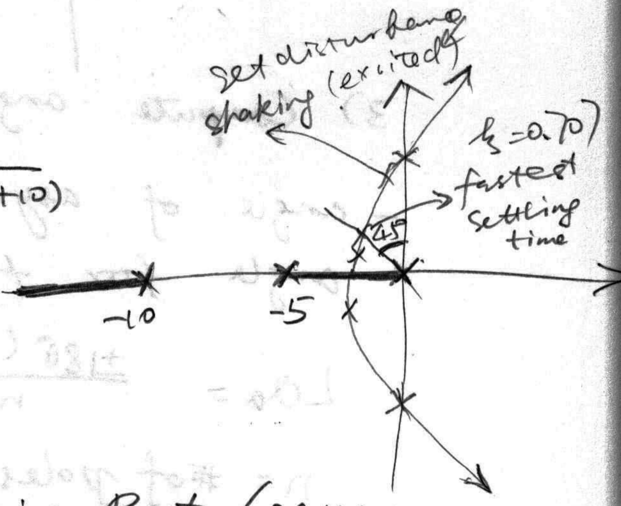
But $H(S)G(S)$ is a complex number, and whenever $KG(S)H(S)$ has a magnitude of 1, and angle of ±180°, we have a ~~possible~~ closed loop pole location.

If $\angle KG(S)H(S) = \pm 180°$ then we have a possible closed loop pole location, and we can find K, such that $|KG(S)H(S)| = 1$.

Let's say $KG(S)H(S) = \dfrac{K}{S(S+5)(S+10)}$

plot open loop pole location.

$\angle KG(S)H(S) = \pm 180°$

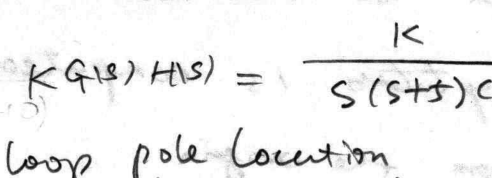


P337-358.

General Rules for constructing Root Locus

(Root locus - Chapter 6 P337-365)

Recall feedback structure

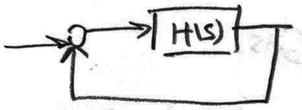


$$\frac{C(S)}{R(S)} = \frac{H(S)}{1 + G(S)H(S)}$$

$$\therefore \angle G(S)H(S) = \pm 180°(2K+1) \quad K = 0, 1, 2 \cdots$$
$$= \pm \pi (2K+1) \text{ radians}.$$

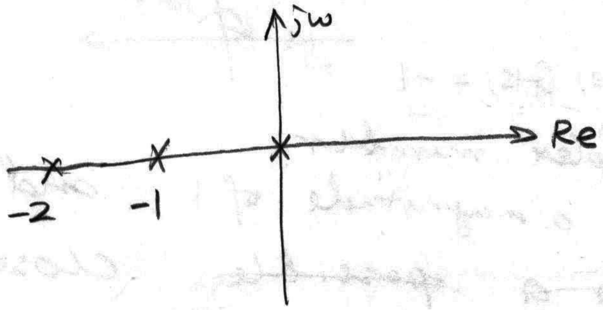
$$|H(s)G(s)| = 1$$

$$H(s) = \frac{K}{s(s+1)(s+2)}$$



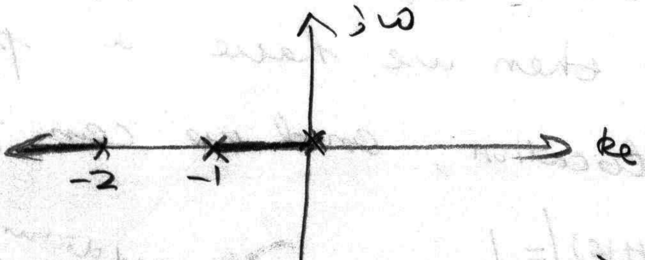
Rules for root locus

1) plot open loop poles and zero's on s-plane



2) plot root locus on the real axis.

(putting a test point -0.5)



3) Compute angle & intersection of asymptotes.

- angle of asymptotes is computed by determine angle ~~for~~ from the origin

$$\angle\theta_a = \frac{\pm 180°(2k+1)}{n-m} \quad k=0,1,2,\ldots$$

n = # of poles. m = # of zero

in this example, n=3, m=0

$$\angle\theta_a = \frac{\pm 180°(2k+1)}{n-m} \quad k=0,1,\ldots$$

$$= \frac{\pm 180°}{3} = \pm 60°$$

$$\text{or } \frac{\pm 180°\times 3}{3} = \pm 180°$$

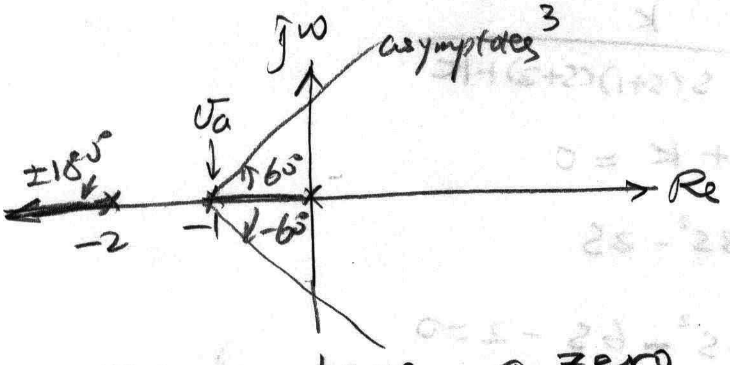
$$\text{or } \frac{\pm 180°\times 5}{3} = \pm 300° = \pm 60°$$

Intersection of the asymptotes

$$\sigma_a = \frac{\Sigma \text{ poles} - \Sigma \text{ zero}}{n-m}$$

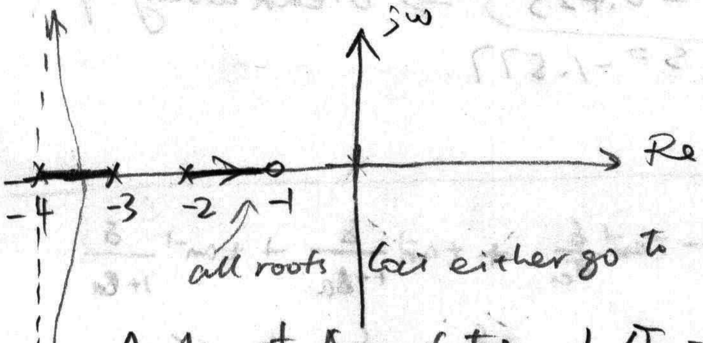
in this case:

$$= \frac{(0-1-2)^{\text{poles}}}{3} = -1$$



If we have a zero

$$H(s) = \frac{k(s+1)}{(s+2)(s+3)(s+4)}$$



all roots can either go to infinity or to open loop zeros

Angle of Asymptotes $\angle \sigma_a = \frac{180°(2k+1)}{n-m}$

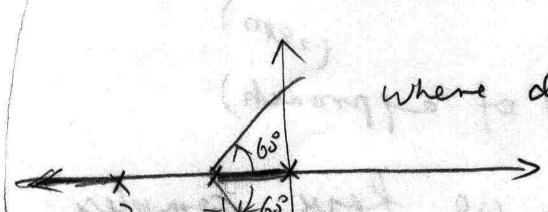
$$= \frac{180°(2k+1)}{3-1} = \pm 90°$$

$$\sigma_a = \frac{-2-3-4-(-1)}{2} = -4$$

4) Compute Breakaway points & Angle of Departure (Angle of approach)

$$H(s) = \frac{k}{s(s+1)(s+2)}$$, we know $\angle \sigma_a = \pm 60°$

$\sigma_a = -1$



where does root locus break away from real axis

Compute $\frac{dk}{ds} = 0$

Compute characteristic equation with unity feedback

$$S_{CL} = \frac{H(s)}{1+H(s)} = \frac{\frac{K}{S(S+1)(S+2)}}{1+\frac{K}{S(S+1)(S+2)}}$$

$$= \frac{K}{S(S+1)(S+2)+K}$$

$$P(S) = S^3 + 3S^2 + 2S + K = 0$$

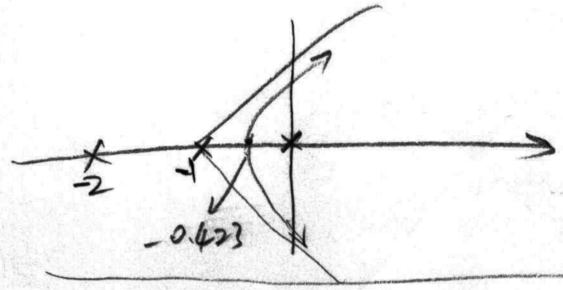
$$K = -S^3 - 3S^2 - 2S$$

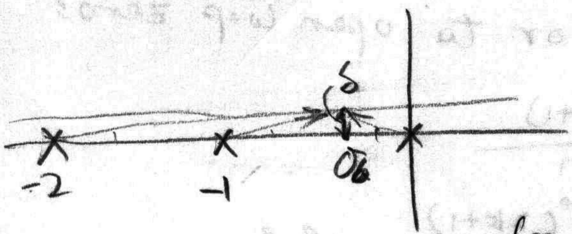
Compute $\frac{dK}{ds} = -3S^2 - 6S - 2 = 0$

$$S^2 + 2S + \tfrac{2}{3} = 0$$

$\boxed{S = -0.423}$ ⇒ Breakaway point

or $S = -1.577$



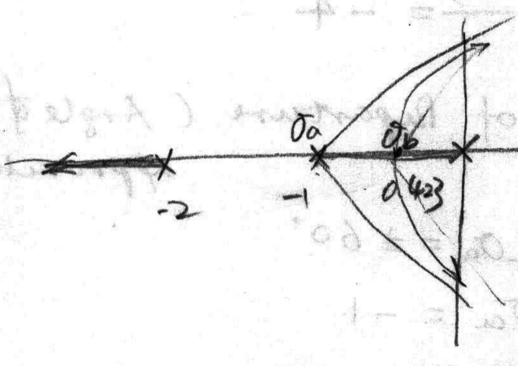


$$180° - \tan^{-1}\frac{\delta}{\sigma_a} + \tan^{-1}\frac{\delta}{1-\sigma_a} + \tan^{-1}\frac{\delta}{1+\sigma_a}$$

$$\approx 180° - \frac{\delta}{\sigma_b} + \frac{\delta}{1-\sigma_b} + \frac{\delta}{1+\sigma_b} = 180°$$

for δ small. δ

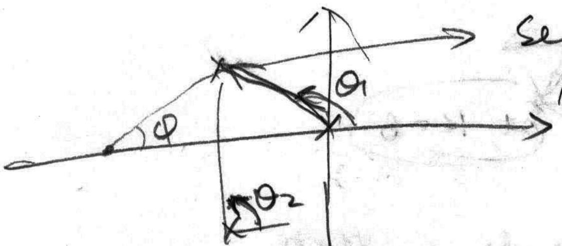
$\sigma_a = 0.423$



Angle of $\overset{\text{(poles)}}{\text{Departure}}$ (Angle of $\overset{\text{(zero)}}{\text{approach}}$)

This is only used when we have complex poles or zeros.

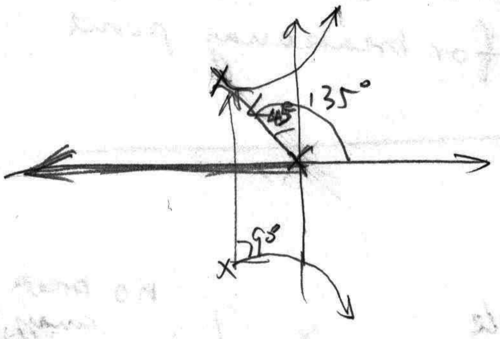
At what angle does the root locus leave the complex pole / zero



search solve for test pts. infinitly close to pole!

$$\pm 180^\circ = \theta_1 + \theta_2 \overset{-\varphi}{+} \angle dep$$

$$\angle dep = 180^\circ - \theta_1 - \theta_2 + \varphi$$



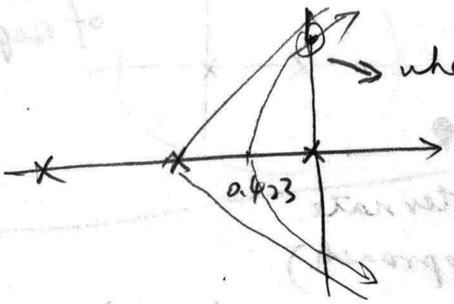
$$135^\circ + 90^\circ + \angle dep = 180$$

$$\angle dep = -45^\circ .$$

5) Crossing the imaginary axis (important)

Use Routh-Hurwitz method.

$$H(s) = \frac{K}{s(s+1)(s+2)}$$



→ what is this point at what value of "K" does it occur at?

solve for closed loop characteristic equation and build Routh-Hurwitz table.

from last time → $P(s) = s^3 + 3s^2 + 2s + K$

s^3	1	2	0
s^2	3	K	0
s^1	$\frac{6-K}{3}$	0	0
s^0	K		

$\Rightarrow \underline{0 < K < 6}$

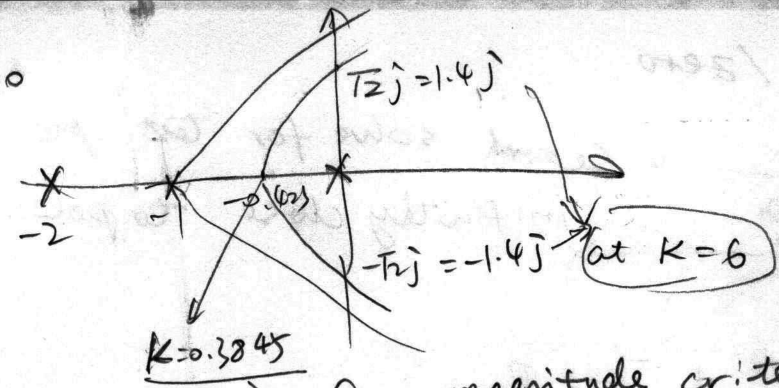
at $K = 6$, solve auxillary equation

$$3s^2 + 6 = 0$$

$$s^2 + 2 = 0 \Rightarrow s = \pm\sqrt{2}\,j$$

This is ✓ crossing of imaginary axis

So



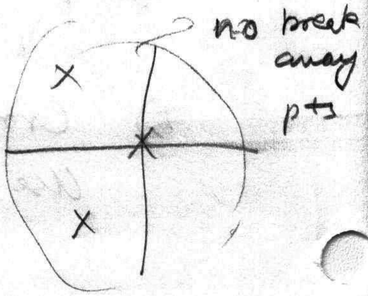
→ from magnitude criteria

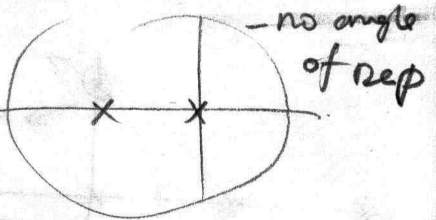
$$\left.\frac{K}{s(s+1)(s+2)}\right|_{s=-0.423} = 1 \Rightarrow K = 0.3845$$

for breakaway point.

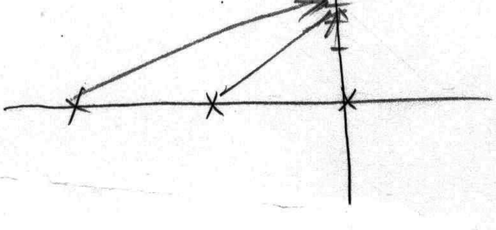
plotting Rules

1) plot pole/zero
2) Locus on Real Axis
3) asymptotes, Intersection & Angle
4) breakaway points (If any)
5) Angle of Departure (If any)
6) Crossing Imaginary axis (Routh-Hurwitz)





Crossing Imaginary axis (Alternate Approach)



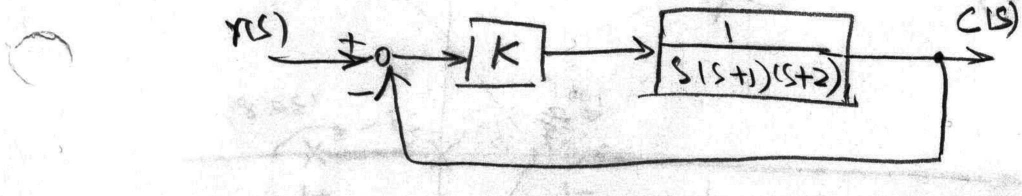
by trial & error using $j\omega$ axis & 180° criteria

I tried: $s = 1.5j$ (from asymptote ±60°

$\angle = 90^\circ + 56.31^\circ + 36.87^\circ$
$= 183.18^\circ > 180^\circ$

$s = 1.45$ 181.35°
1.42 180°
1.41 179.839°

Design using root-locus

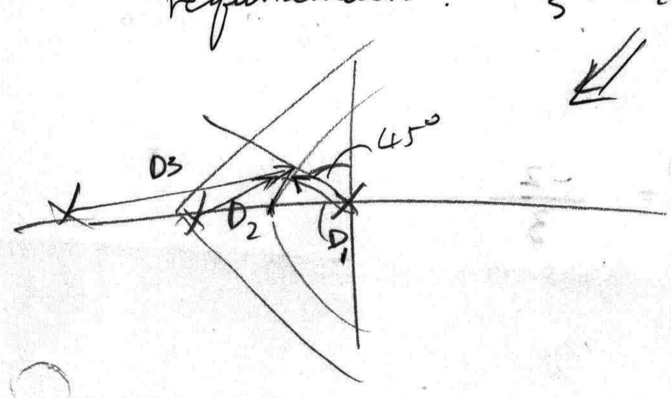


specs. given or transient response

if we need $\frac{1}{2}$ sec for time response. can not overshoot]

s = −2. dominate poles are not in this area.

requirements. $\xi = 0.707$ or $\underline{\underline{\xi = 1}} \Rightarrow$ critical damped at 0.423.



find closed loop poles from 180° criteria. & K from magnitude criteria

From Mag-Criteria

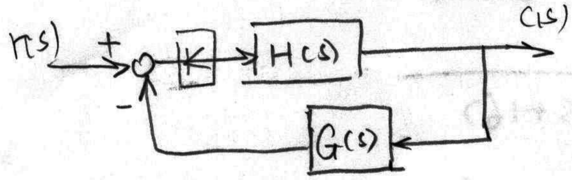
$$\frac{K}{D_1 D_2 D_3} = 1 \Rightarrow K = D_1 D_2 D_3$$

Root Locus Example

Longitudinal Dynamics of an Aeroplane

$$G(s)H(s) = \frac{K(s+1)}{s(s-1)(s^2+4s+16)}$$

Recall Root Locus



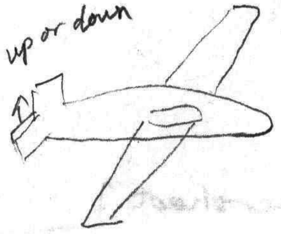
$$\frac{C(s)}{r(s)} = \frac{K H(s)}{1 + G(s)H(s)}$$

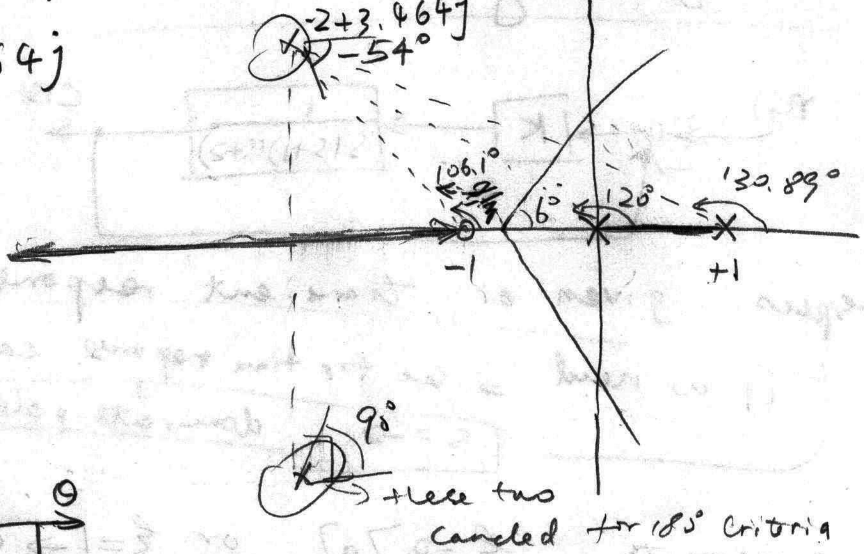
∠ criteria. $\angle G(s)H(s) = \pm 180°(2k+1)$

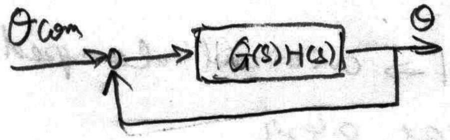
$|G(s)H(s)| = 1$.

$$S^2+4S+16 = (S-P)(S-\bar{P})$$

$$P = -2 \pm 3.464j$$







Intersection of Acymptotes

$$\sigma_a = \frac{-2-2-0+1-(-1)}{\underset{(4)}{n}-\underset{(1)}{m}} = \frac{-2}{3}$$

Angle of the Acyptotes

$$\angle\sigma_a = \frac{\pm 180^\circ(2K+1)}{n=m=3} = \pm 60, \pm 180^\circ$$

Angle of Departure (Arrivals)

Only for complex poles (zero's)

$$\angle Dep = +180^\circ - 90^\circ - 120^\circ - 130.89^\circ + 106.1^\circ$$

$$= -54.8^\circ$$

$$G(s)H(s) = \frac{K(S+1)}{S(S-1)(S-P)(S-\bar{P})}$$

Crossing of Imaginary Axis

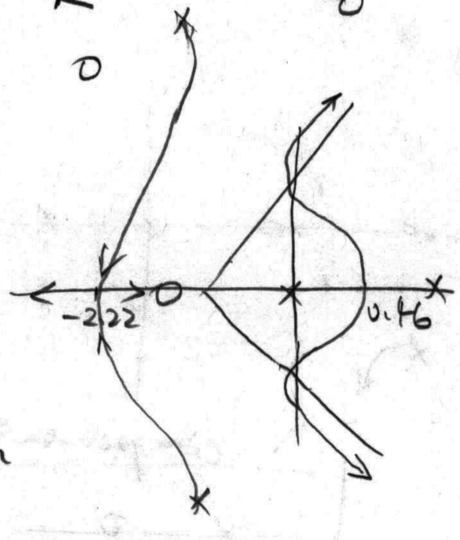
Use Routh-Hurwitz table

$$G(s)H(s) = \frac{K(S+1)}{S(S-1)(S^2+4S+16)}$$

The characteristic closed loop equation is,

$P(s) = s^4 + 3s^3 + 12s^2 + (K-16)s + K = 0$

s^4	1	12	K	0
s^3	3	$K-16$	0	
s^2	$\frac{52-K}{3}$	K		
s^1	$\frac{-K^2+59K-832}{3(52-K)}$	0		
s^0	K			



The s^1 term equals zero when

$K = 23.3$ & $K = 35.7$

↳ These are roots of $-K^2 + 59K - 832$

Now find crossing of imaginary axis from the auxillary equation.

$\frac{52-K}{3} s^2 + K = 0 \;|_{K=23.3}$ $S_{crossing} = \pm 1.56j$

$K = 35.7$ $S_{crossing} = \pm 2.56j$

For $K < 23.3$ → 2 unstable roots

For $23.3 < K < 35.7$ → we have 2 stable roots

For $K > 35.7$ 2 unstable roots

Find the Breakaway and Breakin points

compute $\frac{dK}{ds}$

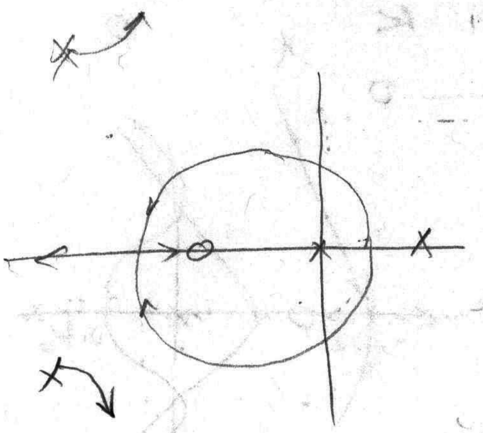
The closed loop T.F. $1 + \frac{K(s+1)}{s(s-1)(s^2+4s+16)} = 0$

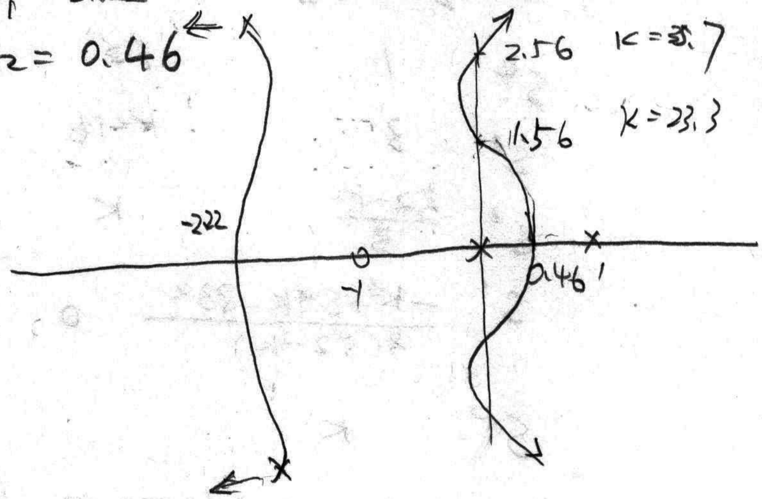
$K = \frac{-s(s-1)(s^2+4s+16)}{s+1}$

$\frac{dK}{ds} = \frac{-3s^4+10s^3+21s^2+24s-16}{(s+1)^2} = 0$

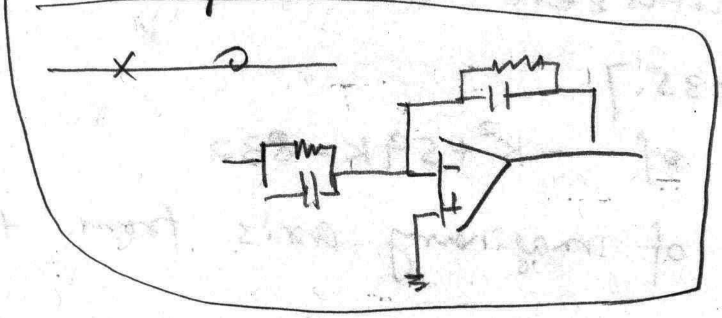
Two roots solve the angle criteria.

