FSK and DPSK over Unresolved Multipath Rayleigh Fading Channels

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Abstract In this paper we present single pulse performance of detection structures for fading multipath channels with common modulation formats such as FSK and DPSK. The multipath components are assumed to be unresolved, with known delays. These delays could have been estimated, for example, by sounding the channel with a pulse of much wider bandwidth than that of the information conveying signal. This paper shows that it is possible to have diversity-like gains over multipath fading channels with unresolved components even if the channel is not fully tracked. However the required SNR to achieve such diversity gains is highly dependent on the modulation formats. Properly designed receivers for unresolved multipath channels may reduce the SNR, where the diversity gains are achieved, and lower significantly the error floors. It is shown that a Quadratic Decorrelation Receiver performs well over unresolved multipath channels.

1. Introduction

In the radio indoor environment characterized as a fading multipath channel, the resolvability assumption [1] cannot be ensured for narrow band systems