The roots are $s_1 = -2.22$
$s_2 = 0.46$

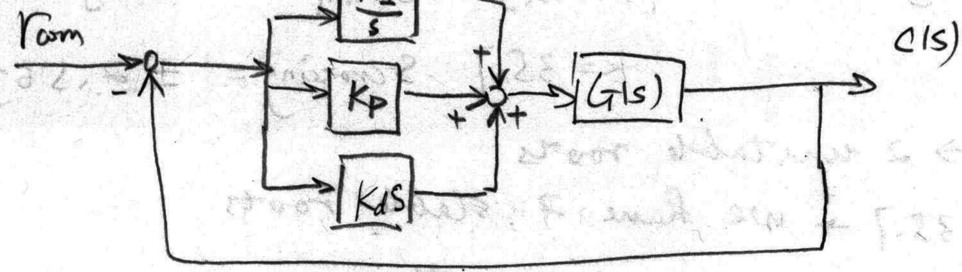




compensator:



PID Controllers



$\frac{K_I}{s}$ disturbances

K_p steady state error.

K_p is proportional gain.

K_D is derivative gain.

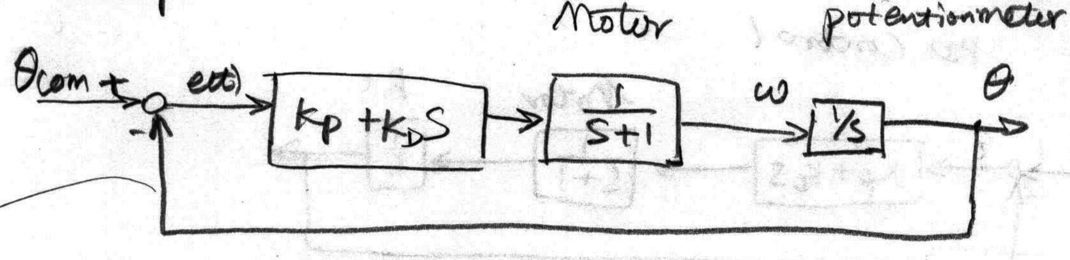
K_I is integral gain.

Case 1 only K_p & K_D.

The transfer function of a PD controller is $K_p + K_D s$

The K_p defines the steady state error and the K_D defines the transient response.

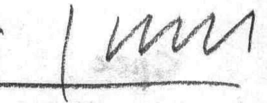
Example.



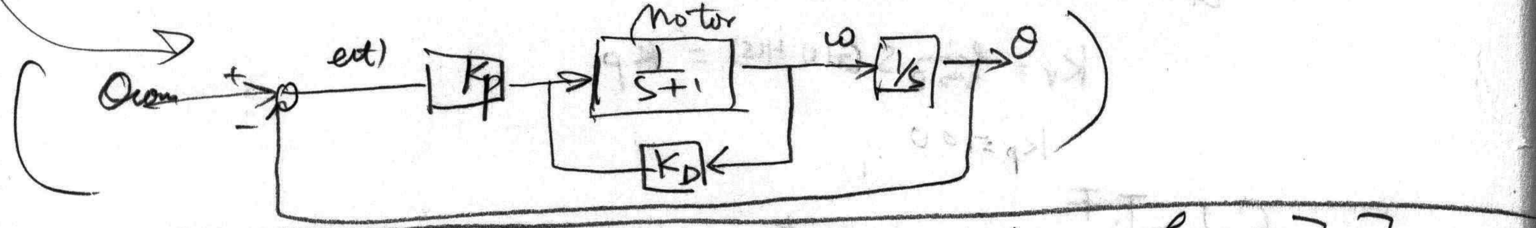
1.00	$\theta(k)$	
1.01	$\theta(k+1)$	
1.02	$\theta(k+2)$	

$\frac{\theta(k+1)-\theta(k)}{0.01}$ — small sample → huge error

$\frac{\theta(k+1)-\theta(k-1)}{0.02}$ → delay, phase



noise → × K_D → motor



The specifications are $e_{ss} \le 0.01$. $\xi = 0.707$

$$K_v = \lim_{s\to 0} S\,G(s)H(s)$$

$$G(s)\,H(s) = \frac{K_p + K_D S}{S(S+1)} = \frac{K_D(S+K_p/K_D)}{S(S+1)}$$

$$K_V = \lim_{s\to 0} \frac{\cancel{s}\cdot K_D(S+K_p/K_D)}{\cancel{s}(S+1)} = K_p$$

$$e_{ss} = \frac{1}{K_v} \le 0.01 \quad = \frac{1}{K_p} \le 0.01$$

$$\boxed{K_p \geq 100}$$

→ The requirement on steady state error sets the K_p value.

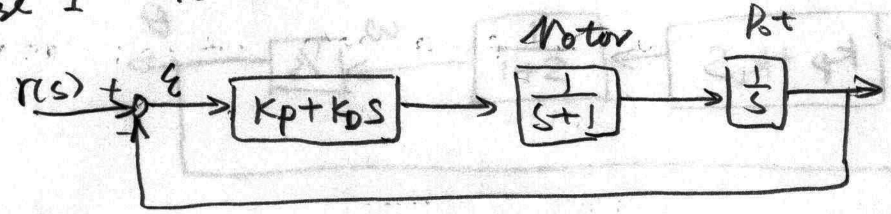
K_D sets Transient response i.e. $\xi = 0.707$

$$\frac{\theta(s)}{\theta_{com}(s)} = \frac{K_D(S+\frac{100}{K_D})}{S^2+(1+K_D)S+100}$$

$$\left.\begin{array}{l} \omega_n = 10 \\ 2\xi\omega_n = 1+K_D \\ \xi = 0.707 = \frac{1}{\sqrt{2}} \end{array}\right\} \Rightarrow K_D = 13.14$$

PID Control P681

Case 1 PD Control



spec $\varepsilon_{ss} = \frac{1}{K_v} \le 0.01$

$$K_v = \lim_{s \to 0} S\,G(s)\,H(s)$$

$$G(s)\,H(s) = \frac{K_D(S + \frac{K_P}{K_D})}{\quad}$$

$$K_v = \lim_{s \to 0} S\,G(s)\,H(s) = K_P$$

$K_p = 100$

C.L.T.F.

$$\frac{\theta(s)}{r(s)} = \frac{K_D(S + \frac{K_P}{K_D}) / S(S+1)}{1 + \frac{K_D(S + \frac{K_P}{K_D})}{S(S+1)}}$$

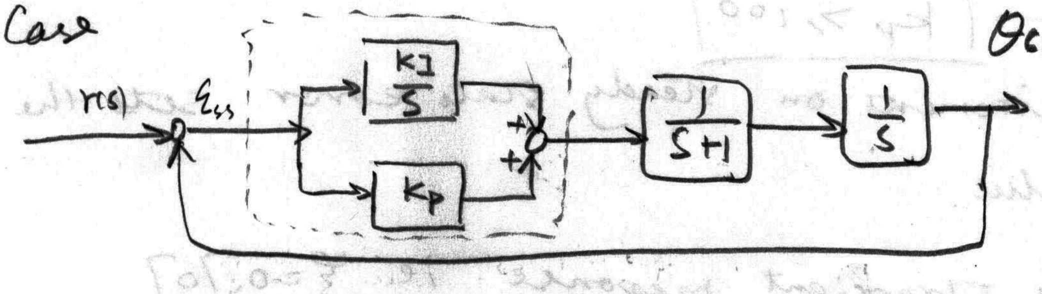
Char. eqn.

$$S^2 + (K_D + 1)S + K_P$$

If $K_P = 100$, $K_D = 13.14$.

NEXT PI control

Case



PI compensator

$$G_{PI} = \frac{K_I}{S} + K_P = \frac{K_I + K_P S}{S} = \frac{K_P(S + \frac{K_I}{K_P})}{S}$$

an increase in system type by 1

The integral term will reject disturbances, and will have better steady state error performance. But, we have another zero pole at the origin and this is bad for transient perf.

The closed Loop Transfer function is

$$\frac{\theta(s)}{r(s)} = \frac{\boxed{\frac{K_p(s+\frac{K_I}{K_p})}{s^2(s+1)}} \rightarrow \text{open Loop T.F.}}{1 + \frac{K_p(s+\frac{K_I}{K_p})}{s^2(s+1)}} = \frac{K_p(s+\frac{K_I}{K_p})}{s^3+s^2+K_p s+K_I}$$

Recall from Routh Hurwitz for a 3rd order system to be stable.

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

Stability $\quad a_1 a_2 > a_0 a_3$

$$K_p > K_I$$

To compare with PD controller.

Let $K_p = 100$, $K_I = 10$.

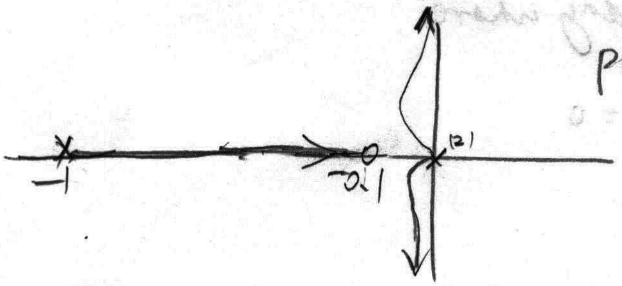
The C.L. pole locations would be $P_{1,2} = -0.45 \pm 9.985j$
$P_3 = -0.1$

↳ lightly damped.

But $K_V = \lim_{s \to 0} s G(s) H(s) = \infty$

$$e_{ss} = \frac{1}{K_V} = 0.$$

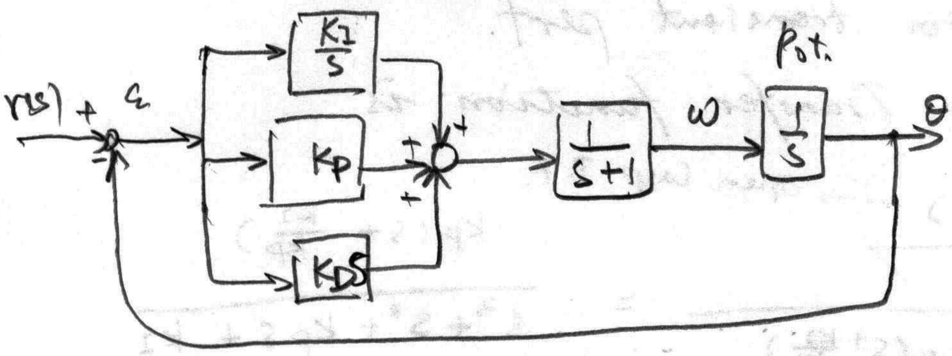
↳ follows ramp. (angular vel.) perfectly.



poor transient response.

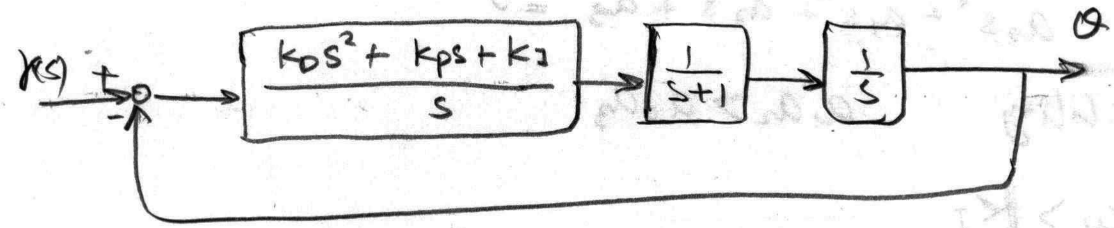
The PID Controller Case

We can now reject disturbances, achieve steady state error performance & good transient response



The transfer function of the P. I. D. compensator is,

$$G_{PID} = \frac{K_I}{s} + K_p + K_D s = \frac{K_D s^2 + K_p s + K_I}{s}$$



The closed loop transfer function becomes.

$$\frac{\theta(s)}{r(s)} = G_{CL} = \frac{K_D s^2 + K_p s + K_I}{p(s) = s^3 + (1+K_D)s^2 + K_p s + K_I}$$

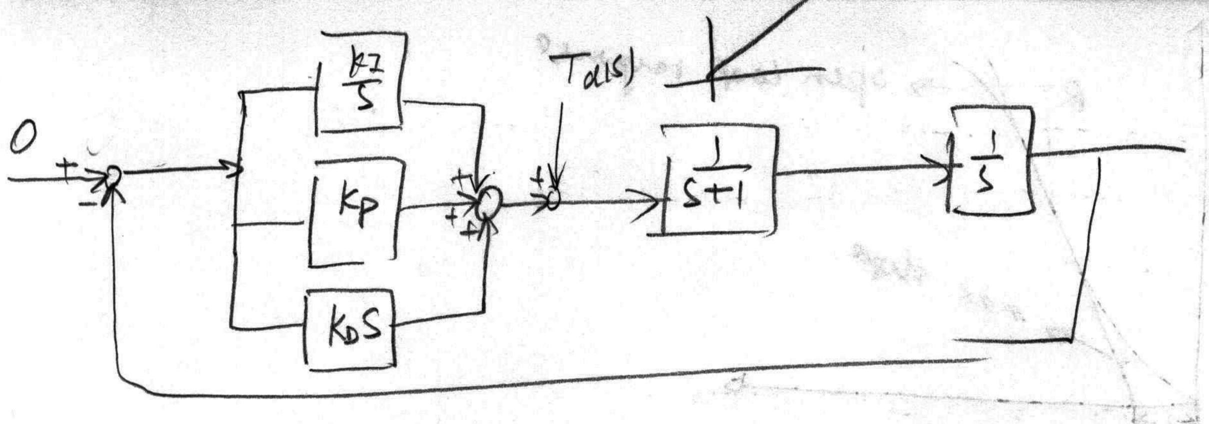
I want all three roots at $p = -5$

Characteristic equation: $p(s) = (s+5)^3 = s^3 + 15s^2 + 75s + 125$

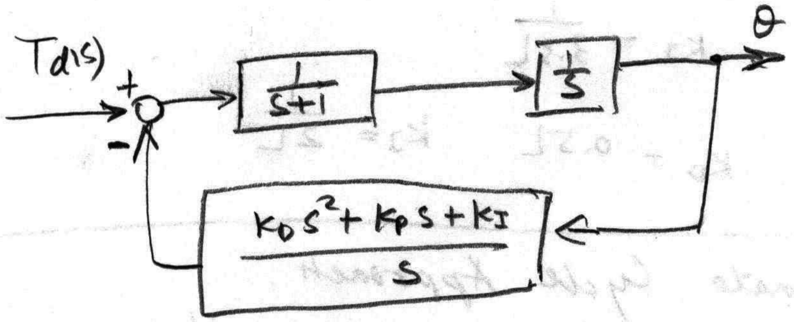
Therefore we get $K_p = 75$, $K_I = 125$, $K_D = 14$

We can put the roots anywhere.

$K_v = \infty$, $E_{ss}|_{ramp} = 0$.



Let's push on the motor



$$\frac{\theta(s)}{T_d(s)} = \frac{\frac{1}{s(s+1)}}{1+\frac{K_D s^2+K_p s+K_I}{s^2(s+1)}} = \frac{s}{s^3+(1+K_D)s^2+K_p s+K_I}$$

For a step input (I push on my robot)

$T_d(s)$ = step.

Use F.V.T.

$$\theta_{ss} = \lim_{t\to\infty}\theta(t) = \lim_{s\to 0} s\, S_{CL}(s) = \lim_{s\to 0} s\,\frac{s T_d/s}{s^3+(1+K_D)s^2+K_p s+K_I}$$

$$= 0$$

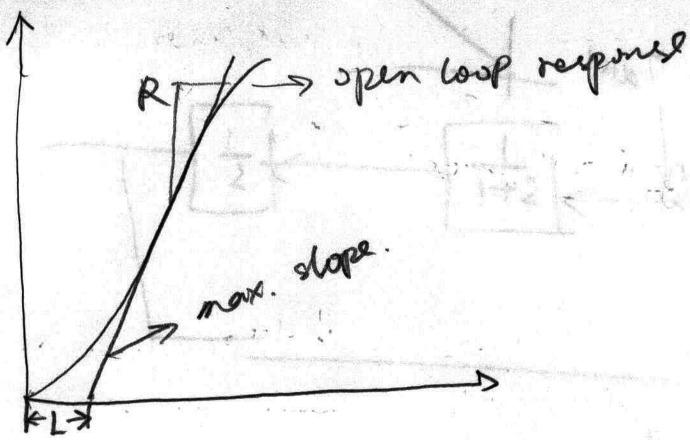
The robot goes back to zero.

Selecting Controller Gains

Industry often use "rule of thumb approach".

- Ziegler-Nichols Rules

Process Reaction Method.



P $k_p = \frac{1}{RL}$

PI $k_p = 0.9/RL$ $\quad k_I = \frac{1}{3.3L}$

PID $k_p = 1.2/RL$ $\quad k_D = 0.5L$ $\quad k_I = \frac{1}{2L}$

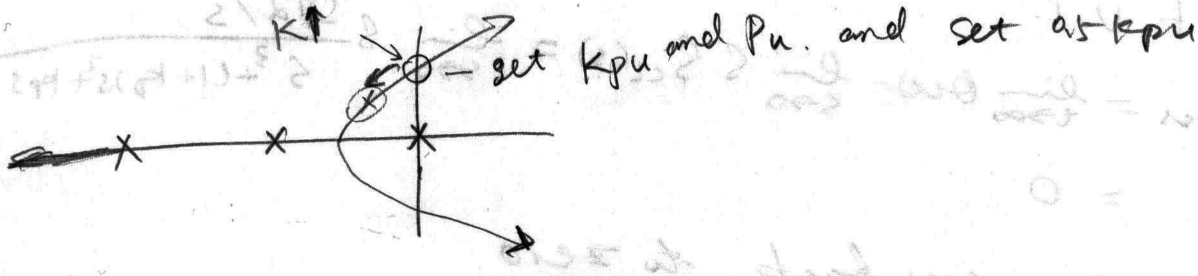
Method 2: Ultimate Cycle Approach

Set $k_D = k_I = 0$, and increase k_p until sustained oscillation. k_{pu} $\quad P_u$ is period

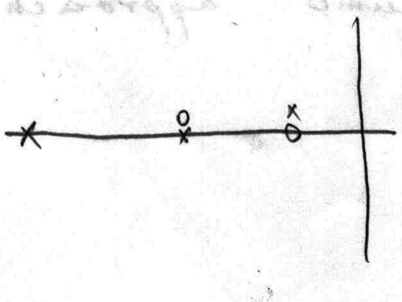
P $k_p = 0.5 k_{pu}$

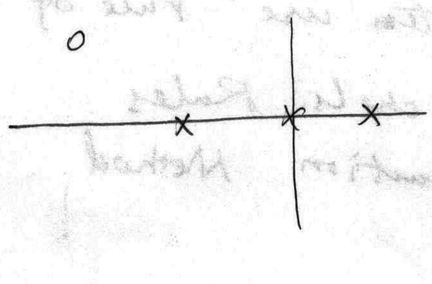
PI $k_p = 0.45 k_{pu}$, $k_I = \frac{1}{0.83 P_u}$

PID $k_p = 0.6 k_{pu}$, $k_I = \frac{2}{P_u}$, $k_D = 0.125 P_u$.



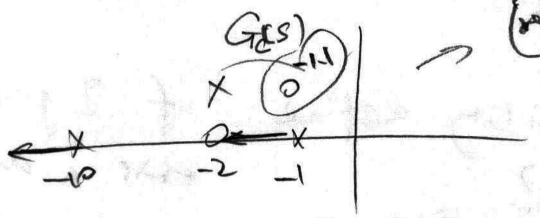
Troubles with pole/zero cancellation.





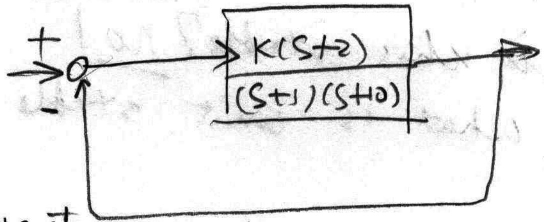
One sometimes tries pole/zero cancellation to improve performance.

residue is very small



$$G(s)H(s) = \frac{s+2}{(s+1)(s+10)}$$

$(s+1)$ → Time constant 1s → can not go faster.



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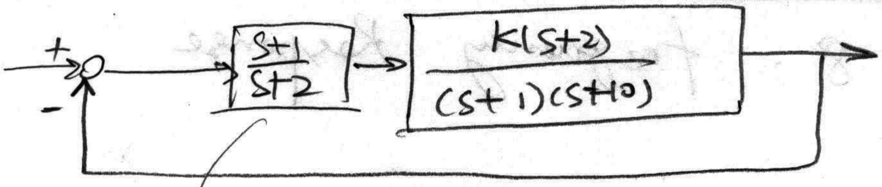
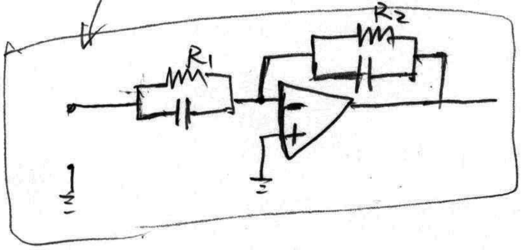
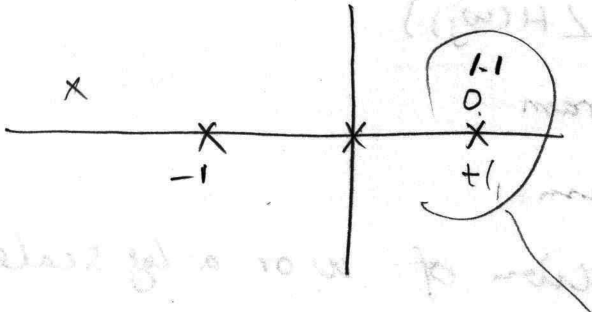
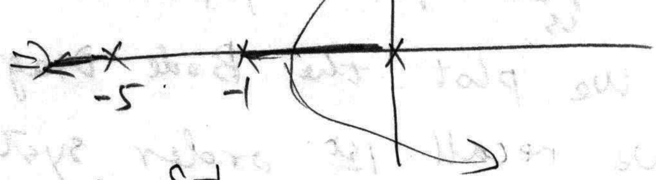


$$\Rightarrow G(s)H(s) = \frac{K}{s+10}$$



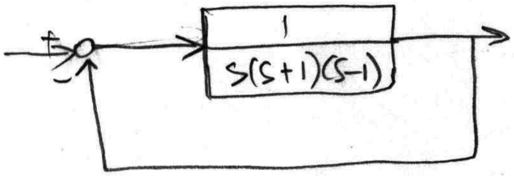
Can not do pole/zero cancellation in right 1/2 plane.





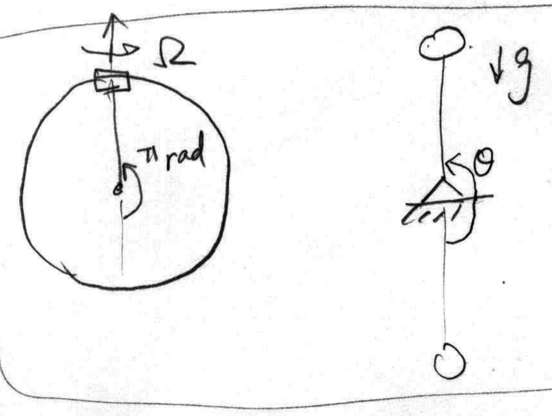
$$G_c(s) = \frac{s-1}{s+5}$$

($G_c(s) = s-1$ × cannot do derivative
Saturation)



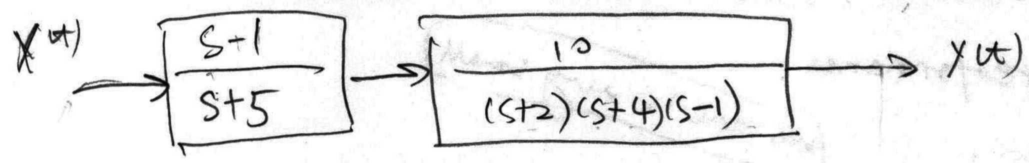
→ I will get a close loop pole stuck here.

$$G(s)H(s) = \frac{K}{s(s+1)(s-1)} \frac{(s-1)}{(s+5)}$$



$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

→ any ~~infinite~~ error can lead to system unstable

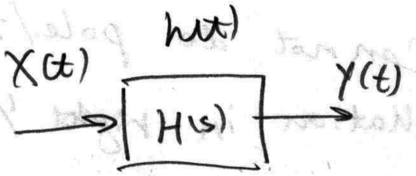


is this stable? no! unstable

stability define: if I give this a disturbance (slightest perturbation or error)

what is unit stable resp ?

Chapter 8 : frequency Response



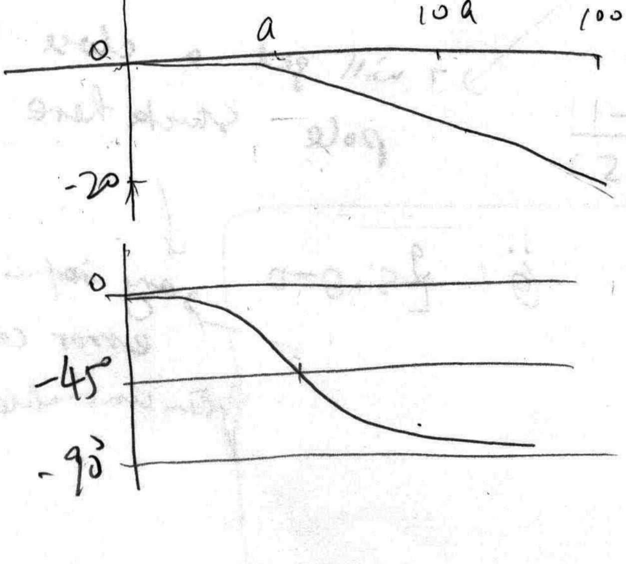
$x(t) = A \sin \omega t$

$y_{ss}(t) = A|H(\omega j)| \sin(\omega t + \angle H(\omega j))$

We plot the Bode Diagram.

We recall 1st order system

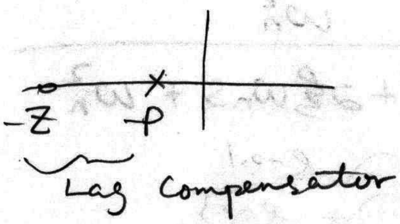
$|H(\omega j)|$ & $\angle H(\omega j)$ as function of ω or a log scale



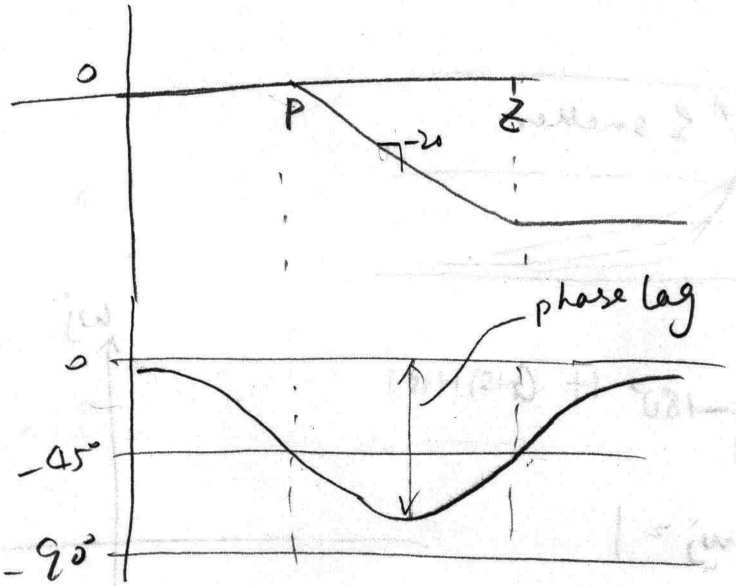
Phase Lag Compensator

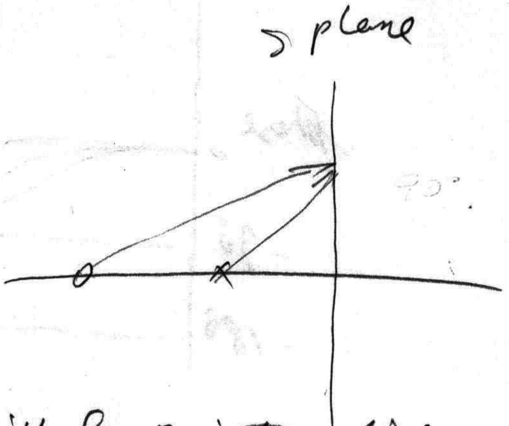
$$H(s) = \frac{p/z\,(s+z)}{s+p}$$

=



Bode Diagram of a Lag Compensator

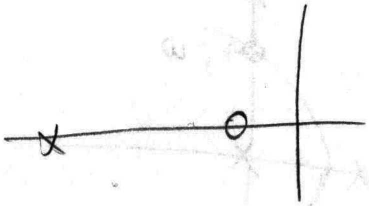


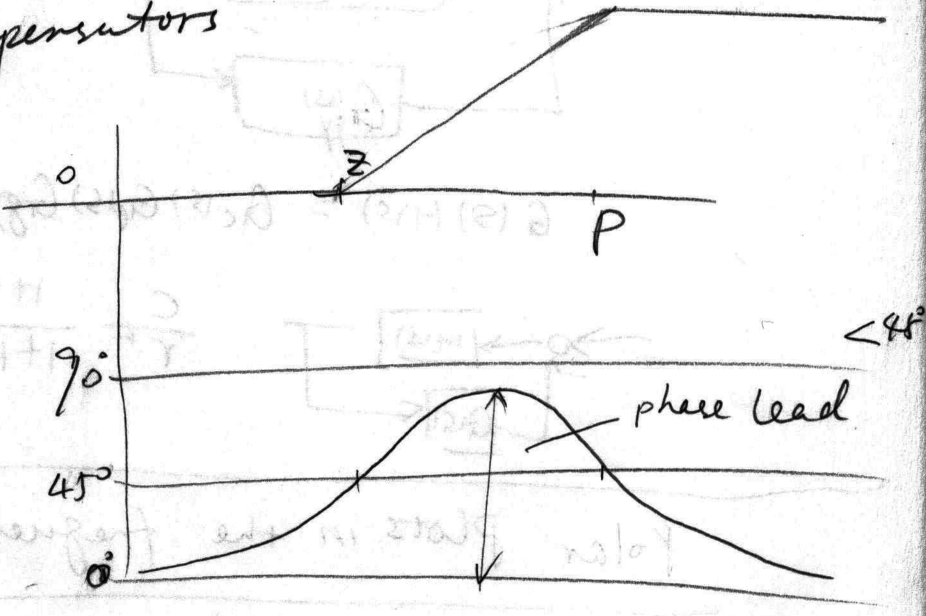


will have interesting steady state properties.

phase lead compensators

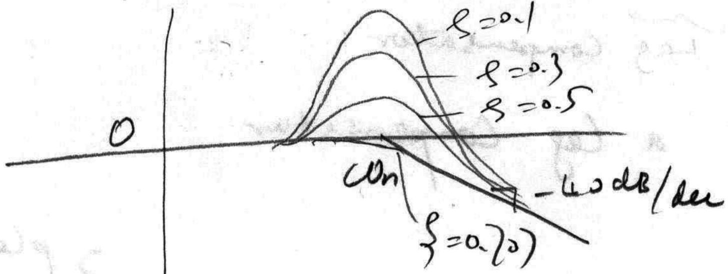
$$H(s) = \frac{P/z\,(s+z)}{s+P}$$

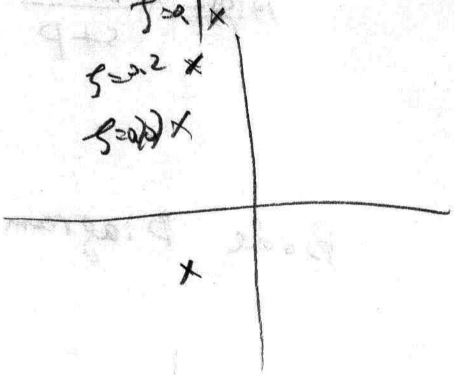


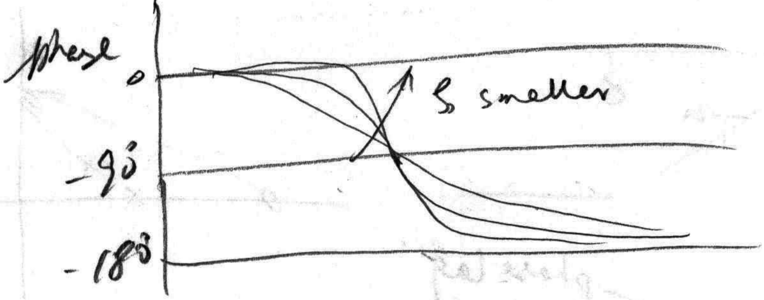


underdamped 2nd order systems

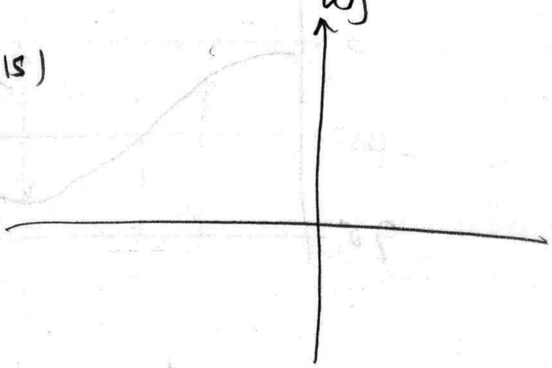
$$H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$



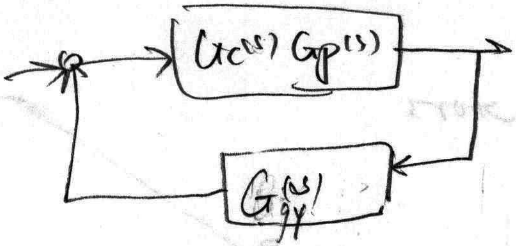




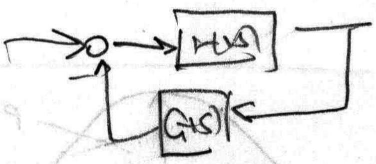
$\angle G(s)H(s)\big|_{s=\omega j} = -180^\circ \quad 1 + G(s)H(s)$



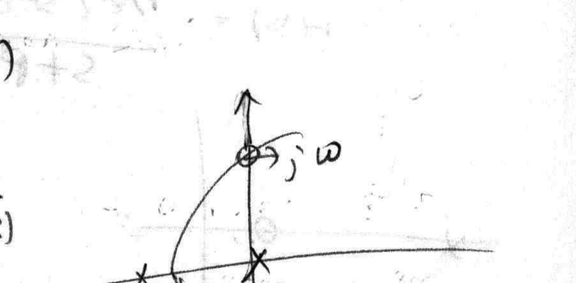
$|G(s)H(s)|_{s=\omega j} = 1.$



$G(s)H(s) = G_c(s)\,G_p(s)\,G_{gy}(s)$

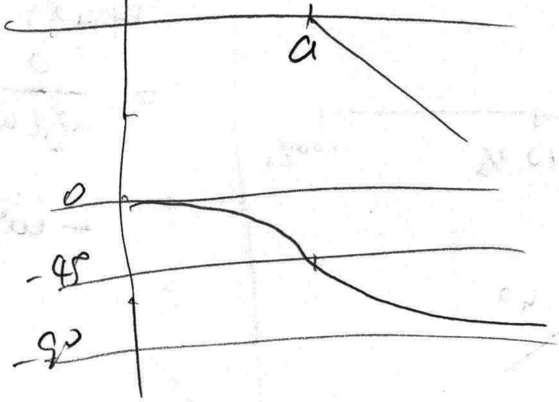


$$\frac{C}{r} = \frac{H(s)}{1 + H(s)G(s)}$$

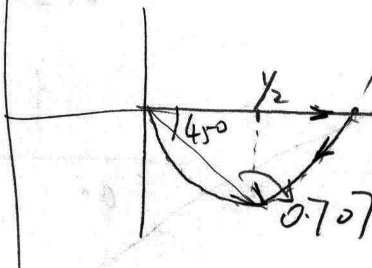


Polar plots in the frequency Domain

We plot the magnitude & phase of $G(s)H(s)$ as a vector $H(s) = \frac{a}{s+a}$

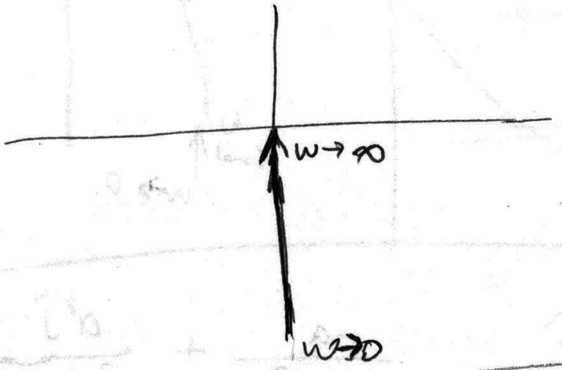


Polar plot of $H(s) = \frac{a}{s+a}$.



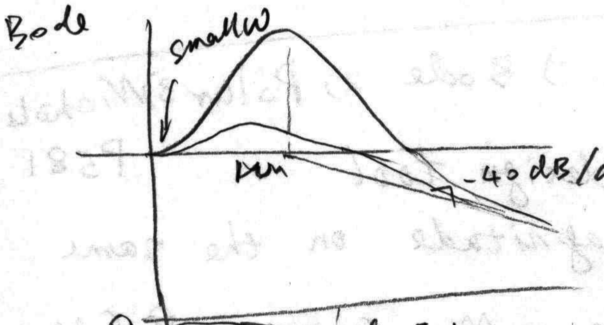
This draws out a semicircle.

Polar plot of $H(s) = 1/s$.

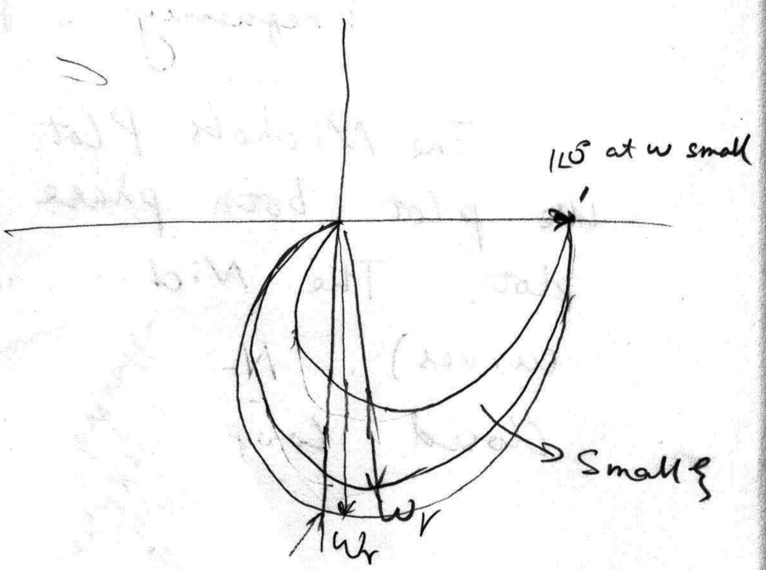


straight line coming into the origin.

2nd order systems

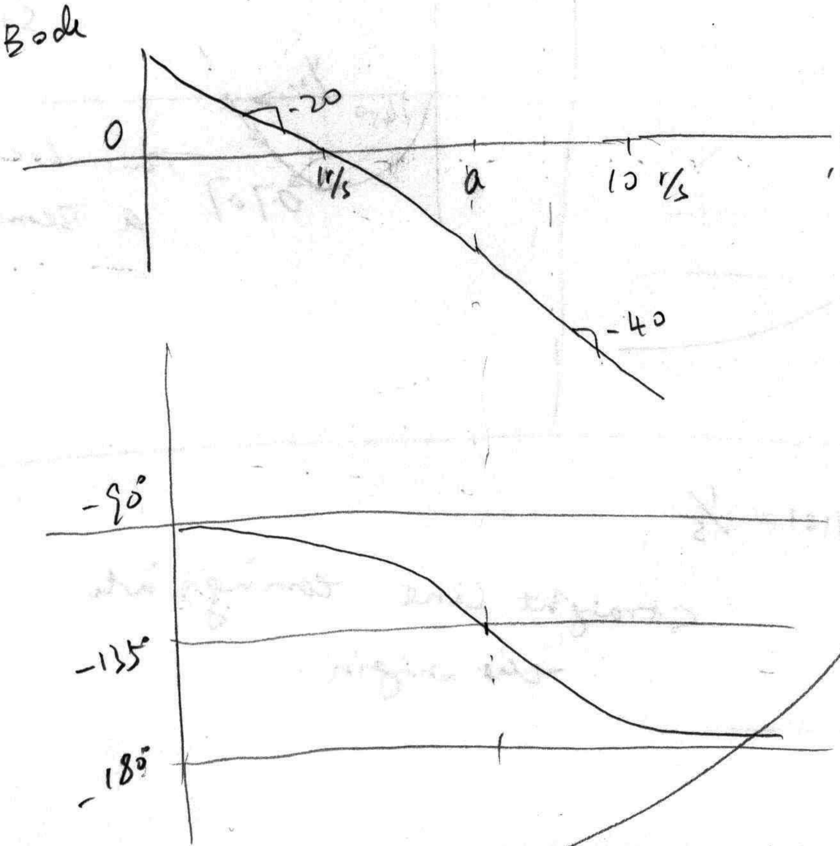


Polar plot of $H(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s\omega_n}$



Example $H(s) = \frac{a}{s(s+a)}$

Bode

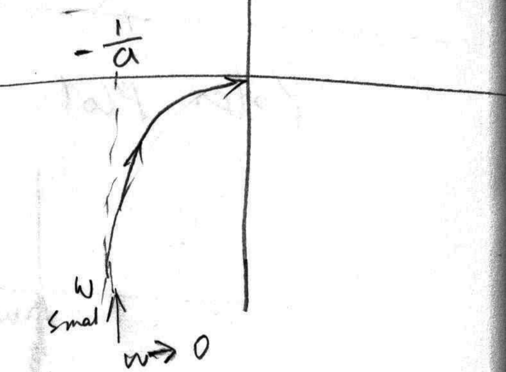


Polar

$$H(\omega j) = \frac{a}{\omega j(\omega j+a)}$$

$$= \frac{a}{-\omega^2+a\omega j} \quad \text{(now normalize it)}$$

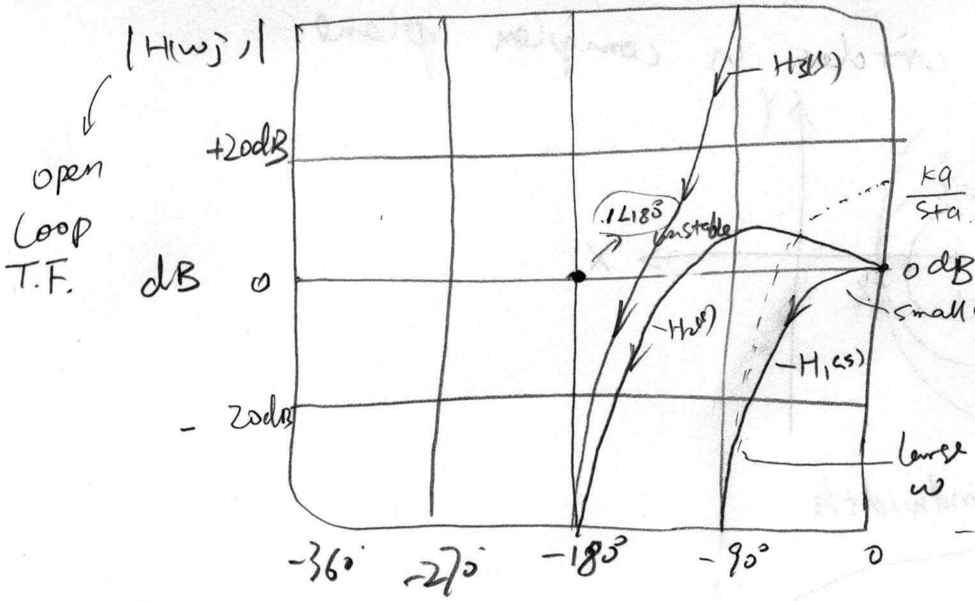
$$=$$



$$\frac{-a\omega^2}{+a^2\omega^2+\omega^4} + \frac{+a^2\omega j}{+a^2\omega^2+\omega^4} = \frac{-a}{a^2+\omega^2} + \frac{a^2 j}{a^2\omega+\omega^3}$$

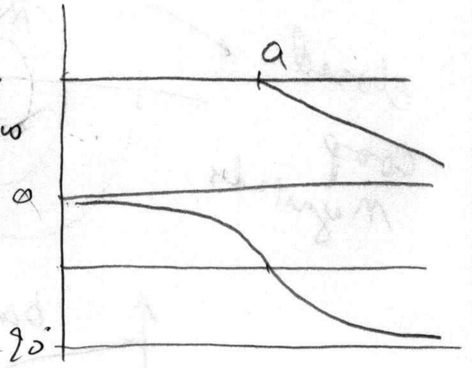
Frequency Plots 1) Bode 2) Polar 3) Nichols

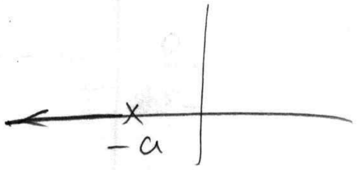
The Nichols Plot is a design tool. P581 We plot both phase and magnitude on the same plot. The Nichols also has M- curves & (N- curves). N- curves are curves of equivalent closed loop magnitude.



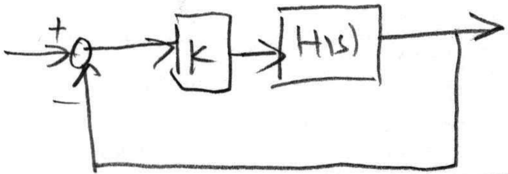
$H_1(s) = \frac{a}{s+a}$

Bode





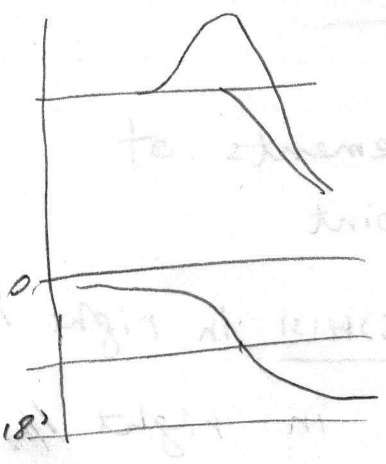
Never become unstable whenever K.



$$S_{CL} = \frac{KH(s)}{1+KH(s)}$$

↓

$H_2(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ → will not cross center point $1\angle 180°$



$H_3(s) = \frac{a}{s(s+a)}$ → close to $1\angle 180°$

One Also has plotted M-curves on the Nichols Chart

G(H) = open Loop T.F.

$$M = \frac{G(s)}{1+G(s)}$$

write $G(s) = X+Yj$

$$M = \frac{|X+Yj|}{|1+X+Yj|}$$

↳ constant

$M\sqrt{(1+X)^2+Y^2} = \sqrt{X^2+Y^2}$

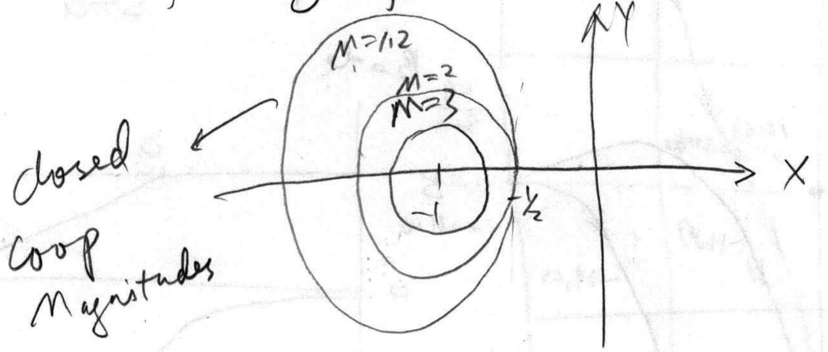
square both sides

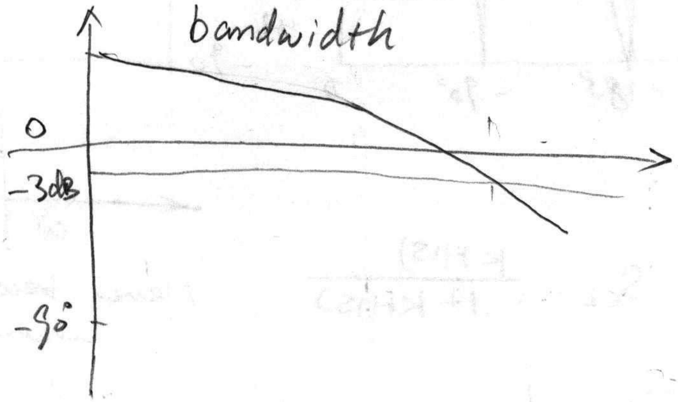
$M^2((1+X)^2+Y^2) = X^2+Y^2$

and do lots of algebra

We get the equation (P.577) $\left(X - \frac{M^2}{1-M^2}\right)^2 + Y^2 = \left[\frac{M}{(1-M^2)}\right]^2$

family of circles in complex plane





Nyquist Stability Criteria

$N = Z - P$

N = # of clockwise encirclements of the $-1+0j$ point.

(C.L poles)

Z = # of zeros of $1+G(S)H(S)$ in right ½ plane.

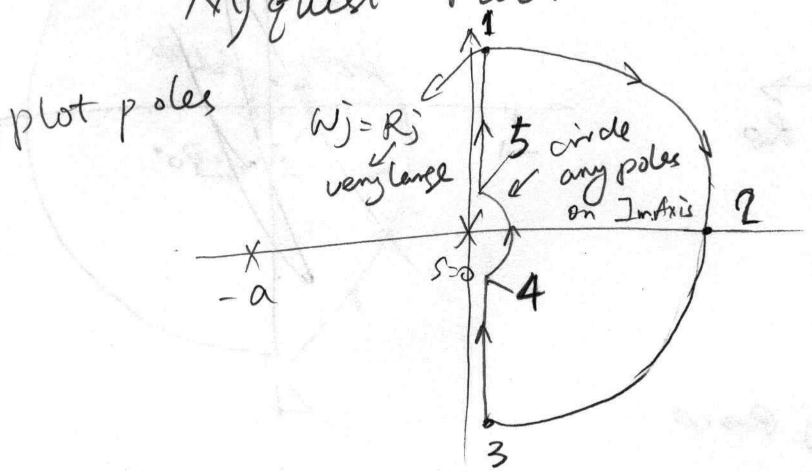
p = # of open loop poles in right ½ plane

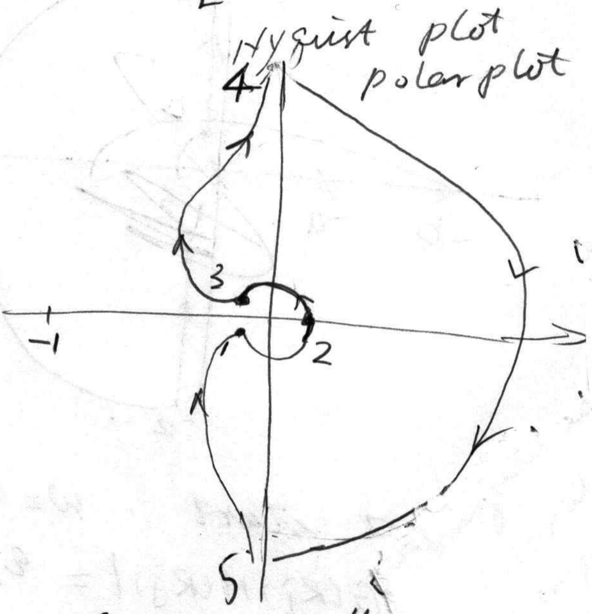
This criteria allows us to determine the closed loop stability of a system from open loop transfer function (frequency response)

$\boxed{\text{Typically we want } N = P = Z = 0.}$

Example 1 P540 $\frac{K}{s(s+a)} = G_{OL}(s)$

Nyquist Path





At 1, ω is very large

$|G(\omega j)| = \left|\frac{K}{s(s+a)}\right|_{s=\omega j} = \varepsilon$ very small

$\angle -180°$

At 2, ω is very large

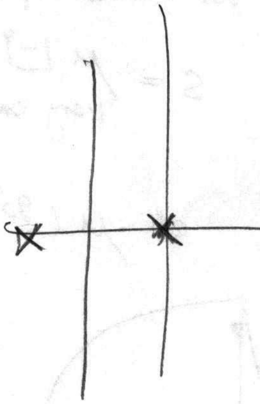
$|G(\omega j)| = \varepsilon \angle 0°$

At 3. ω is very large

$|G(\omega j)| = \varepsilon \angle +180°$

At 4 $|G(\omega j)| \approx \infty$

$\measuredangle G(\omega j) = +90°$



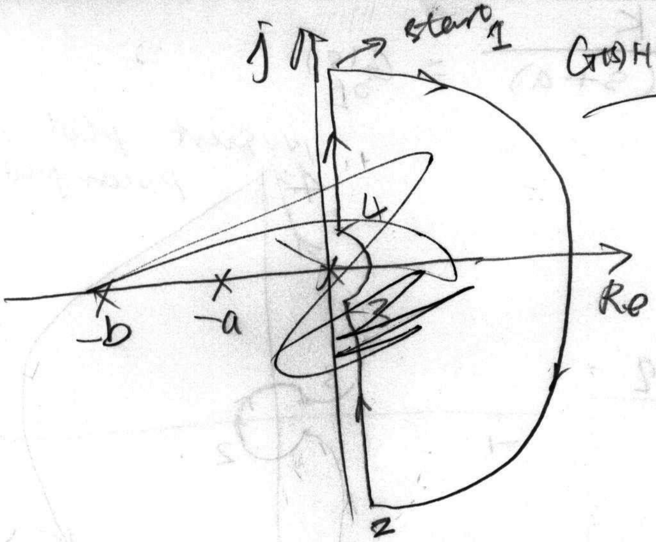
Example 2

$G(s)\,H(s) = \frac{K}{s(s+a)(s+b)}$

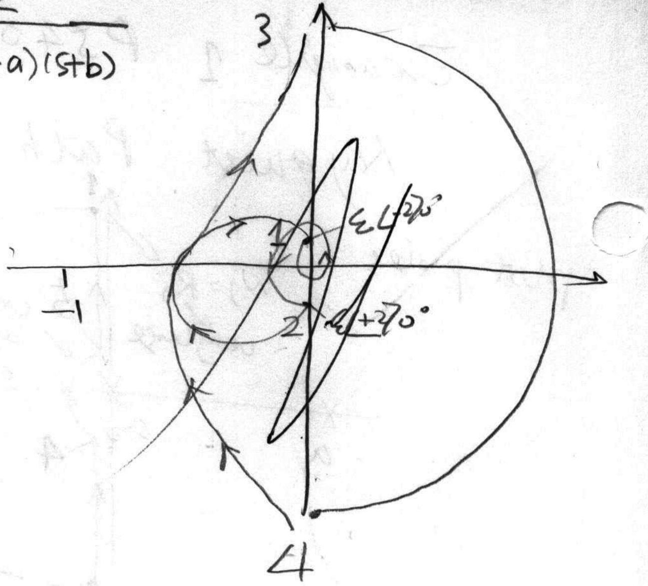
Evaluate stability as K increases by using Nyquist stability Criteria.

$K = ab(a+b) \rightarrow$ cross the imaginary axis

Draw the Nyquist Path



$$G(s)H(s) = \frac{K}{s(s+a)(s+b)}$$



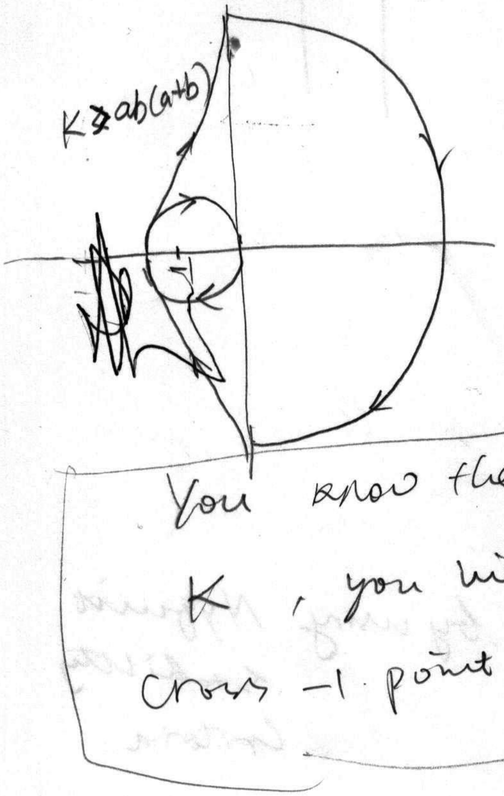
1) Angle at start , $w = Rj \quad R \to \infty$

$|G(Rj)H(Rj)| = \varepsilon_{\approx}$ verysmall $\angle -270^\circ$

2) we have $|G(s)H(s)| = \varepsilon \to 0. \quad \angle +270^\circ$

3) $s = \underset{\text{very small}}{\mu} \angle -90^\circ \quad |G(s)H(s)|\Big|_{s=\mu\angle -90^\circ} \Rightarrow \infty \quad \angle G(s)H(s) \mid = 90^\circ$

4) $s = \mu \angle 90^\circ \quad |G(s)H(s)| \Rightarrow \infty \quad \angle G(s)H(s) = -90^\circ$



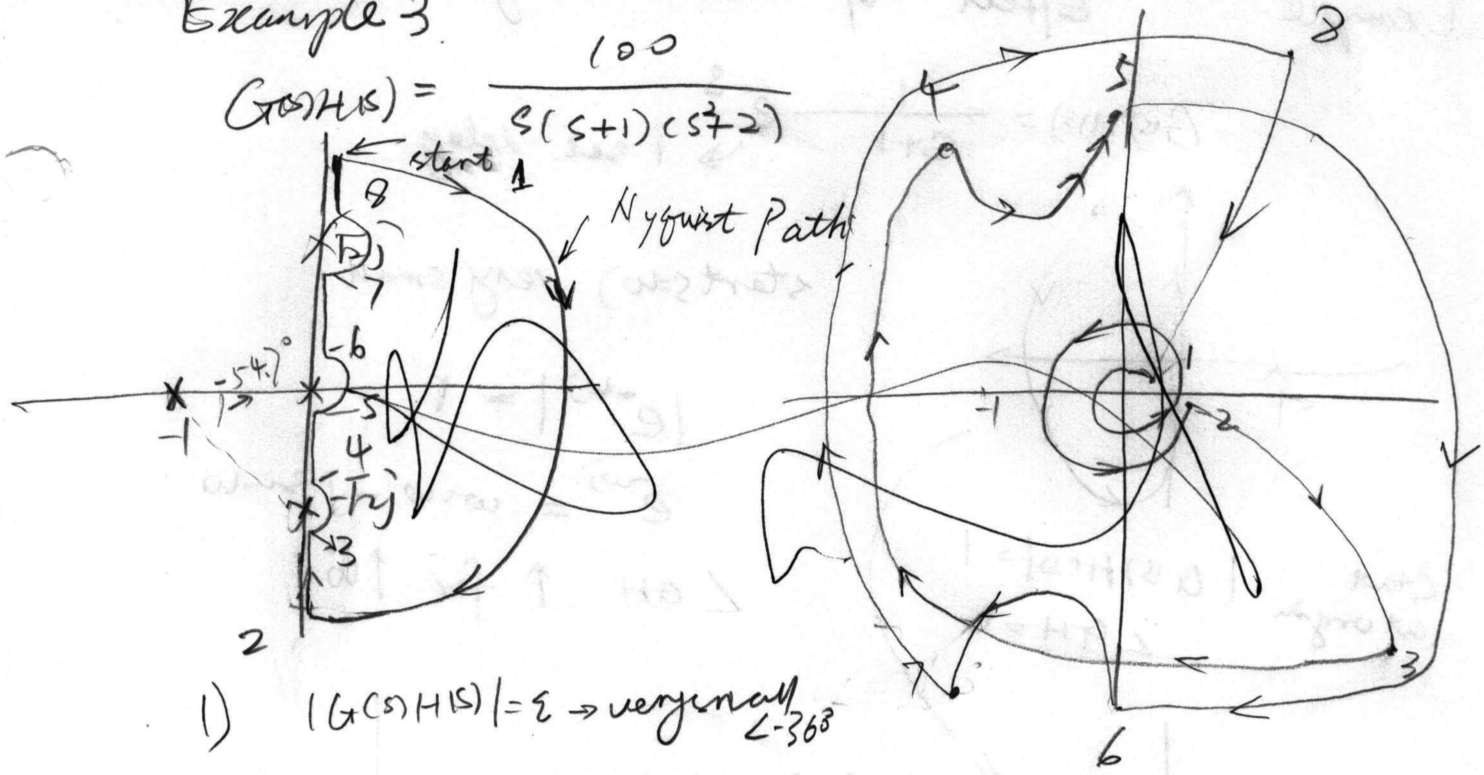
two closewise encirclements of -1 point for $K > ab(a+b)$

$$\underset{\overset{\|}{2}}{N} = Z - \underset{\overset{\|}{0}}{P} \Rightarrow Z = 2$$

You know the frequency response, and you adjust K, you will find at what value of K, Nyquist plot cross -1 point.

Example 3

$$G(s)H(s) = \frac{100}{s(s+1)(s^2+2)}$$



1) $|G(s)H(s)| = \varepsilon \rightarrow$ very small $\angle -360$

2) $s = -Rj$ $|GH| = \varepsilon$ $\angle +360$

3) $s = (-\sqrt{2} - \varepsilon)j$ $+90° + 90° + 90° + 54.7° = 324.7°$

$|GH| \approx \infty$

$-90° + 90° + 90° + 54.7° = 144.7°$

4) $s = (\sqrt{2} + \varepsilon)j$

5) $s = -\mu j$ $\angle GH = 90°$ $|GH| \approx \infty$

6) $s = \mu j$ $\angle GH = -90°$ $|GH| \approx \infty$

7) mirror of 4

8) mirror of 3

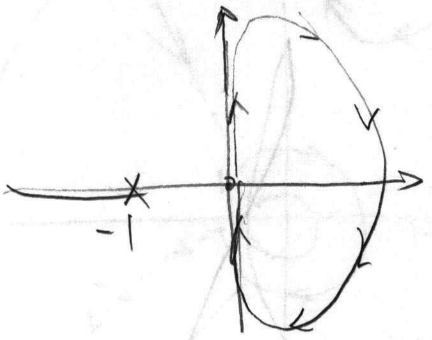
2 enclosements of -1 pt.

$$\underset{2}{\overset{}{N}} = Z - \underset{0}{P} \Rightarrow Z = 2.$$

Example   Effect of time delay

$$G(s)H(s) = \frac{1}{s+1} e^{-s}$$

$\rightarrow$ 1 sec. delay



start $s=\omega j$ very small

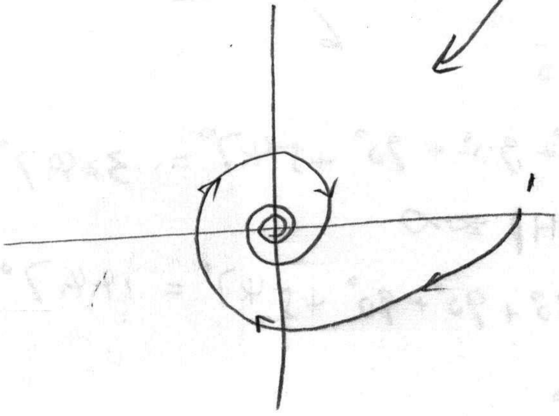
$$|e^{-\omega j}| = 1$$

$$e^{-\omega j} = \cos\omega - j\sin\omega$$

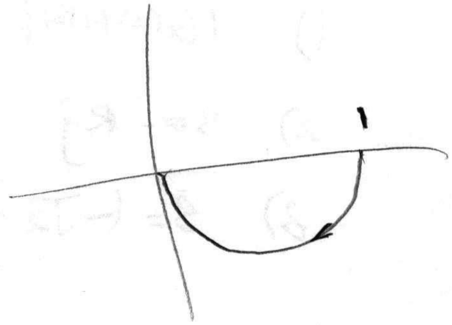
$\angle GH \uparrow$ for $\uparrow \omega$

Start at origin $|G(s)H(s)| = 1$

$\angle GH = 0$

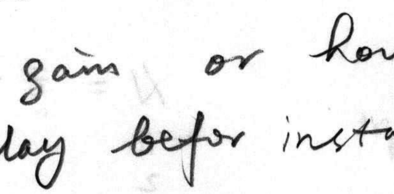


instead of



P562-567   Frequency specifications.

Specifications on how much we can increase the O.L. gain or how much more the phase delay befor instability occurs. These are similar to margin of stability will effect performance of system



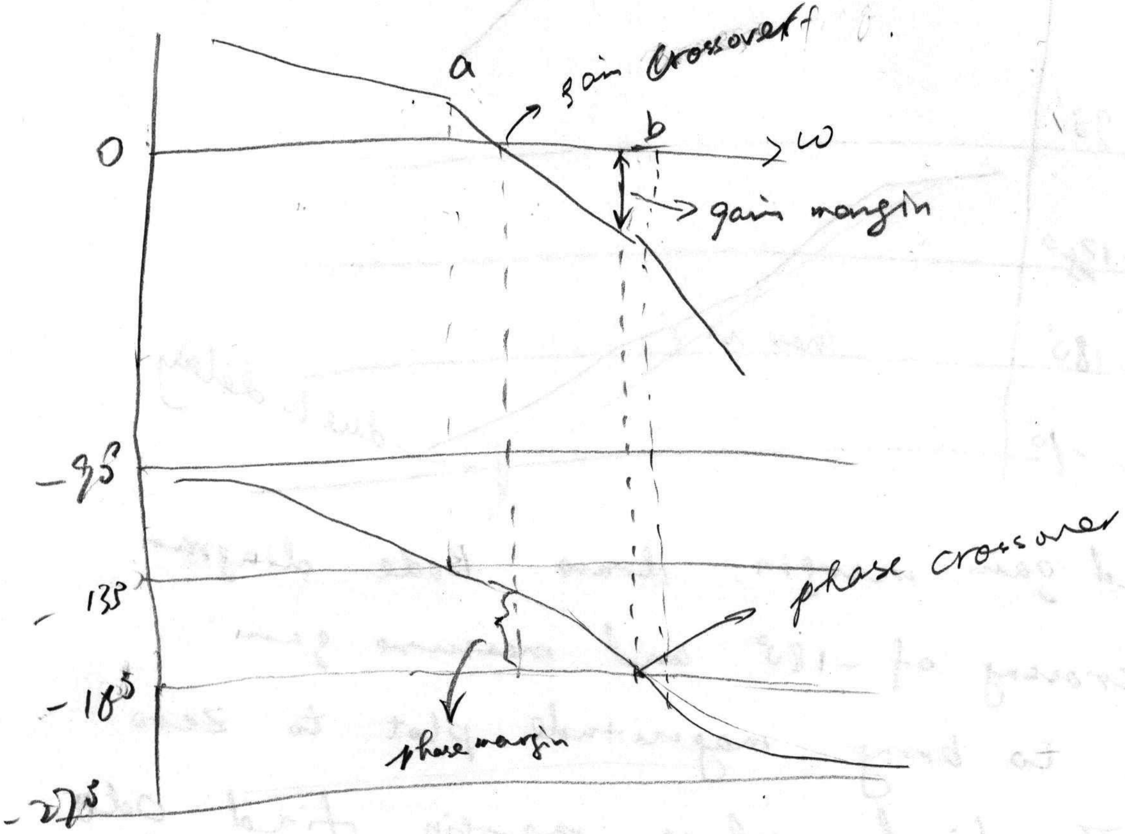
$\angle KGH|_{s=\omega j} = \pm 180$ $\Rightarrow$ unstable at ω

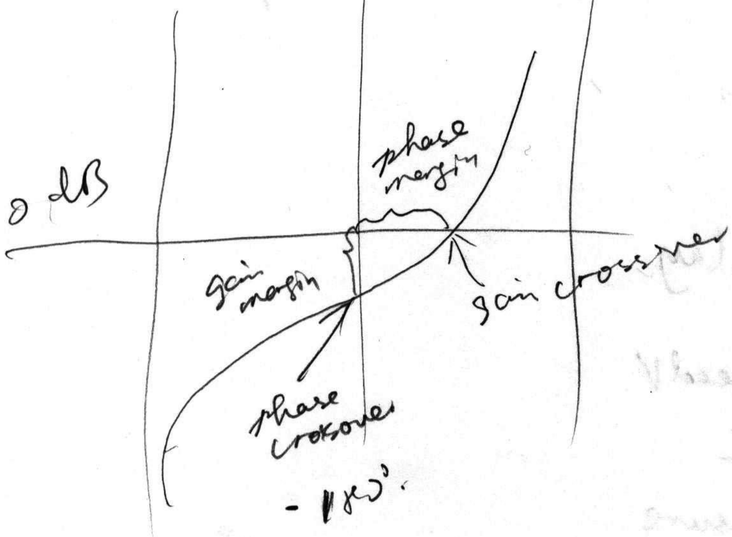
$|KGH|\,|_{s=\omega j} = 1$

strength = 1.25 x Dead weight
Light X 2.25

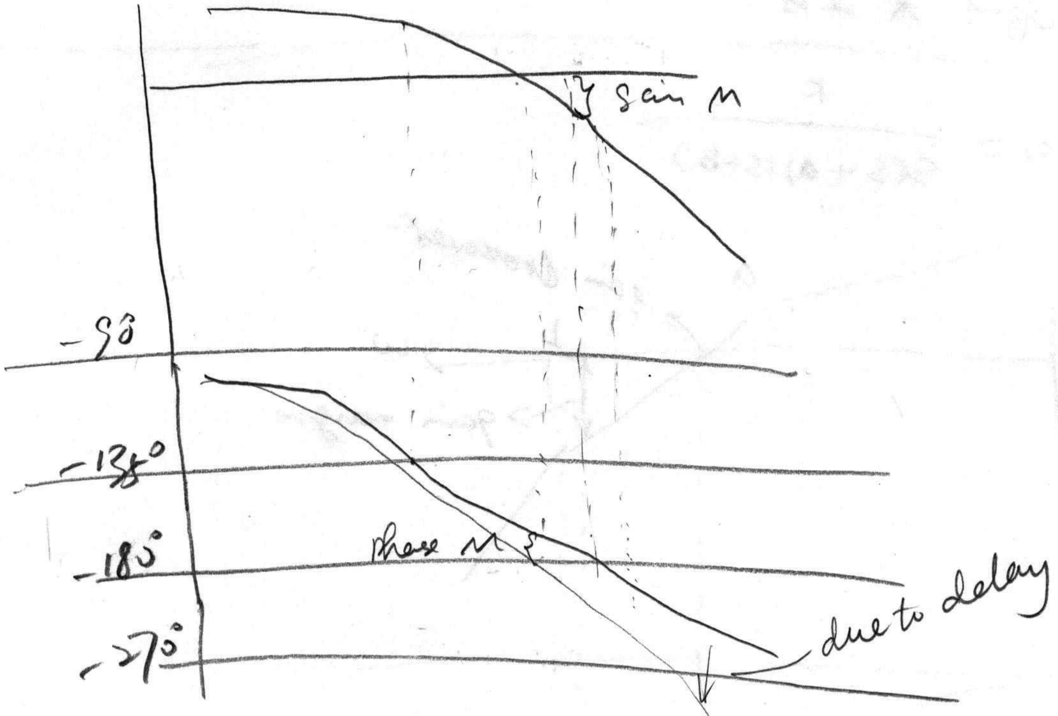
civil engineering for building house

$$H(s) = \frac{K}{s(s+a)(s+b)}$$





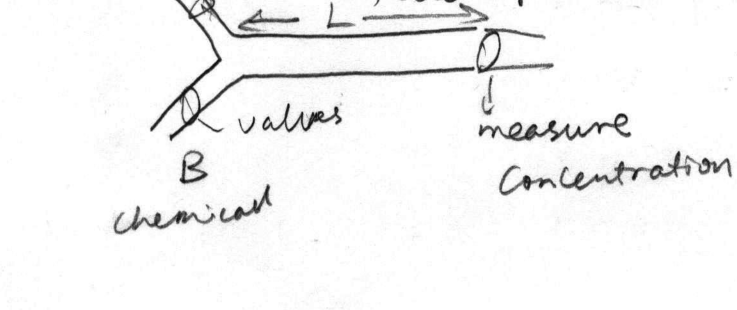
Phase Margin & Gain Margin

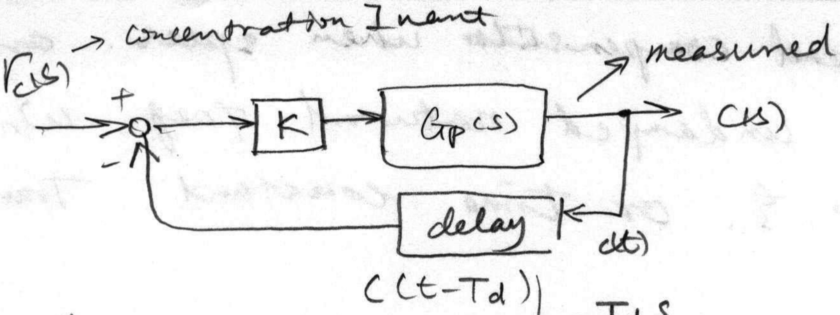


To find gain margin, draw Bode diagram find crossing of -180° and measure gain in dB, to bring magnitude plot to zero dB. To find phase margin find 0dB crossing on magnitude and measure phase need to get -180°.

Advanced Ideas

Effect of Time Delay





$C(t-T_d)$ ↓ $e^{-T_d s}$

$$\frac{L^{m}}{V_{m/s}} = T_d \text{ sec}$$

The open loop Transfer function is

$$G_{OL} = K G_p(s) e^{-T_d s}$$

If we plot magnitude

$$|G_{OL}(wj)| = K|G_p(wj)|\overbrace{|e^{-T_d wj}|}^{1} = \overbrace{K|G_p(wj)|}^{\text{open loop magnitude is same regardless}} \text{ of delay}$$

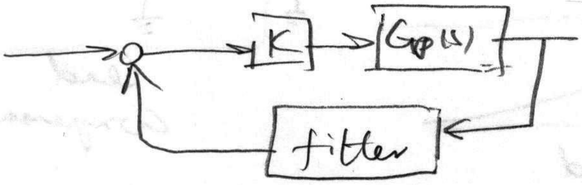
shower example

$$\angle G_{OL}(wj) = \angle G_p(wj) + \angle e^{-T_d wj}$$

$$= \angle G_p(wj) - \underbrace{T_d w}_{\text{added phase shift}}$$

Linear with freq

Time delay "eats away" my phase margin.



↓ lowpass filter

They average

Compensator Design

We cannot achieve specified dynamic using only feedback, we need some compensator.

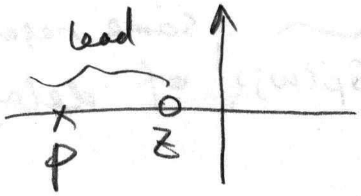
We will start by designing lead compensators using root-locus techniques.

One uses a lead compensator when specs. are in terms of, undamped natural freq, ω_n, damp ratio, ξ, or time constant. Transient Dynamics.

A lead compensator takes the form.

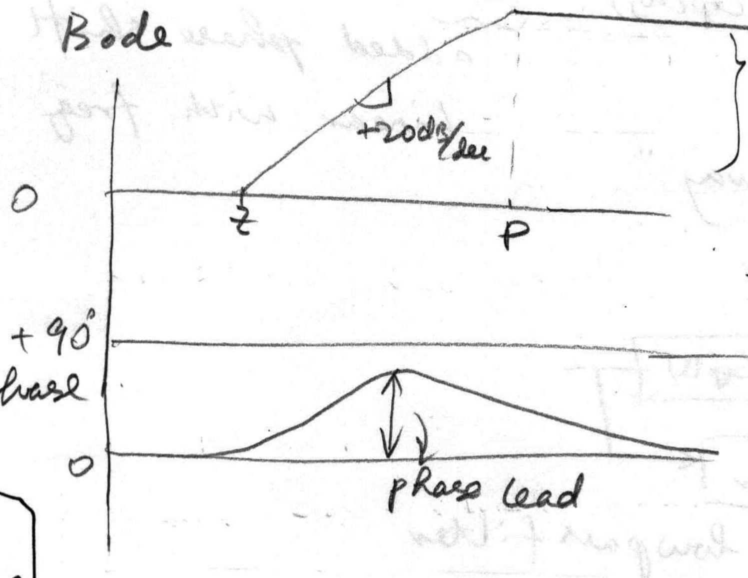
$$G_{Lead}(s) = \frac{K_c(s+Z)}{s+P} \qquad |Z|<|P|$$

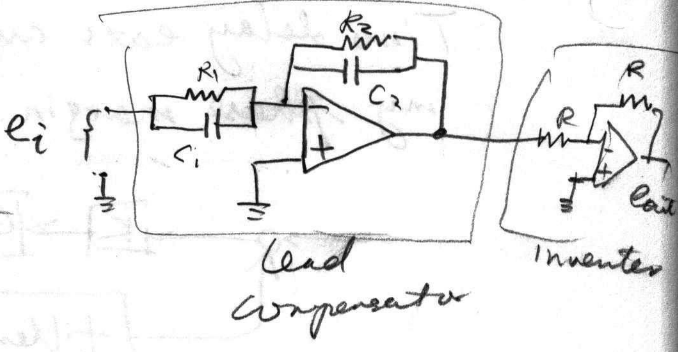
$$= \frac{aTs+1}{Ts+1} \qquad a>1$$



$$= \frac{Ts+1}{\alpha Ts+1} \qquad \alpha<1$$

(phase lead)





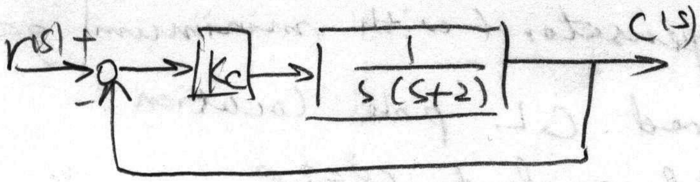
Lead Example

Let's say we have a system with O.L. T.F.

$$G_{OL} = \frac{1}{s(s+2)}$$

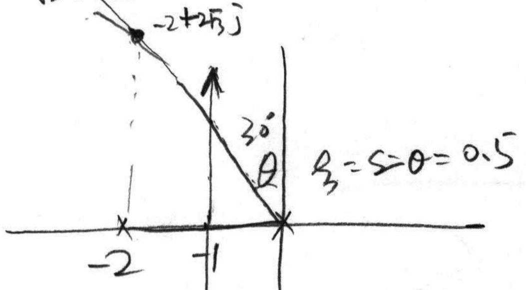
Specs. The closed loop undamped natural freq. must be $\omega_n = 4$ r/s & damping ratio $\xi = 0.5$.

Can we do this with feedback alone.



$$G_{CL}(s) = \frac{C(s)}{R(s)} = \frac{\frac{K_c}{s(s+2)}}{\underbrace{1 + \frac{K_c}{s(s+2)}}_{} \rightarrow 1 + G_{OL}(s)}$$

Draw Root Locus



$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$ spec.

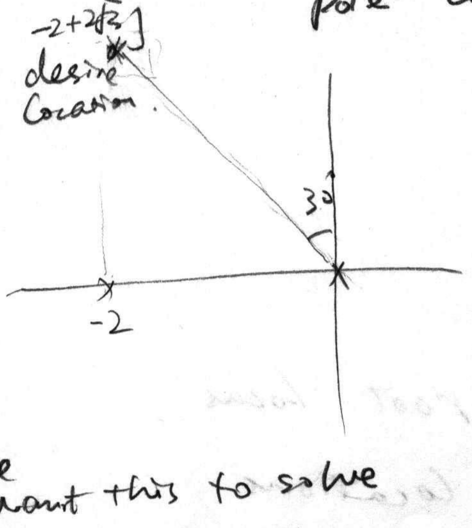
for $\xi < 1$, poles are

$$p, \bar{p} = -\xi\omega_n \pm \omega_n\sqrt{1-\xi^2}\, j$$
$$= -2 \pm 2\sqrt{3}\, j \text{ (for specs)}$$

Design of lead compensator

1) check if feedback alone can achieve spec. C.L. pole locations.

2) Determine angle deficiency at spec. C.L. pole location



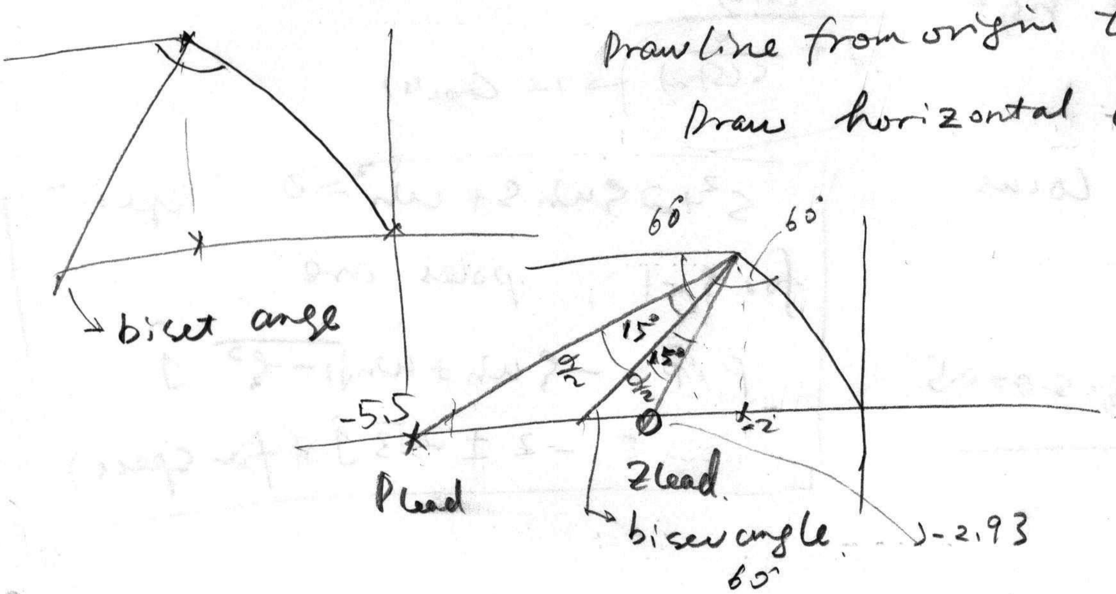
The angle deficiency is,
$-180° - (-210°) = 30°$, we need to add 30° with a lead compensator.

we want this to solve −180° criteria.

$$\angle G_{OL}(s)\Big|_{s=2+2\sqrt{3}j} = \angle \frac{K_c}{s(s+2)} = -120° - 90° = -210°$$

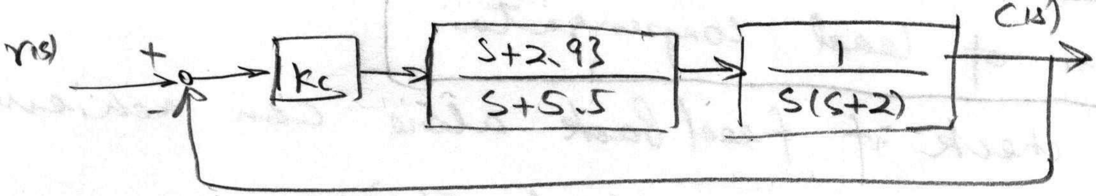
3) compute Lead compensator (with minimum gain) to achieve desired C.L. pole location

Need 30° of phase lead $\underline{\phi^\circ = 30^\circ}$

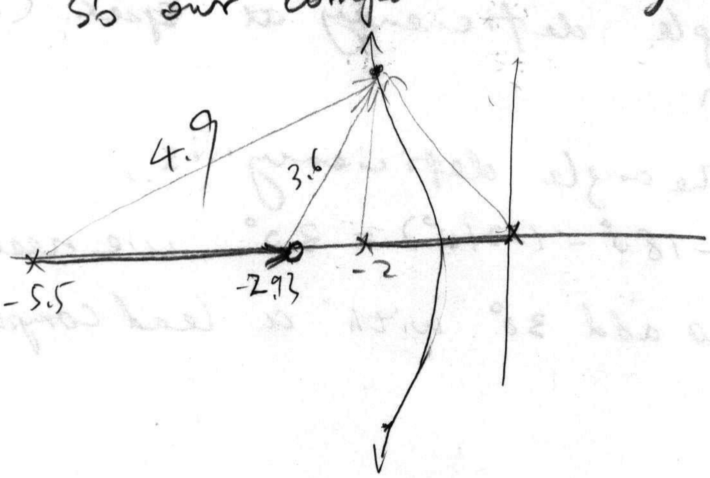


Draw line from origin to C.L. pole

Draw horizontal line



So our compensated system is



Using the gain criteria from root locus, find K_c to achieve C.L. pole locations

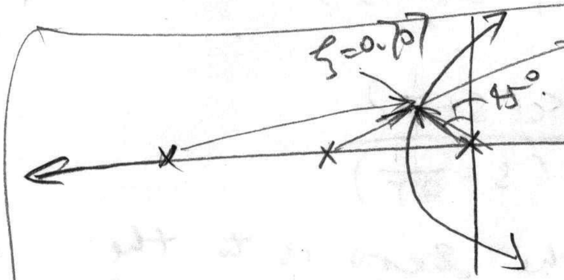
$$\left| \frac{K_c\,(s+2.93)}{(s+5.5)(s)(s+2)} \right| = 1 = \frac{K_c(3.6)}{4.9(3.46)(4.1)} \Rightarrow$$

$$K_c = 19.3$$

The lead compensator is

$$G_{Lead}(s) = \frac{19.3\,(s+2.93)}{s+5.5}$$

$-\underline{\underline{182^\circ}}$ $\quad$ $\underline{-0.38+0.38j}$ ✓



$\underline{-0.4+0.4j}$

1) search for 45° line

$\underline{-\sigma \pm \sigma j}$

2) work out the magnitude

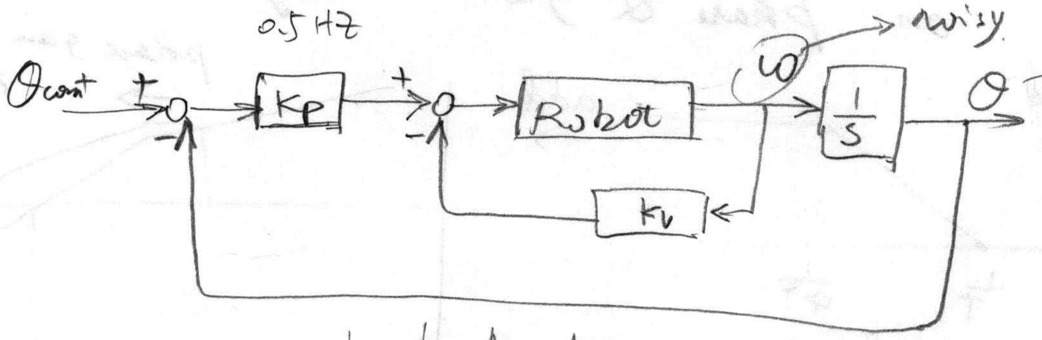
$$\frac{K}{D_1 D_2 D_3} = 1 \Rightarrow \text{find } K.$$

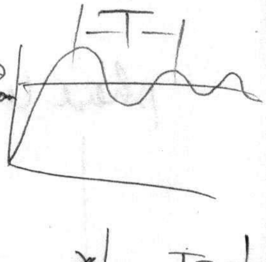
* Designing Lead Compenentors in the freq. domain

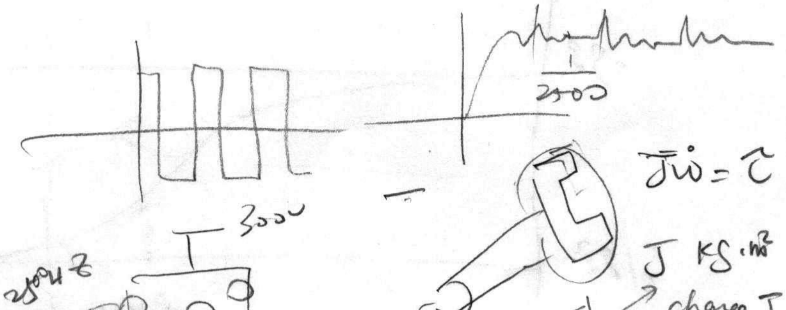
Now we will be given specifications on phase & gain margins & steady state error.

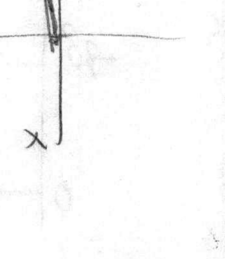
P621 - Phase Lead in frequency Domain

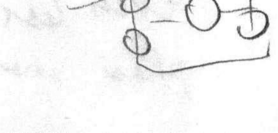
One uses a phase lead compensator when specifications on steady state error and phase margin & gain margain are given.











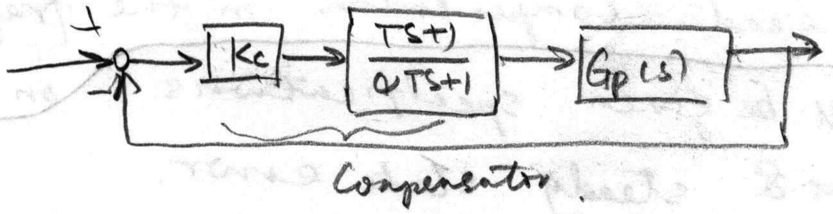
Phase Lead

$$G_c(s) = K_c\left(\frac{Ts+1}{\alpha Ts+1}\right) = K\left(\frac{aTs+1}{Ts+1}\right)$$

(Ogata $\alpha<1$) $\quad$ ($a>1$ others)

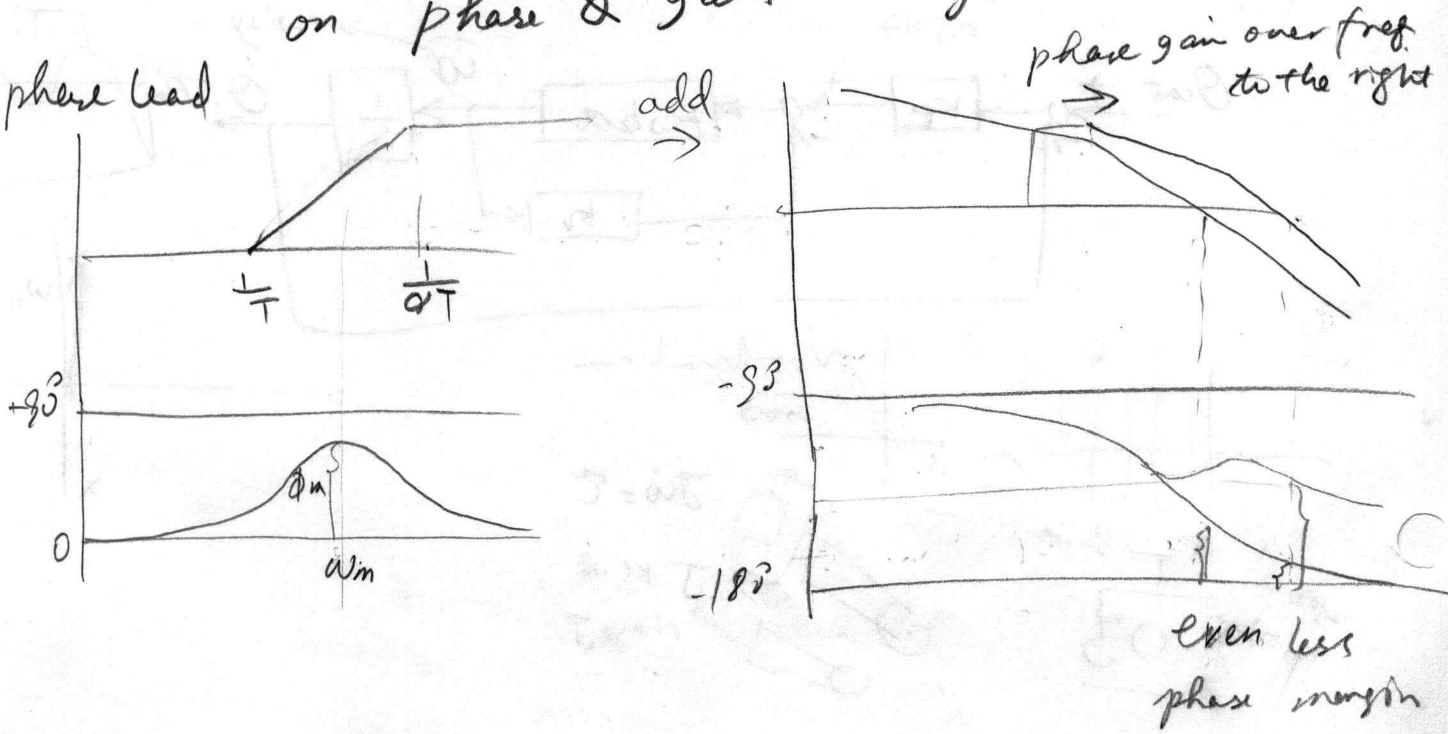
$$G_c(s) = K_c\left(\frac{T(s+\frac{1}{T})}{\alpha T(s+\frac{1}{\alpha T})}\right) = \frac{K_c(s+\frac{1}{T})}{\alpha(s+\frac{1}{\alpha T})}$$

Withi a lead compensator the zero is to the right of the pole.



Step 1) Find Gain necessary to meet steady state error specifications.

Step 2) Given the gain found in step 1) Draw the O.L. Bode diagrams & see if specs on phase & gain margin are met.



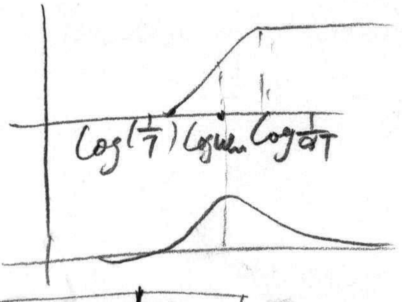
step 3) Compute what the phase margin deficiency and 5°–12° extra phase lead to compensate for gain X-over freq. shift to the right.

step 4) Calculate α from result that

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha} \Big\} \text{ Ogata}$$

step 5) Place the corner freq. of lead Compensator on either side of new gain crossover freq.

ω_m will ½ way between $\frac{1}{T}$ $\frac{1}{\alpha T}$.



$$\log\omega_m = \frac{1}{2}\left(\log\frac{1}{T} + \log\frac{1}{\alpha T}\right)$$

$$= \frac{1}{2}\left(\log\frac{1}{T} + \log\frac{1}{\alpha} + \log\frac{1}{T}\right)$$

$$= \log\frac{1}{T} + \frac{1}{2}\log\frac{1}{\alpha}$$

$$= \log\left(\frac{1}{T}\right) + \log\frac{1}{\sqrt{\alpha}} = \log\frac{1}{\sqrt{\alpha}\,T}$$

Therefore, $\boxed{\omega_m = \frac{1}{\sqrt{\alpha}\,T}}$

what is high freq. gain?

$$\left|G_{Lead}(s)\right|_{s=\text{large }\omega} = \left|\frac{Ts+1}{\alpha Ts+1}\right| \approx \frac{1}{\alpha}$$

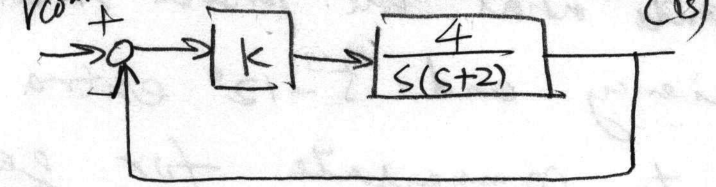
Lead freq.

$\boxed{\text{Example}}$: $H(s) = \frac{4}{s(s+2)}$

Specs. are phase margin > 50°

Coefficient of steady state error due to a ramp is = 20 sec^{-1} $K_v = \lim_{s\to 0} sG(s)H(s)$

step1)

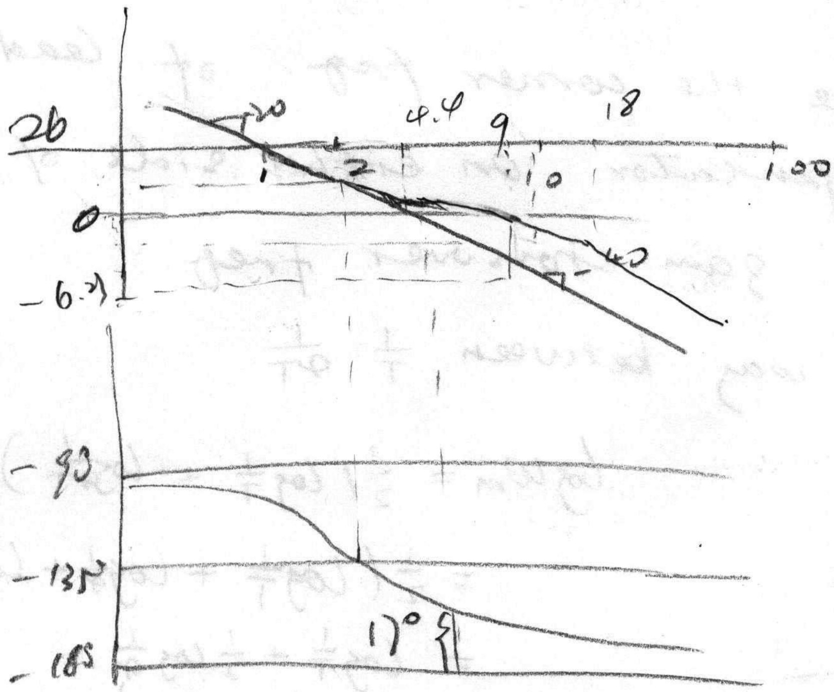


$$K_v = \lim_{s \to 0} s \cdot K \frac{4}{s(s+2)} = 20 \Rightarrow 2K = 20 \Rightarrow K = 10.$$

step2) Plot Bode Diagram $G_{OL} = \frac{40}{s(s+2)}$

$\times$ Dc gain $\frac{40}{2}$

$20 \log 20 = 26$



We have a 17° phase margin $17 < 50$ by 33°

To compensate for the shift to the right and 5° & use a phase lead of 38°.

step 4) Find a from $\sin \phi_m = \frac{1-a}{1+a}$ $\quad \sin 38° = 0.616$

$$0.616 + 0.616\, a = 1 - a \Rightarrow a = 0.238$$

The high freq. gain is $20 \log\left(\frac{1}{a}\right) = 12.46$ dB

There fore, the new gain x-over freq. is where

The uncompensator Bode Diagram crosses

$-\frac{12.46}{2} = -6.23$ dB Line

This occurs at 9 r/s.

$\omega_m = 9 = \frac{1}{\sqrt{a}T} = \frac{1}{\sqrt{0.238}\,T}$, solving for T = 0.23.

The lead compensator is,

$$G_{Lead}(s) = \frac{0.23S+1}{0.055S+1} = \frac{4.2(S+4.4)}{(S+18.2)} = \frac{TS+1}{aTS+1}$$

Lag Compensator Design

One use a lag compensator when the transient response is good but we have not met steady state error requirements.

$$G_{lag} = \frac{TS+1}{\beta TS+1}\ (\beta>1), \text{ or } G_{clag} = \frac{aTS+1}{TS+1}\quad a<1$$

P630-639

In pole zero form,

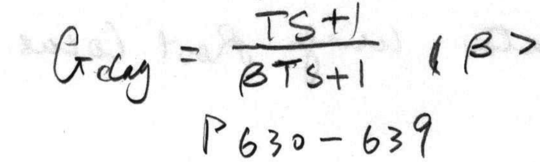
$$G_{c\,lag} = \frac{T(s+\frac{1}{T})}{\beta T(s+\frac{1}{\beta T})}\quad \beta>1$$

$$= \frac{s+\frac{1}{T}}{\beta(s+\frac{1}{\beta T})}$$

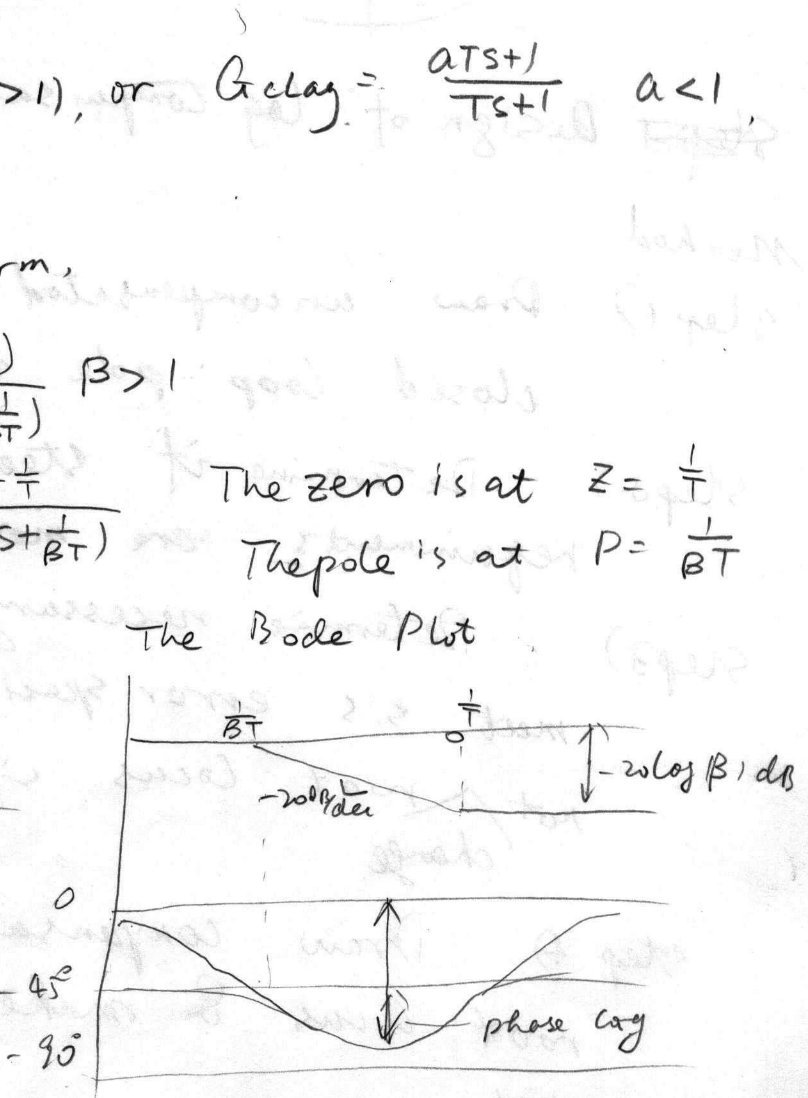
The zero is at $Z = \frac{1}{T}$

The pole is at $P = \frac{1}{\beta T}$

plotting on the s-plane

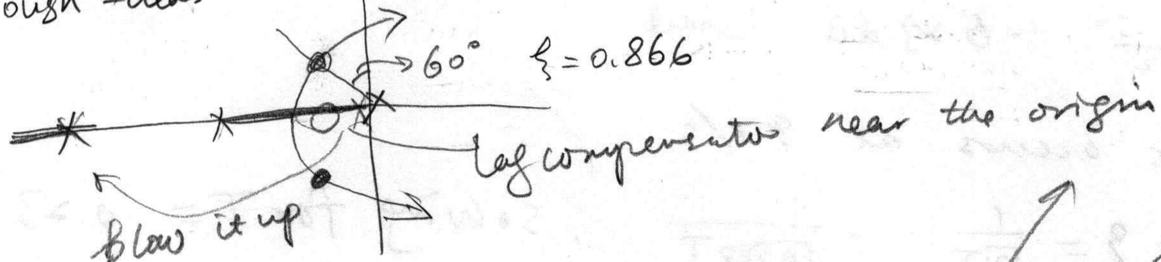


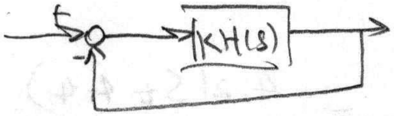
The Bode Plot

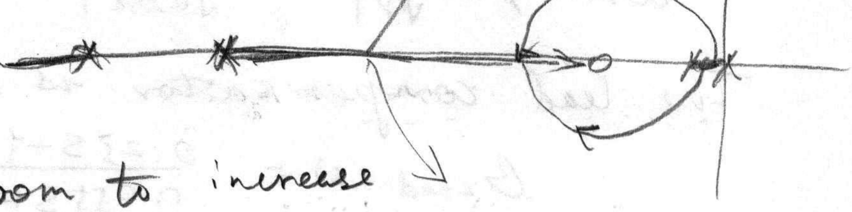


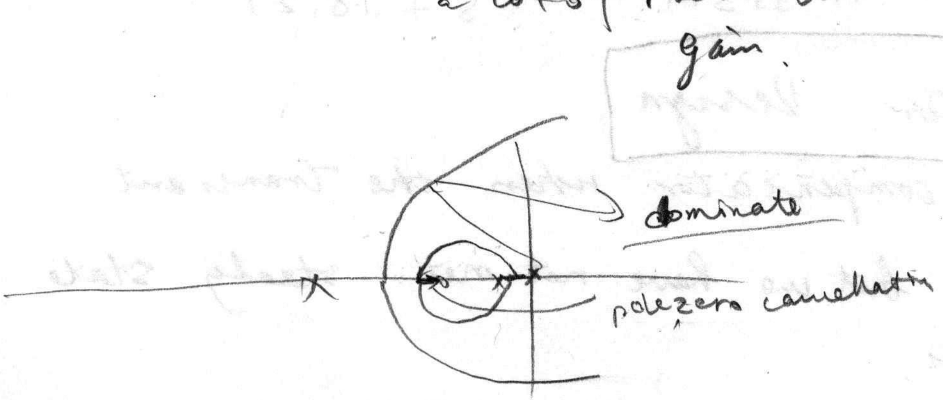
The root-locus Method of Lag compensator Design

Rough Ideas









~~step~~ Design of lag Compensator using Root Locus

Method

step 1) Draw uncompensated root locus, find closed loop pole locations.

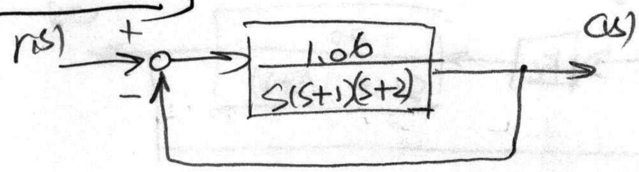
step 2) Determine if steady state error requirements are met

step 3) Determine necessary lag compensator to meet s.s error specifications, but does not change root locus significantly.

step 4) Draw compensated and uncompensated root locus & make sure all specs. are met.

Example

lag trans. & ss Example



Happy with transient response as is. Specification on $K_v = 5$ sec^{-1}

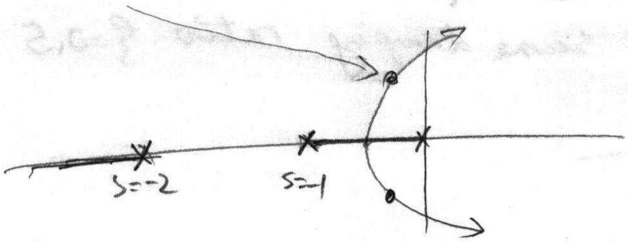
Find close loop pole locations.

Reduce block diagram

$$\frac{C(s)}{R(s)} = \frac{\frac{1.06}{s(s+1)(s+2)}}{1 + \frac{1.06}{s(s+1)(s+2)}} = \frac{1.06}{s^3 + 3s^2 + 2s + 1.06}$$

The roots of $s^3 + 3s^2 + 2s + 1 = 0$, are $P_1 = -2.33$.

$$\underbrace{P_2, \bar{P} = -0.33 \pm 0.58j \quad \} \quad \zeta = 0.5, \ \omega_n = 0.667}_{\text{I Like this transient response}}$$



$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} \frac{s\,1.06}{s(s+1)(s+2)}$$

$$= \frac{1.06}{1 \times 2} = 0.53$$

$K_v = 0.53 << 5.0$

to meet this requirement, we will use a Lag compensator.

Chose the following Lag compensator to give a "gain" of 10. I need to increase K_v by 10.

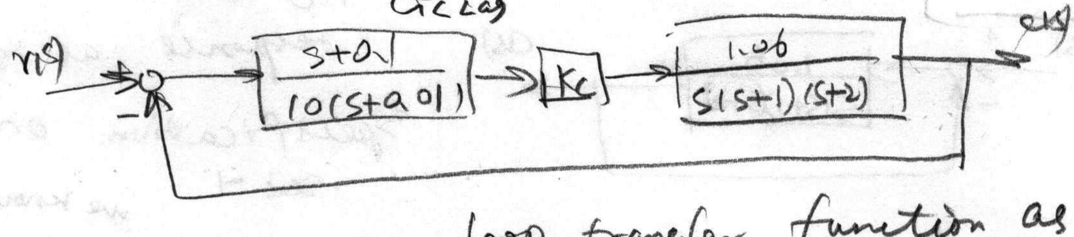
$$G_c(s) = \frac{10s+1}{100s+1}$$

$\beta = 10$ set by increase needed in K_v.

$$G_c(s) = \frac{10s+1}{100s+1} = \frac{10(s+0.1)}{100(s+0.01)}$$

We set $T = 10$, placing zero at $Z = -0.1$ & $p = -0.01$

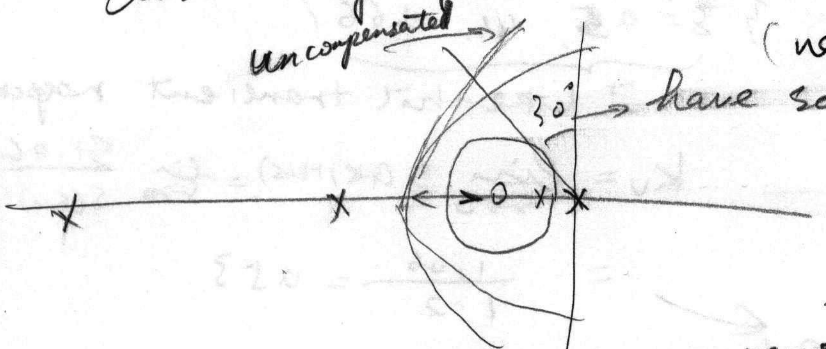
So that the root locus is not severely affected, and the closed loop system is not overly slow.



Let's write the open loop transfer function as,

$$S_{OL} = \frac{K(s+0.1)}{\gamma(s+0.01)\quad s(s+1)(s+2)} \quad \text{where } K = \frac{K_c \cdot 1.06}{\beta}$$

Let us draw the compensated root locus and Find gain K to achieve approx. the uncompensated Closed loop pole locations.



(we require the same ξ)

→ have same damping ratio $\xi = 0.5$

We have draw compensated root locus & for $\xi = 0.5$ the closed loop pole Locations (i.e. meets 180° criteria) are $p, \bar{p} = -0.28 \pm 0.51j$

recall the uncomp. poles were $p_u, \bar{p}_u = -0.33 \pm 0.58j$

Use magnitude criteria to find the gain

$$\left.\frac{K(s+0.1)}{s(s+0.01)(s+1)(s+2)}\right|_{s=-0.28\pm0.51j} = 1 \Rightarrow K = \left|\frac{s(s+0.01)(s+1)(s+2)}{s+0.1}\right|_{s=-0.28\pm0.51j}$$

≈ 1.0

The gain K, will be approximately equal to 1

$K = K_c \frac{1.06}{\beta}$, Infact, it was exactly equal to $K = 0.98$

to achieve C.L. $\xi = 0.5$. $K_c \approx 10$ (9.25)

$$K_V = \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} \frac{9.25\times 1.06\,(s+0.1)}{10(s+0.01)(s+1)(s+2)} = \frac{9.25\times 1.06\times 0.1}{10\times 0.01\times 1\times 2}$$

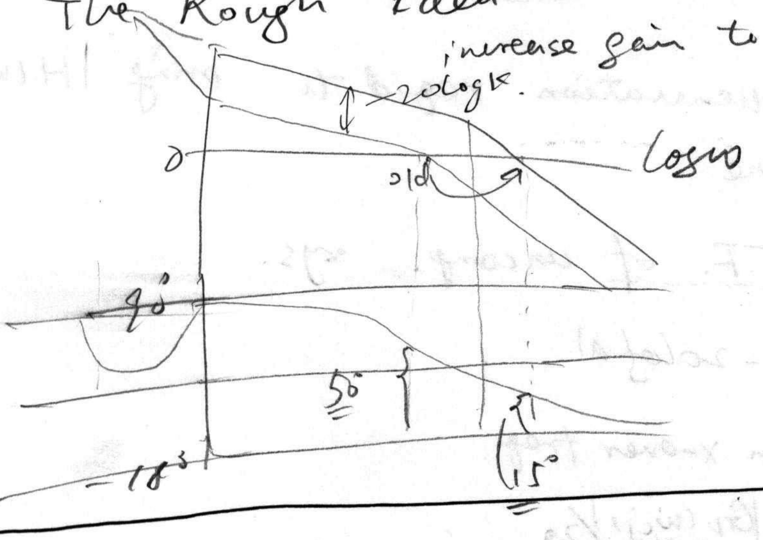
$$\approx 5 \text{ sec}^{-1}$$

The specs. are met.

Lag Compensator Design in the frequency Domain.

Use this technique when specs are given in terms of phase margin, gain margin & steady state error specifications.

The Rough idea

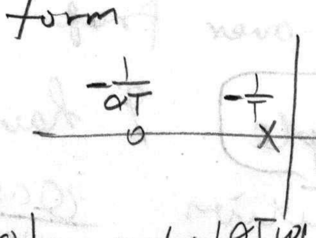


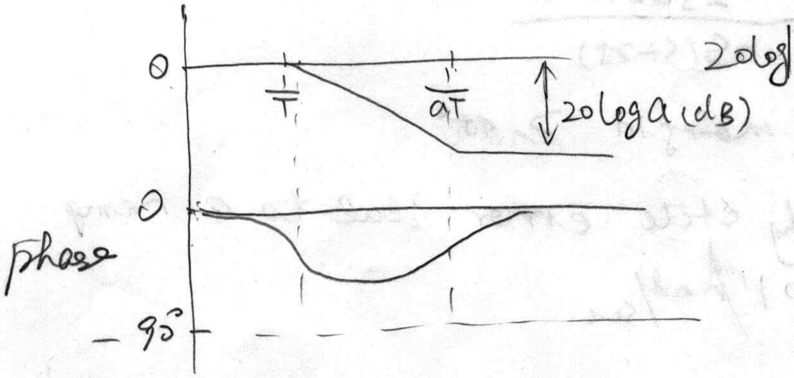
Lag Compensation in the frequency Domain   P631.

Use to achieve specified phase margin by providing attentuation at high frequency

The Lag compensator has the form

$$G_c(s) = \frac{\alpha T s + 1}{Ts + 1} \quad (\alpha < 1)$$



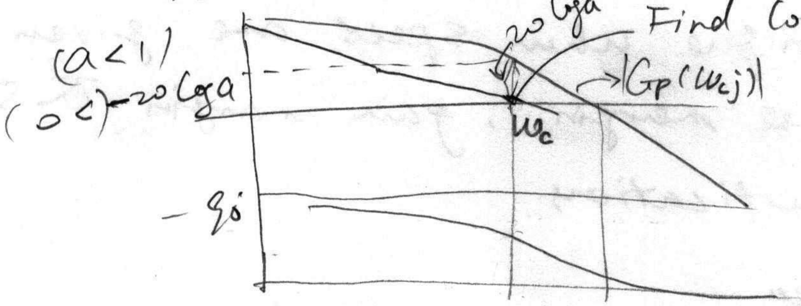


$20\log|G_c(s)| = 20\log\left|\frac{\alpha T\omega}{T\omega}\right|$ for Large W.

for Large ω, 20 log a is the magnitude

step1) Determine gain from steady state error spec.

step2) Plot open Loop uncompensated Bode diagram & determine if specified phase margin is met.



Find Location of new gain x-over freq. ⇒ find this from phase margin

step 3) Determine attenuation req'd to bring |H(ωj)| down to 0 dB Line

Let the O.L. T.F. of uncomp. sys.

$|G_p(\omega_c j)| = -20\log(a)$

↓

the new gain x-over freq.

$$\therefore \quad a = 10^{-\underbrace{|G_p(\omega_c j)|}_{\text{in dB}}/20}$$

step4) Set T such that the zero of the Lag compensator is one decade below new gain x-over freq.

Log freq. Example

Example. I have a system with open Loop Transfer function $\frac{\Theta(s)}{\alpha(s)} = \frac{2500K}{S(S+25)}$

The spec The phase margin ≥ 45°

The steady state error due to a ramp is 0.01 $\overset{rad}{}$ rad/sec

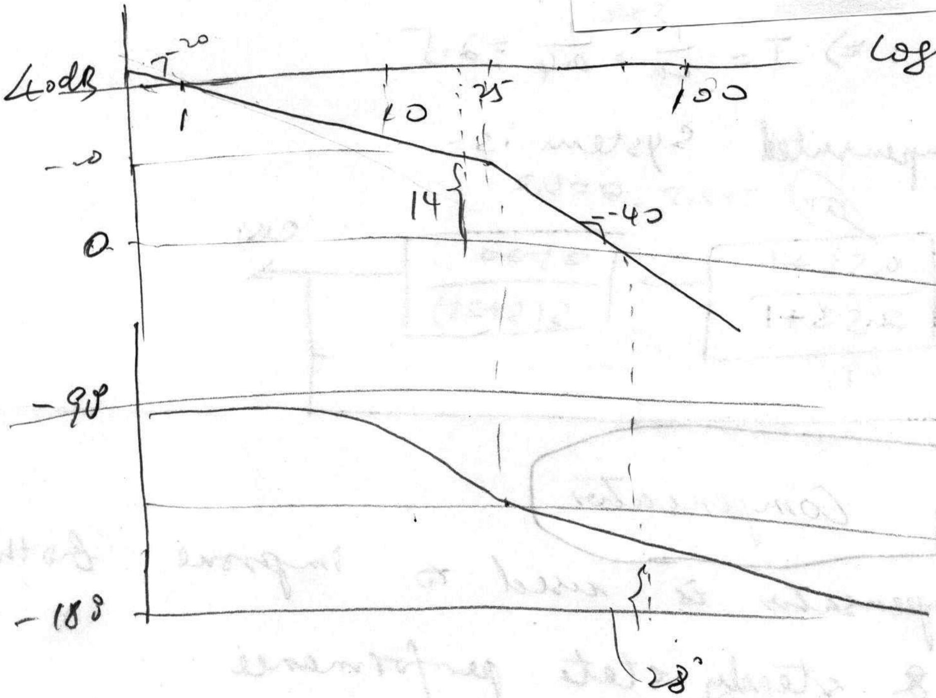
$E_{ss} = \frac{1}{K_V} = 0.01 \Rightarrow K_V = 100$ ($K_V = \lim_{s \to 0} s\,G(s)$ =

$\lim_{s \to 0} \frac{2500K}{s+25} = 100K$.

$\Rightarrow K = 1$

Step2. Plot uncompensated Bode Diagram

$G(s) = \frac{2500}{s(s+25)}$



We can recognize that a new gain x-over freq. at $w_c = 25$ r/s would gave a phase margin of 45°, However, the Lag compensator will add a few degrees of Lag at the new gain x-over freq.

To compensate for this extra Lag. bring new x-over freq. to $w_c = 20$ r/s, Now read from Bode Diagram, how much attenuation acquired to achieve new gain x-over at $w_c = 20$ r/s at $w_c = 20 r/s$, $G(w_cj) = 14dB$, then compute

$$a = 10^{-|G(w_cj)|/20} = 10^{-0.7} = 0.2$$

$$G_c(s) = \frac{aTs+1}{Ts+1}$$

$$z = \frac{1}{aT}$$

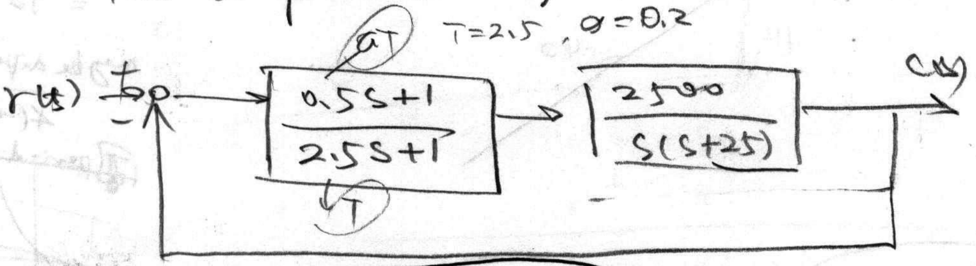
$$p = \frac{1}{T}$$

We now set the zero one decade below the new gain x-over freq.

(i.e. $w_c = 20 r/s$) we want $z = \frac{1}{aT} = \frac{20}{10} = 2$

$\frac{1}{aT} = 2 \Rightarrow T = \frac{1}{2a} = \frac{1}{0.4} = 2.5$

The compensated system is



Lead / Lag Compensator

This compensator is used to improve both transient & steady state performance.

$$G_c(s) = K_c \underbrace{\left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right)}_{\text{Lead}} \underbrace{\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right)}_{\text{Lag}}$$

Matlab. Eispac 78-84

↳ Found Eigenvalues

Numerical Integration to solve D.E.

Runga. Kutta Method. 4th/5th order solver

ODE45. — Simulink estimations.

Navy

↳ ABS Marine

↳ Bectel

↳ Raytheon

SNC-Lavlin! (canada)

Newton Euler solver $P = \cancel{Vt+}\frac{1}{2}at^2$

Root Locus design of Lead lag Compensator

The lead/lag compensator has the form

$$G_c(s)=K_c \underbrace{\frac{(s+\frac{1}{T_1})}{(s+\frac{\gamma}{T_1})}}_{\text{Lead}} \underbrace{\frac{(s+\frac{1}{T_2})}{(s+\frac{1}{\beta T_2})}}_{\text{Lag}} \qquad \gamma, \beta > 1$$

One uses lead/lag compensators to improve both the transient & steady state response.

We will design T_1, γ & K_c for the lead compensator and T_2 & β for lag compensator.

step 1) Find desired Closed loop pole locations and find the angle deficiency.

step 2) Find T_1 & γ/T_1; pole & zero to meet angle criteria

step 3) find K_c to meet magnitude criteria

step 4) Choose β to meet steady state error.

step 5) choose T_2 Large

Lead/lag
trans.
s.s.

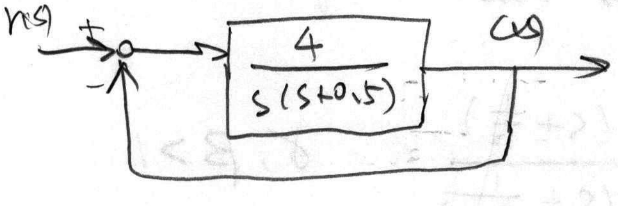
Example

$$G(s) = \frac{4}{s(s+0.5)}$$

It is specified that $\xi = 0.5$ & $\omega_n = 5$ r/s

The coefficient ~~of~~ error is $K_v = 80\ \text{sec}^{-1}$

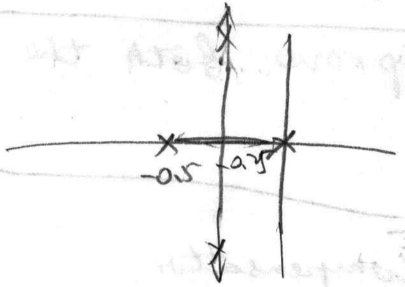
With unity feed back alone



$$\frac{c(s)}{r(s)} = \frac{4}{s^2+0.5s+4}$$

$$p, \bar{p} = -0.25 \pm 1.98j$$

What are the specified closed loop pole locations?

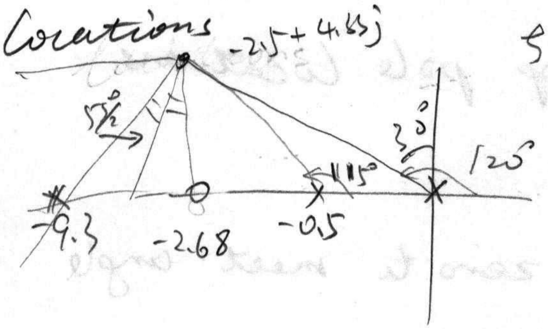


$$\text{desired: } p, \bar{p} = -2.5 \pm 5\sqrt{1-0.5^2}$$

$$= -2.5 \pm 4.33j$$

$$= -\xi w_n \pm w_n\sqrt{1-\xi^2}\,j$$

Compute angle deficiency at desired C.L. pole locations



$$\angle def = 115^\circ + 120^\circ - 180^\circ = 55^\circ$$

The lead zero is at $z = -2.68$, the pole is at $p = -9.3$. Use magnitude criteria to determine the gain

$$\left| K_c \frac{(s+2.68)}{(s+9.3)} \frac{4}{s(s+0.5)} \right| \Bigg|_{s=-2.5+4.33j} = 1$$

$$\left| K_c \frac{4.3 \times 4}{8 \times 5 \times 4.8} \right| = 1 \Rightarrow K_c = 11$$

Therefore $\frac{1}{T_1} = 2.68 \Rightarrow T_1 = 0.37$

$$\frac{\gamma}{T_1} = 9.3 \Rightarrow \gamma = 3.5$$

$$K_c = 11$$

Check coefficient of steady state error

Recall, the lead lag compensator has the form,

$$G_c(s) = K_c \frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}} \cdot \frac{s+\underbrace{\frac{1}{T_2}}_{-0.2}}{s+\underbrace{\frac{1}{\beta T_2}}_{-0.064}}$$

The K_v coefficient is,

$$K_v = \lim_{s \to 0} s\, G_c(s)\, G(s)$$

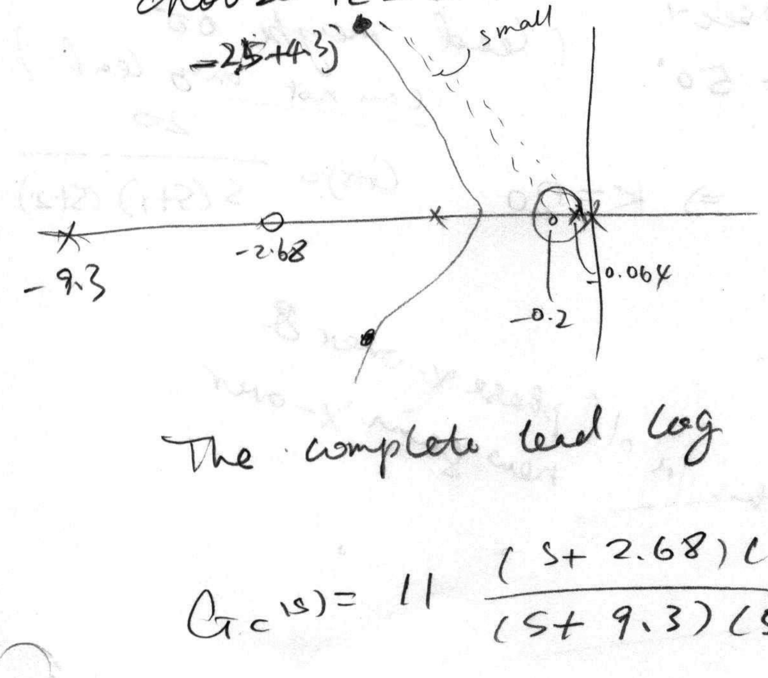
$$K_v = 80 = \lim_{s \to 0} \cancel{s}\, K_c \frac{\left(s+\overset{2.68}{\frac{1}{T_1}}\right)}{\left(s+\underset{9.3}{\frac{\gamma}{T_1}}\right)} \frac{\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_2}\right)} \frac{4}{\cancel{s}(s+0.5)}$$

$$= \overset{\checkmark}{K_c} \frac{\overset{\checkmark}{\beta}}{\gamma} \cdot \frac{4}{0.5} = 11 \frac{\beta}{3.5} \cdot \frac{4}{0.5} = 80$$

$\beta = 3.1$

Now we make T_2 big such that angle contribution to C.L. poles $\approx 5°$ or less. (+the gain as well)

choose $T_2 = 5$



The complete lead lag compensator is

$$G_c(s) = 11 \frac{(s+2.68)(s+0.2)}{(s+9.3)(s+0.06)}$$

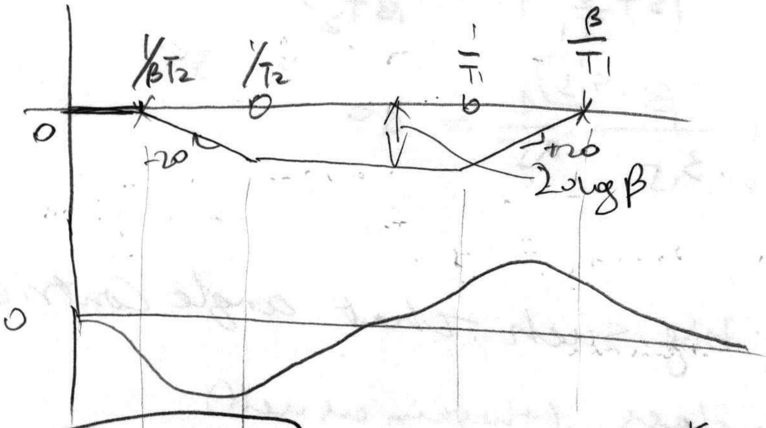
P641 Lead/Lag Compensation in the frequency domain.

The compensator transfer function has the form

$$G_c(s) = \frac{(s+\frac{1}{T_1})(s+\frac{1}{T_2})}{(s+\frac{\beta}{T_1})(s+\frac{1}{\beta T_2})} \quad \Big\} \text{ steady state gain} = 1 \quad (\beta>1,\ T_2 \text{ large})$$

Only 3 terms to design T_1 β & T_2

The Bode Diagram has the form



Example

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

Specifications: $K_v = 10\,\text{sec}^{-1}$

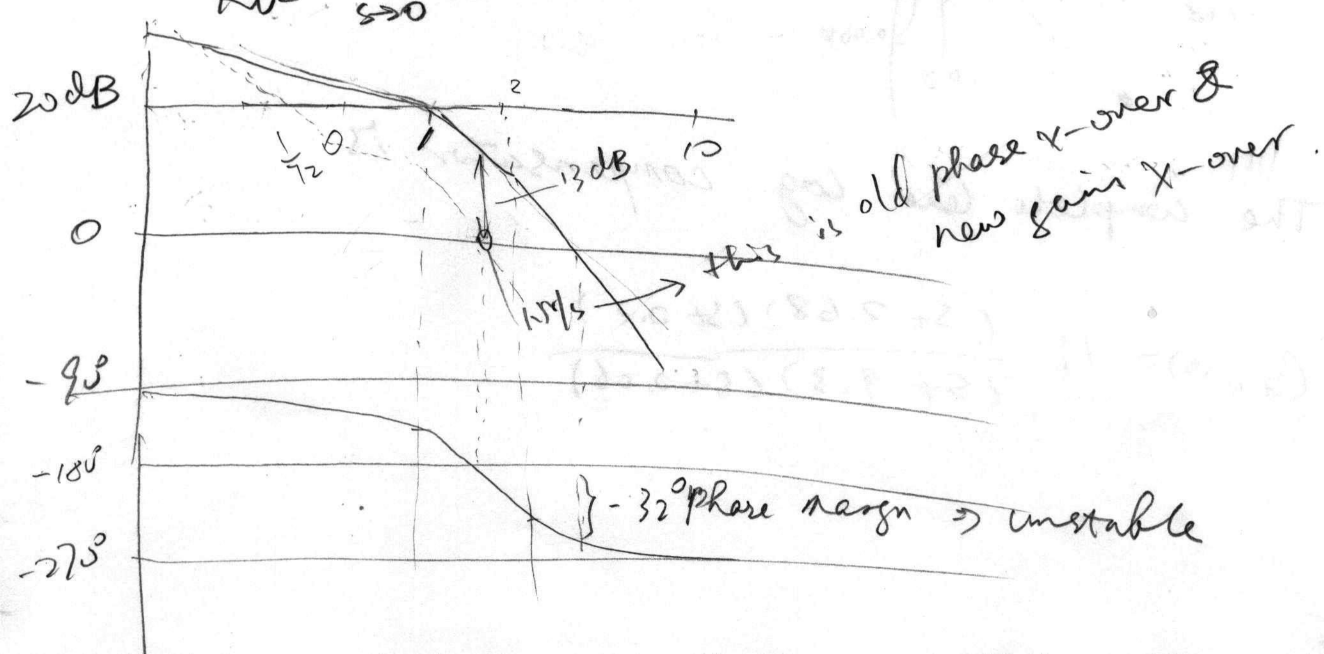
Phase = 50°

(lead maybe 60°

can not only lead)

$$G(s) = \frac{20}{s(s+1)(s+2)}$$

$$K_v = \lim_{s\to 0} s\,G(s) = 10 \Rightarrow K = 20$$



We need 82° of Lead, this cannot be done with a lead compensator alone, we will use a Lag compensator to bring gain x-over freq. to phase x-over (-180°) freq, then add phase Lead. At phase x-over freq. we need at least 50° of Lead.

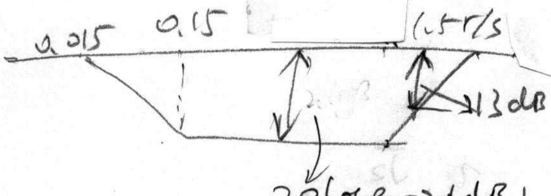
$\sin \phi_m = \frac{\beta-1}{\beta+1}$ } we remember this formula from lead comp. design.

set $\phi_m = 55°$, Xtra 5° due to comp. for lag.

This gives $\beta = 10$, The old phase x-over freq. is $\omega_c = 1.5$ r/s. New desired gain x-over freq. Place the Lag Zero 1 decade down from new gain x-over freq. $\frac{1}{T_2} = 0.15$

$\frac{1}{T_2} = 0.15 \Rightarrow T_2 = 6.7$

At $G(1.5j) = -13$ dB. We want the compensator $G_c(1.5j) = -13$ dB to achieve desired gain x-over freq.



Then draw a line with +20 dB slope going through 1.5 r/s and -13dB. where it crosses -20 dB is $\frac{1}{T_1}$ & 0 dB is $\frac{\beta}{T_1}$,

$$G_c(s) = \frac{(s+0.)}{(s+)} \frac{(s+0.15)}{(s+0.015)}$$

Recap Lead / Lag Compensator In freq. Domain

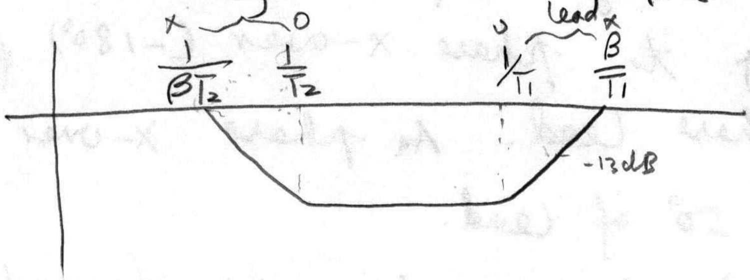
$$G_c(s) = \frac{K_c(s+\frac{1}{T_1})(s+\frac{1}{T_2})}{(s+\beta/T_1)(s+\frac{1}{\beta T_2})}$$

(only need)

β - Phase Lead

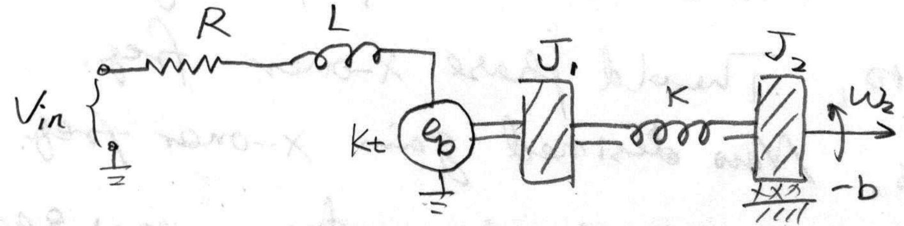
T_2 - 1 dec back
Next x-over

T_1 - chosen from
-13 dB



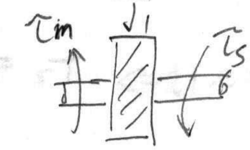
Chapter 11 Modern Control Engineering

State Space Technique



Let's model in state variable / state space form.

4th order system. Energy storage devices L, J_1, K, J_2
$i, \omega_1, \tilde{\theta}, \omega_2$

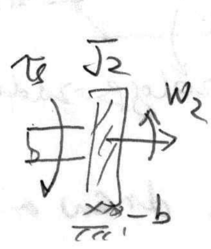


$$V_{in} = Ri + L\dot{i} + e_b$$

$$= Ri + L\dot{i} + K_t\omega_1$$

① $\dot{i} = -\frac{R}{L}i - \frac{K_t}{L}\omega_1 + \frac{1}{L}V_{in}$

$$J_1\dot{\omega}_1 = \tau_m - \tau_s = K_t i - K\tilde{\theta}$$



② $\dot{\omega}_1 = \frac{K_t}{J_1}i - \frac{K}{J_1}\tilde{\theta}$

③ $\dot{\tilde{\theta}} = \omega_1 - \omega_2$

$$J_2\dot{\omega}_2 = K\tilde{\theta} - b\omega_2$$

④ $\dot{\omega}_2 = -\frac{b}{J_2}\omega_2 + \frac{K}{J_2}\tilde{\theta}$

In matrix form

$$\begin{bmatrix}\dot{i}\\ \dot{\omega}_1\\ \dot{\tilde{\theta}}\\ \dot{\omega}_2\end{bmatrix} = \overbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{K_t}{L} & 0 & 0\\ \frac{K_t}{J_1} & 0 & -\frac{K}{J_1} & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & \frac{K}{J_2} & -\frac{b}{J_2}\end{bmatrix}}^{A} \begin{bmatrix} i\\ \omega_1\\ \tilde{\theta}\\ \omega_2\end{bmatrix} + \underbrace{\begin{bmatrix}\frac{1}{L}\\ 0\\ 0\\ 0\end{bmatrix}}_{B} V_{in}$$

This is state space form

$$Y(t) = \omega_2 = \underbrace{\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}}_{C} \begin{bmatrix} i\\ \omega_1\\ \tilde{\theta}\\ \omega_2\end{bmatrix}$$

Let's write this as $\dot{\bar{X}}(t) = A\bar{x}(t) + B\bar{u}(t)$

$\bar{Y}(t) = C\bar{x}(t)$

$\bar{x}^T = [i \quad \omega_1 \quad \tilde{\theta} \quad \omega_2]$, $u = V_{in}$

$A \in R^{n\times n}$, $\bar{X} \in R^n$, $\bar{U} \in R^m$

m is the number of inputs. $m = 1$ no. of inputs

$n = 4$ no. of states

$B \in R^{n\times m}$, $C \in R^{k\times n}$ k = outputs

Solving the matrix D.E.

$$\dot{\bar{X}} = A\bar{x} + Bu$$

Case 1 unforced Response, Homogeneous Solution

$u = 0$, $\bar{X}(0) = X_0$

$$\dot{\bar{X}} = A\bar{X} + B\bar{u}$$

Case 1 $X(0) = \bar{X}_0$, $\bar{u} = 0$

Assume a solution of the form $\bar{X}(t) = A^{At}C$

Let's define (by definition)

$$\boxed{\begin{matrix}A\\ \text{symbol}\end{matrix}} \rightarrow e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$\frac{de^{At}}{dt} = 0 + A + A^2t + \frac{A^3t^2}{2!} + \frac{A^4t^3}{3!} + \cdots$$

$$= A + A^2t + \frac{A^3t^2}{2!} + \frac{A^4t^3}{3!} + \cdots$$

Factor out A

$$\frac{de^{At}}{dt} = A(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots)$$

$$= Ae^{At}.$$

Assuming $\bar{X}(t) = e^{At}C$

$\dot{\bar{X}}(t) = Ae^{At}C$ $\quad$ $\dot{\bar{X}} = A\bar{X}(t)$, $\bar{X}(0) = \bar{X}_0$

$Ae^{At}C = Ae^{At}C$ ✓ solves.

solving for I.C.

$\bar{X}(0) = e^{A0}\cdot C = \bar{X}_0$ $\quad$ $e^{A\cdot 0} = I$.

$\Rightarrow C = \bar{X}_0$

The homogeneous solution,

$\bar{X}(t) = e^{At}\bar{X}_0$

The term e^{At} is the state transition matrix.

$\Phi(t) = e^{At}$

In matlab, $e^{At} = expm(At)$

$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$

The forced Response,

Case 2 $\quad$ $\bar{X}(0) = 0$, $\quad$ $\bar{u}(t)$ is some input

Just like in the scalor case,

Assume a solution of the form,

$\bar{x}(t) = e^{At}c(t)$

(1) $\quad$ $\dot{\bar{x}}(t) = A\bar{x} + B\bar{u}$

$\dot{\bar{x}}(t) = Ae^{At}c(t) + e^{At}\dot{c}(t)$

substitite into equation (1)

$\cancel{Ae^{At}c(t)} + e^{At}\dot{c}(t) = A\bar{x} + B\bar{u} = \cancel{Ae^{At}c(t)} + B\bar{u}$

$e^{At}\dot{c}(t) = B\bar{u}$ $\quad$, $\quad$ $\dot{c}(t) = e^{-At}B\bar{u}(t)$

Integrate assuming zero I.C.

$$C(t) = \int_0^t e^{-A\lambda} B u(\lambda)\, d\lambda \quad (\lambda \text{ is dummy variable})$$

$\therefore$ The solution is (convolution integral)

$$\bar{X}(t) = e^{At} \int_0^t e^{-A\lambda} B u(\lambda)\, d\lambda$$

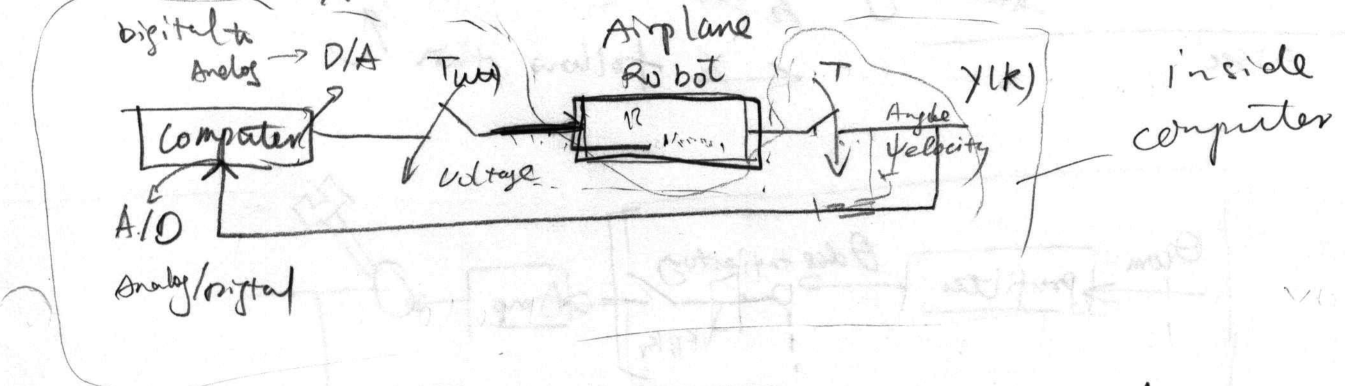
$$\bar{X}(t) = \int_0^t e^{A(t-\lambda)} B\, u(\lambda)\, d\lambda$$

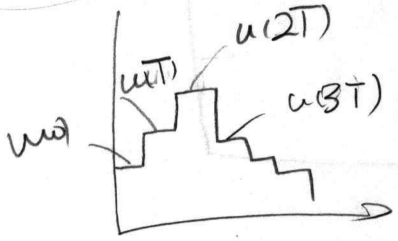
And the complete solution is

$$\bar{X}(t) = e^{At} \bar{X}(0) + \int_0^t e^{A(t-\lambda)} B\, u(\lambda)\, d\lambda$$

In general

$$\bar{X}(t) = e^{A(t-\tau)} \bar{X}(\tau) + \int_\tau^t e^{A(t-\lambda)} B u(\lambda)\, d\lambda$$





T is my sampling period.

Let's solve this for 1 period "T"

$$\bar{X}(T) = e^{AT} X(0) + \int_0^T e^{A(T-\lambda)} d\lambda\, B \bar{u}(0)$$

$\uparrow$ constant over integral

$\hookrightarrow$ At the next time integral

$$\bar{X}(2T) = e^{AT} X(T) + \int_T^{2T} e^{A(2T-\lambda)} d\lambda\, B u(T)$$

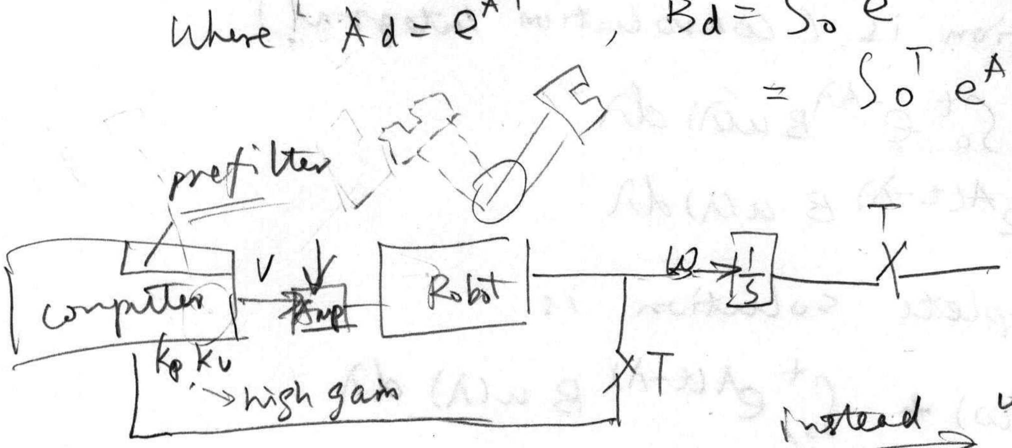
$$v = T - \lambda \rightarrow \int_0^T e^{A(T-v)}\, dv$$

We then get a discrete time equation of the computer controlled system

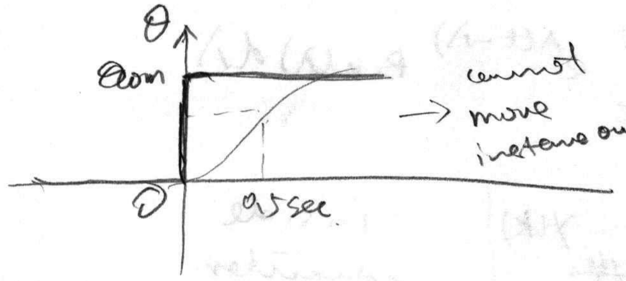
$$\bar{X}(k+1) = A_d \bar{X}(k) + B_d \bar{u}(k)$$

Where: $A_d = e^{AT}$, $B_d = \int_0^T e^{A(T-\lambda)} d\lambda \, B$

$$= \int_0^T e^{A\lambda} d\lambda B$$



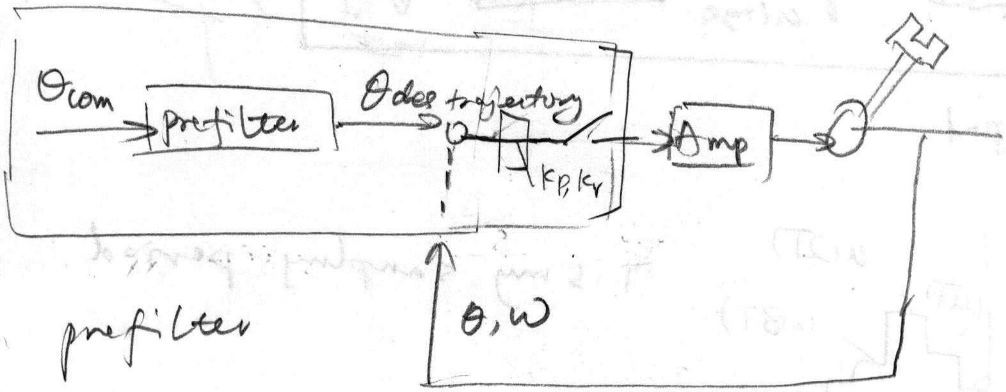
instead use $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$



→ cannot move instaneously

the constant 1/2 sec ← $\zeta = 1$ $\omega_n = 2$ r/s

← X follow this



Match prefilter Bandwidth to Robot controller Bandwidth

State space Techniques

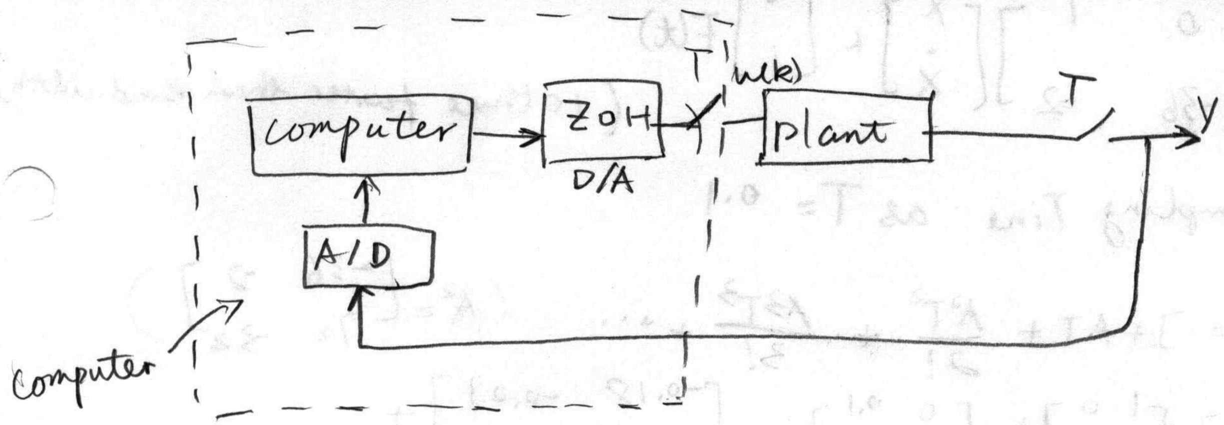
A system is described as

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

$$\bar{y} = C\bar{x}$$

The complete solution is $\bar{x}(t) = e^{At}x(0) + \int_0^t e^{A(t-\lambda)} B u(\lambda) d\lambda$

Therefore, we can write a discrete time version as,



$$\bar{x}(k+1) = A_d \bar{x}(k) + B_d \bar{u}(k)$$

Let's take a sampling time T.

$$A_d = e^{AT} = I + AT + \frac{A^2T^2}{2!} + \cdots$$

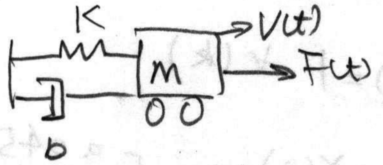
$$B_d = \int_0^T e^{A(T-\lambda)} d\lambda B = \int_0^T e^{A\lambda} d\lambda B$$

$$= \int_0^T (I + A\lambda + \frac{A^2\lambda^2}{2!} + \cdots) d\lambda B$$

$$= (\lambda I\Big|_0^T + \frac{A\lambda^2}{2}\Big|_0^T + \frac{A^2\lambda^3}{3!}\Big|_0^T + \cdots + \int_0^T) B$$

$$= (TI + \frac{AT^2}{2!} + \frac{A^2T^3}{3!} + \cdots) B = A^{-1}(e^{AT} - I) B$$

Example



I know the dynamics are given by

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}F(t)$$

Let's take $m = 1 (kgm^2)$ $k = 36 (N/m)$, $b = 2$ $N/(m/s)$

$$\ddot{x} + 2\dot{x} + 36x = F(t) \qquad \frac{X(s)}{F(s)} = \frac{1}{s^2+2s+36}$$

We know something about behaviour

$W_n = 6 r/s \approx 1 Hz$

$\zeta = 1/6$ } underdamped

$$\begin{bmatrix} \dot{X} \\ \ddot{X} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -36 & -2 \end{bmatrix} \begin{bmatrix} X \\ \dot{X} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F(t)$$

(10 times faster than Bandwidth)

Set sampling Time as $T = 0.1$

$$A_d = I + AT + \frac{A^2T^2}{2!} + \frac{A^3T^3}{3!} + \cdots \quad \left(A^2 = \begin{bmatrix} -36 & -2 \\ 72 & -32 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0.1 \\ -3.6 & -0.2 \end{bmatrix} + \begin{bmatrix} -0.18 & -0.01 \\ 0.36 & -0.16 \end{bmatrix} +$$

rapid convergence

$$\begin{bmatrix} 0.012 & -0.0053 \\ 0.192 & 0.0227 \end{bmatrix} = \begin{bmatrix} 0.832 & 0.0847 \\ -3.048 & 0.6627 \end{bmatrix}$$

3-6 terms

On Computer, $\text{expm}(A) = \begin{bmatrix} 0.8364 & 0.0853 \\ -3.0707 & 0.6658 \end{bmatrix}$

$$B_d = \begin{bmatrix} 0.0045 \\ 0.0853 \end{bmatrix}$$

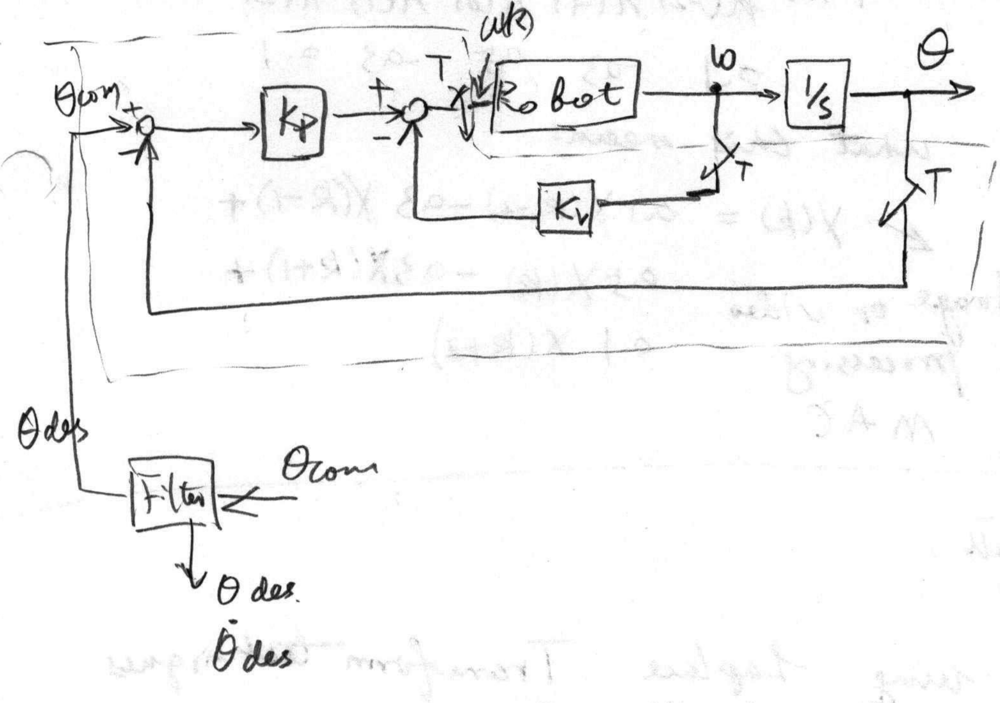
The Descrete time version becomes,

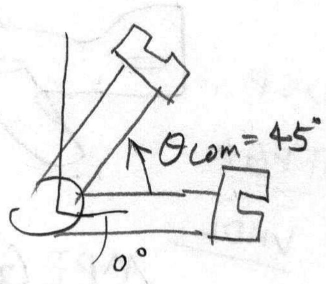
$$\begin{bmatrix} X(k+1) \\ \dot{X}(k+1) \end{bmatrix} = \begin{bmatrix} 0.8364 & 0.0853 \\ -3.0707 & 0.6658 \end{bmatrix} \begin{bmatrix} X(k) \\ \dot{X}(k) \end{bmatrix} + \begin{bmatrix} 0.0045 \\ 0.0853 \end{bmatrix} F(k)$$

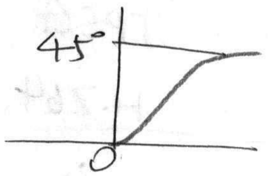
start at $\bar{X}(0) = \emptyset$, $F(k) = 1 \ \forall (k)$

$$\begin{bmatrix} X(1) \\ \dot{X}(1) \end{bmatrix} = \begin{bmatrix} 0.0045 \\ 0.0853 \end{bmatrix}, \begin{bmatrix} X(2) \\ \dot{X}(2) \end{bmatrix} = \begin{bmatrix} 0.0456 \\ 0.1281 \end{bmatrix}$$

$$\begin{bmatrix} X(3) \\ \dot{X}(3) \end{bmatrix} = \begin{bmatrix} 0.0285 \\ 0.1226 \end{bmatrix}$$







0.5 HZ Bandwidth

We picked Kp & kv to give me a desired bandwidth

Wnd = 0.5 HZ

ξ = 1

Characteristic eqn desired for prefilter is

$$(s+3)^2 = s^2 + 6s + 9$$

Start in continuous time

unity gain, ie. T.F. needs to be $H(s) = \frac{9}{s^2+6s+9}$

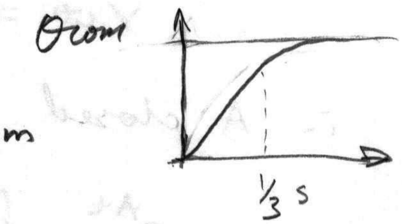
Continuous time IIR filter

$$\begin{bmatrix} \dot\theta_{des} \\ \ddot\theta_{des} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} \begin{bmatrix} \theta_{des} \\ \dot\theta_{des} \end{bmatrix} + \begin{bmatrix} 0 \\ 9 \end{bmatrix} \theta_{com}$$

$$\ddot\theta_{des} + 6\dot\theta_{des} + 9\theta_{des} = 9\theta_{com}$$

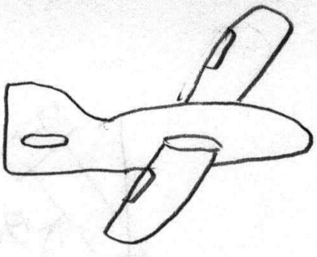
(Input is θ com)

Set T=0.1



$$\begin{bmatrix} \theta_{des}(k+1) \\ \dot\theta_{des}(k+1) \end{bmatrix} = [A_d] \begin{bmatrix} \theta_{des}(k) \\ \dot\theta_{des}(k) \end{bmatrix} + B_d\,\theta_{com}$$

(C2D in matlab)



DSP
DFPGA

VHDL

FIR

$h(-2)$	$h(-1)$	$h(0)$	$h(1)$	$h(2)$
0.1	-0.3	0.5	-0.3	0.1

what this means

$$y(k) = 0.1\,x(k-2) - 0.3\,x(k-1) + 0.5\,x(k) - 0.3\,x(k+1) + 0.1\,x(k+2)$$

MPEG
JPEG
H.264

Image or video processing
MAC

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

$$\bar{y} = C\bar{x}$$

Solve this using Laplace Transform techniques

$$sI\bar{x}(s) = A\bar{x}(s) + B\bar{u}(s)$$

Take unforced case 1st $\bar{u}(t)=0$, $\bar{u}(s)=0$

$$\left(\mathcal{L}\{\dot{x}_1\} = s\,X_1(s) - X_1(0) \right)$$

$$sI\bar{x}(s) = A\bar{x}(s) + \bar{x}(0)$$

$$(sI - A)\bar{x}(s) = \bar{x}(0)$$

$$\bar{x}(s) = (sI - A)^{-1}\bar{x}(0)$$

Take inverse Laplace Transform

$$\bar{x}(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}\}\bar{x}(0)$$

Recall the solution to the IC Response is

$$\bar{x}(t) = e^{At}\bar{x}(0)$$

∴ A closed form solution for e^{At} is

$$e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}.$$

The forced Response $\bar{x}(0)=0$

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}, \quad y = C\bar{x}$$

Taking Laplace Transform

$$SI\bar{x}(s) = A\bar{x}(s) + B\bar{u}(s)$$

$$(SI-A)\bar{x}(s) = B\bar{u}(s)$$

$$\bar{x}(s) = (SI-A)^{-1} B u(s)$$

$$\bar{y}(s) = \overset{k\times n}{C}\,\overset{n\times n}{(SI-A)^{-1}}\,\overset{n\times m}{B}\,u(s)$$

$G(s) = C(SI-A)^{-1}B$ ⇒ matrix of transfer functions, relating each input to each output

State space form

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

$$\bar{y} = C\bar{x}$$

In discrete form

$$\bar{x}(k+1) = A_d\bar{x}(k) + B_d\bar{u}(k)$$

$$A_d = e^{AT} \qquad B_d = \int_0^T e^{A\lambda}\,d\lambda\, B$$

Using Laplace Transform Methods,

$$\mathcal{L}\{\dot{\bar{x}}(t)\} = \mathcal{L}\{A\bar{x}(t) + B\bar{u}(t)\}$$

$C \in R^{k\times n}$ k outputs

$B \in R^{n\times m}$ m inputs

$G(s) \in \mathbb{C}^{k\times m}$

$$SI\bar{x}(s) = A\bar{x}(s) + B\bar{u}(s)$$

$$(SI-A)\bar{x}(s) = B\bar{u}(s)$$

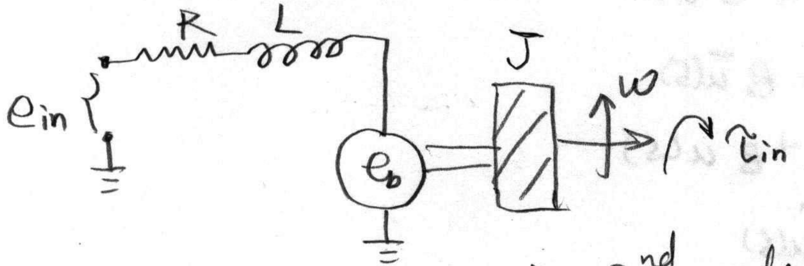
$$\bar{x}(s) = (SI-A)^{-1}B\bar{u}(s)$$

$$\bar{y}(s) = C(SI-A)^{-1}Bu(s)$$

We define the Transfer function matrix as

$$G(s) = C(SI-A)^{-1}B = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1m}(s) \\ G_{k1}(s) & & \cdots & G_{km}(s) \end{bmatrix}$$

Each Transfer function in $G(s)$, $G_{ij}(s)$ represents the transfer function from j th input to the i th output.



$$\begin{cases}\begin{bmatrix}\dot{i}\\ \dot{\omega}\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} i\\ \omega\end{bmatrix} + \begin{bmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\begin{bmatrix}e_{in}\\ \tau_{in}\end{bmatrix} \\ \bar{y} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix} i\\ \omega\end{bmatrix}\end{cases}$$

This system is 2^{nd} order

State variables are i & ω

In this case

$G(s)$ becomes a 2×2 complex matrix.

$$G(s) = \begin{bmatrix} G_{ie_{in}}(s) & G_{i\tau_{in}}(s) \\ G_{\omega e_{in}}(s) & G_{\omega\tau_{in}}(s)\end{bmatrix}$$

Recall $\mathcal{L}^{-1}[(sI-A)^{-1}] = e^{At}$ $\qquad M \in \mathbb{C}^{n\times n}$

How do I compute $M^{-1} = \frac{1}{\det(M)} \cdot Adj(M)$.

$$(sI-A)^{-1} = \frac{1}{\det(sI-A)} Adj(sI-A)$$

$$G(s) = \frac{C\, Adj(sI-A)\, B}{\det(sI-A)} \begin{array}{l} \rightarrow \text{polynomial in "s" of order at most "n-1"} \\ \rightarrow \text{polynomial of order "n"} \end{array}$$

$$= \frac{Z(s)}{D(s)} \begin{array}{l} \rightarrow \text{matrix of zero polynomial} \\ \rightarrow \text{Denominator scalar complex polynomial in } s^n \end{array}$$

$$s^n + a_{n-1}s^{n-1} + \cdots + a_0 =$$

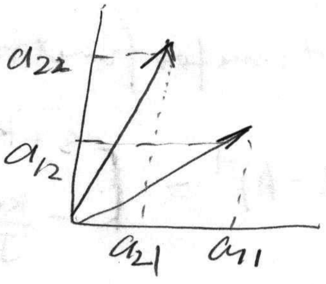
The eigen values of "A" are the poles/roots / dynamic modes of the system.

If "A" is an nxn matrix, then we have n eigenvalues representing the n poles of the system.

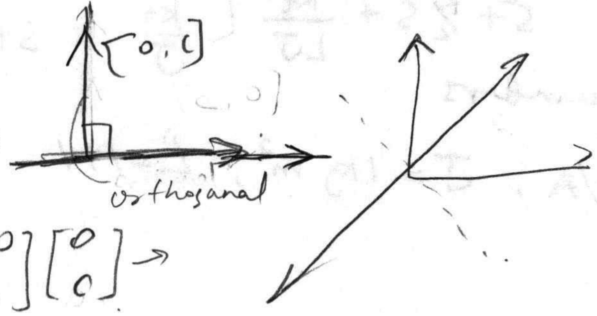
We now have the matrix $\det(SI-A)|_{S=\lambda_i}=0$

$\det(\lambda_i I - A)=0$

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \det(M)=0$$



$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$



$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ c \end{bmatrix} \rightarrow$

If $(SI-A)|_{S=\lambda_i}$ does not span the space, then there must exist an eigenvector (that sits in the null space of $(SI-A)|_{S=\lambda_i}$ that is orthogonal to $(SI-A)|_{S=\lambda_i}$

$$(\lambda_i I - A)C_i = 0. \qquad \text{for } C_i \neq 0$$

$\hookrightarrow$ must exist.

Using the above example:

$$e_{in}(t) = Ri + L\dot{i} + k_t\omega$$

$$\boxed{\dot{i} = -\frac{R}{L}i - \frac{k_t}{L}\omega + \frac{1}{L}e_{in}(t)}$$

$$J\dot{\omega} = k_t i + \tau_{in}$$

$$\boxed{\dot{\omega} = \frac{k_t}{J}i + \frac{1}{J}\tau_{in}}$$

$$\begin{bmatrix}\dot{i}\\ \dot{\omega}\end{bmatrix} = \begin{bmatrix} -R/L & -k_t/L \\ \frac{K_t}{J} & 0\end{bmatrix}\begin{bmatrix} i\\ \omega\end{bmatrix} + \begin{bmatrix} 1/L & 0\\ 0 & \frac{1}{J}\end{bmatrix}\begin{bmatrix} e_{in}\\ \tau_{in}\end{bmatrix}$$

The output is $\omega = [0 \quad 1]\begin{bmatrix} i\\ \omega\end{bmatrix}$

The transferfunction matrix is, $G(s) = C(sI-A)^{-1}B$

$$(sI-A)^{-1} = \begin{bmatrix} s+R/L & \frac{K_t}{L} \\ -\frac{K_t}{J} & s\end{bmatrix}^{-1} = \frac{1}{s^2+\frac{R}{L}s+\frac{K_t^2}{LJ}}\begin{bmatrix} s & -\frac{K_t}{L} \\ \frac{K_t}{J} & s+\frac{R}{L}\end{bmatrix}$$

Let's give this some numbers

$R = 5\Omega$, $K_t = 2\ N\cdot m/A$, $J = 1\,kg\cdot m^2$, $L = 1H$.

$$\det(sI-A) = s^2+5s+4$$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{s}{s^2+5s+4} & \frac{-2}{s^2+5s+4} \\ \frac{2}{s^2+5s+4} & \frac{s+5}{s^2+5s+4}\end{bmatrix}$$

The eigenvalues are $\lambda_1 = -4$, $\lambda_2 = -1$

$$(s^2+5s+4) = (s+4)(s+1)$$

$$\mathcal{L}^{-1}[(sI-A)^{-1}] = e^{At} = \phi(t) \rightarrow \text{state transition matrix}$$

$$\phi(t) = \begin{bmatrix} -\frac{1}{3}e^{-t}+\frac{4}{3}e^{-4t} & -\frac{2}{3}e^{-t}+\frac{2}{3}e^{-4t} \\ \frac{2}{3}e^{-t}-\frac{2}{3}e^{-4t} & \frac{4}{3}e^{-t}-\frac{1}{3}e^{-4t}\end{bmatrix}$$

$$= I + At + \frac{A^2t^2}{2!} + \cdots$$

We are find the transfer function matrix.

$G(s) = C(sI-A)^{-1}B$ $\quad$ eig(A) $\quad$ [if A is hermitte or stable ⇒ eig(A) < 0]

$$A = \begin{bmatrix} -5 & -2 \\ 2 & 0\end{bmatrix}$$

$$y = w = [0 \quad 1]\begin{bmatrix} v \\ w \end{bmatrix}$$

$$G(s) = [0 \quad 1]\begin{bmatrix} \frac{s}{s^2+5s+4} & \frac{-2}{[\quad]} \\ \frac{2}{[\quad]} & \frac{s+5}{[\quad]} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{s^2+5s+4} & \frac{s+5}{s^2+5s+4} \end{bmatrix}$$

Recall

$$y(s) = C(sI-A)^{-1}B\,u(s)$$

where $u(s) = \begin{bmatrix} V_{in}(s) \\ \tau_{in}(s) \end{bmatrix}$

$$y(s) = \frac{2}{s^2+5s+4} V_{in}(s) + \frac{s+5}{s^2+5s+4} \tau_{in}(s)$$

$$\bar{X}(t) = \phi(t)\,\bar{X}(0)$$

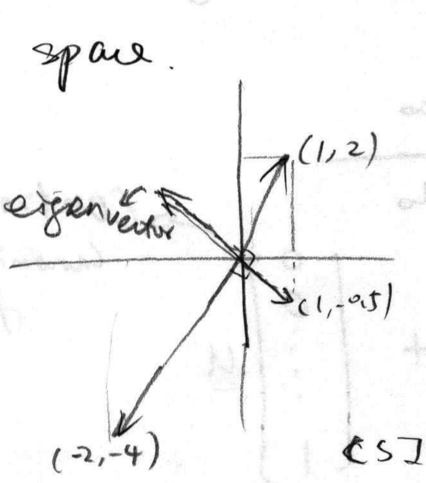
Compute eigenvalues of A.

$$\det(sI-A) = 0 = \begin{bmatrix} s+5 & 2 \\ -2 & s \end{bmatrix} = s^2+5s+4 = (s+4)(s+1)$$

eigenvalues are $\lambda_1 = -4$, $\lambda_2 = -1$.

Find eigenvectors (only have direction) we do not care about magnitude:

Sub. in λ_1 $\quad (sI-A)\big|_{\substack{s=\lambda_1 \\ (s=-4)}} = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} \rightarrow$ vectors do not span 2-D space.



$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ c_4 \end{bmatrix} = 0 \Rightarrow c_4 = -0.5$$

$$v_4 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

eigenvector

$$(sI-A)\big|_{s=-1} = \begin{bmatrix} 4 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ c_1 \end{bmatrix} \Rightarrow c_1 = -2$$

$$v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad (\lambda_2 = -1)$$

$$\phi(t) = \begin{bmatrix} -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t} & -\frac{2}{3}e^{-t} + \frac{2}{3}e^{-4t} \\ \frac{2}{3}e^{t} - \frac{2}{3}e^{-4t} & \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t} \end{bmatrix}$$

point along eig. vector

$$Y(s) = \frac{2}{s^2+5s+4} V_{in}(s)$$

I can write the T.F. $\omega = Y(s) = \frac{2}{s^2+5s+4} V_{in}(s)$ as a D.E.

$$\ddot{\omega}(t) + 5\dot{\omega}(t) + 4\omega(t) = 2 V_{in}(t)$$

We can write this in a different state space.

$$\begin{bmatrix} \dot{\omega} \\ \ddot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} \omega \\ \dot{\omega} \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} V_{in}$$

Last time we had it as

$$\begin{bmatrix} \dot{i} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} i \\ \omega \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cancel{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} V_{in}$$

represent same dynamics

In general, I can take any arbitrary Transfer function.

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

→ Controllable canonical form

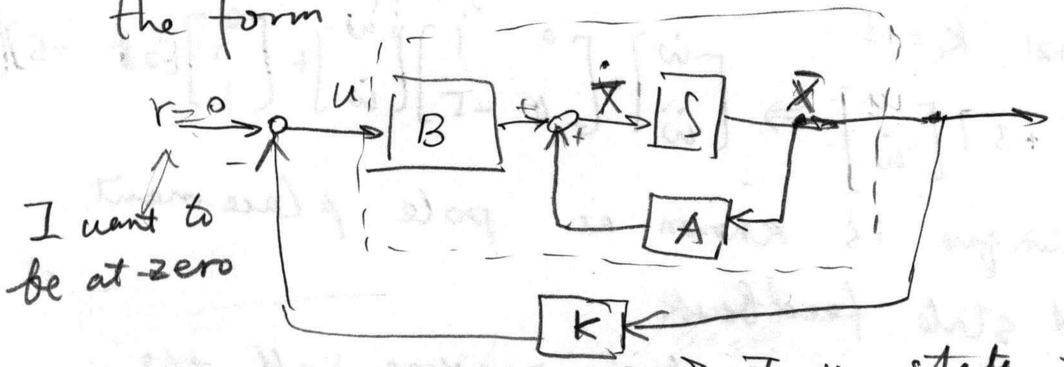
$$y = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_m & 0 & 0 \cdots 0 \end{bmatrix} \bar{X}$$

Let's say I have my system in controllable canonical form. (we can under certain conditions transform into this controllable form)

$$\dot{\bar{X}} = A_C\bar{X}_C + B_C u \leftarrow \text{single input (scalar)}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ -a_0 & -a_1 & -a_2 & & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

We have a feedback control system of the form:



→ Full state feedback

The control law is

$$u = -K\bar{X} \quad K \in R^{1\times n} \quad \bar{X} \in R^{n\times 1}$$

$$K = [k_0 \quad k_1 \quad \cdots \quad k_{n-1}]$$

Substitute for $u = -K\bar{X}$

$$\dot{\bar{X}} = A_C\bar{X} - BK\bar{X} = \underbrace{(A_C - BK)}_{\text{close loop pole locations}}\bar{X}$$

What does BK look like

$$BK = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} [k_0 \quad k_1 \quad \cdots \quad k_{n-1}] = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \\ k_0 & k_1 \cdots & k_{n-1} \end{bmatrix}$$

$$A_C - BK = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ -a_0-k_0 & -a_1-k_1 & \cdots & & -a_{n-1}-k_{n-1} \end{bmatrix}$$

$$(S+5)^3 = S^3 + 15S^2 + 75S + 125 \rightarrow \text{characteristic equation}$$

$$H(s) = \frac{N(s)}{s^3+15s^2+75s+125}$$

$$A_c - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -125 & -75 & -15 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0-k_0 & -a_1-k_1 & -a_2-k_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{w} \\ \ddot{w} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} w \\ \dot{w} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_{in}$$

$-25 \quad -10 \rightarrow -a_0-k_0 = -25, \; -a_1-k_1 = -10 \Rightarrow$

$k_0 = +21, \; k_1 = +5$

$$V_{in} = -\begin{bmatrix} +21 & +5 \end{bmatrix}\begin{bmatrix} w \\ \dot{w} \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{w} \\ \ddot{w} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} w \\ \dot{w} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} -21 & -5 \end{bmatrix}\begin{bmatrix} w \\ \dot{w} \end{bmatrix}$$

This technique is known as pole placement using full state feedback

The problem, it is hard to measure all the states, may be expensive or impractical.

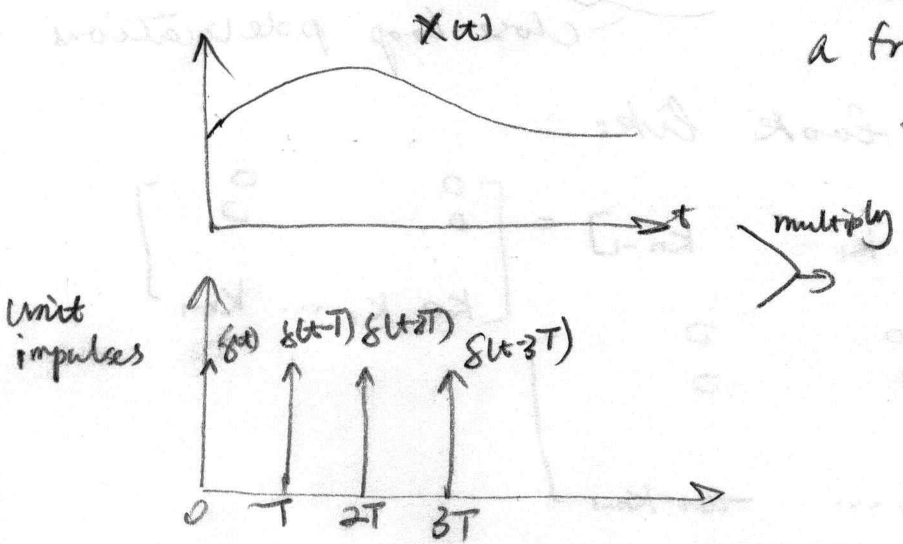
Let's look at discrete time again.

$$\bar{X}(k+1) = A_d \bar{X}(k) + B_d \bar{U}(k)$$

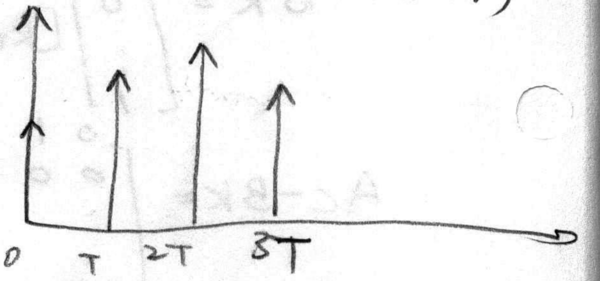
$A_d = e^{AT}$ $T \Rightarrow$ sampling time

$B_d = \int_0^T e^{A\lambda} d\lambda \, B$

Let's write out $x(t)$ as a train of impulses



$$X^*(t) = \delta(t)\,X(0) + \delta(t-T)\,X(T) + \delta(t-2T)\,X(2T) + \cdots$$



Now Let's take Laplace Transforms.

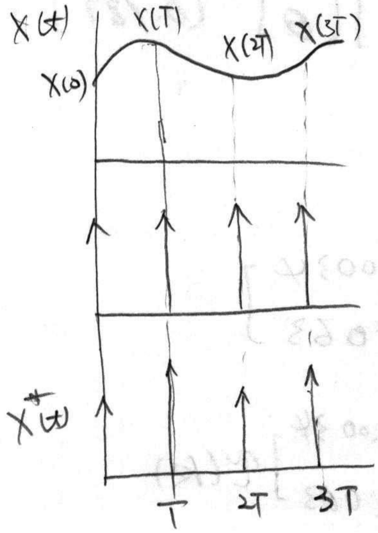
$$X^*(s) = X(0) + X(T)e^{-sT} + X(2T)e^{-s2T} + \cdots$$

Define (some short hand) $Z = e^{sT}$

Define Z-transform as:

$$X(Z) = \sum_{k=0}^{\infty} X(k) Z^{-k}$$

The Z-transform & the Difference Eqn



$$X^*(t) = X(0)\delta(t) + X(T)\delta(t-T) + X(2T)\delta(t-2T) + \cdots$$

$$= \sum_{k=0}^{\infty} X(kT)\delta(t-kT)$$

Taking Laplace Transform

$$X^*(s) = X(0) + X(T)e^{-sT} + X(2T)e^{-s2T} + \cdots$$

$$= \sum_{k=0}^{\infty} X(kT) e^{-sTk}$$

Define $Z = e^{sT} \rightarrow$ Z-transform

The Z-transform of a signal is

$$X(Z) = \sum_{i=0}^{\infty} X(i) Z^{-i}$$

$$= X(0) + \frac{X(1)}{Z} + \frac{X(2)}{Z^2} + \cdots$$

Just like in the s-domain, we had $\mathcal{L}\{\dot{X}(t)\} =$

$sX(s) - X(0)$

$Z_1\{X(t)\} = X(Z)$

$\downarrow$ discrete

$$Z\{X(t+1)\} = X(1) + \frac{1}{Z}X(2) + \frac{X(3)}{Z^2} + \cdots$$

$$= ZX(Z) - ZX(0)$$

This can be extended to

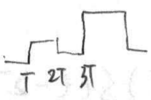
$$Z\{X(t+2)\} = Z^2X(Z) - Z^2X(0) - ZX(1)$$

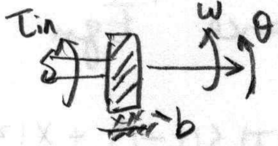
If we neglect I.C., we can think of Z as the time shift operator.

$$Z\{X(t-1)\} = Z^{-1}X(z)$$

In "S" domain, $\dot{\bar{X}} = A\bar{X}$, $SIX(s) = AX(s)$ / "z" domain $\bar{X}(k+1) = A_d\bar{X}(k)$, $zI\bar{X}(z) = A_d\bar{X}(z)$

Conversion from continuous Time to difference eqn.





$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -4.6 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.787 \end{bmatrix} u(t)$$

$$\theta = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

Set sampling time $T = 0.1$ sec

$$e^{AT} = \begin{bmatrix} 1 & 0.08015 \\ 0 & 0.6313 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0.0034 \\ 0.063 \end{bmatrix}$$

$X(k+1)$ $X(k)$

$$\begin{bmatrix} \theta(k+1) \\ \dot{\theta}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.08 \\ 0 & 0.63 \end{bmatrix} \begin{bmatrix} \theta(k) \\ \dot{\theta}(k) \end{bmatrix} + \begin{bmatrix} 0.0034 \\ 0.063 \end{bmatrix} \tau(k)$$

$$Y(k) = C \begin{bmatrix} \theta(k) \\ \dot{\theta}(k) \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Taking the z-domain Transfer function, we get

$$zI\,X(z) = A_d X(z) + B_d U(z)$$

We will compute

$$(zI - A_d)\,X(z) = B_d U(z)$$

$$X(z) = (zI - A_d)^{-1} B_d U(z)$$

The discrete time z-domain transfer function is,

$$Y(z) = C(zI - A_d)^{-1} B_d U(z)$$

$$\frac{Y(z)}{\tau(z)} = C(zI - A_d)^{-1} B_d$$

$$zI - A_d = \begin{bmatrix} (z-1) & -0.08 \\ 0 & (z-0.63) \end{bmatrix}^{-1} = \frac{1}{(z-1)(z-0.63)} \begin{bmatrix} (z-0.63) & 0.08 \\ 0 & (z-1) \end{bmatrix}$$

$$C(zI - A_d)^{-1} = \begin{bmatrix} \frac{z-0.63}{(z-1)(z-0.63)} & \frac{0.08}{(z-1)(z-0.63)} \end{bmatrix}$$

$$C(zI - A_d)^{-1} B_d = \frac{0.0034(z-0.63) + 0.08 \times 0.063}{(z-1)(z-0.63)}$$

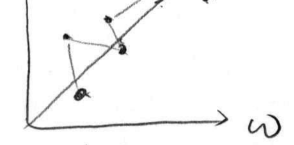
$$= \frac{0.0034z + 0.003}{z^2 - 1.63z + 0.63} = \frac{\theta(z)}{\tau(z)}$$

I remember that z (sometime will be ξ, especially in signal processing)

$$(z^2 - 1.63z + 0.63)\,\theta(z) = (0.0034z + 0.003)\,\tau(z)$$

Now convert back to time domain,

$$\boxed{\begin{aligned} &\theta(k+2) - 1.63\,\theta(k+1) + 0.63\,\theta(k) = 0.0034\,\tau(k+1) + \\ &0.003\,\tau(k) \end{aligned}}$$

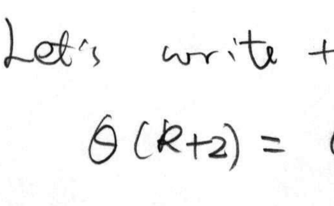


$$J\dot{w} = \tau_{in} - bw$$

$$\dot{w} = \boxed{-\frac{b}{J}} w + \boxed{\frac{1}{J}} \tau_{in}$$

$$\dot{\theta} = w, \ \ddot{\theta} = \dot{w}$$

if we can measure w, $\dot{w}$, we can get $-\frac{b}{J}$, $\frac{1}{J}$ just by 2 eqns



just sample θ. solve for those 4 parameters.

Let's write this as a Linear regression

$$\theta(k+2) = \phi^T(t)\,P$$

P are unknown parameters.

regression vector

$$\phi(k+2) = \begin{bmatrix} \theta(k+1) \\ \theta(k) \\ \tau(k+1) \\ \tau(k) \end{bmatrix}, \quad P = \begin{bmatrix} 1.63 \\ -0.63 \\ 0.0034 \\ 0.003 \end{bmatrix}$$

Identify this parameter vector using least square

time step	θ	τ
0	0	1
1	0.01	1.2
2	0.013	0.9
3	0.019	0.8
4	0.02	0.3
5	0.035	2.1
⋮	⋮	⋮
100	2.94	⋮
⋮		
1000	a1	

The Least square solution:

$$J = \frac{1}{2} \sum (\theta(k+2) + \phi^T \hat{p})^2$$

$$\frac{dJ}{d\hat{p}} = 0$$

$$\hat{p} = (\sum \underbrace{\overset{4\times1}{\phi} \overset{1\times4}{\phi^T}}_{\to \text{+ve def. sym } (4\times4)})^{-1} \sum \phi \, \theta(k+2)$$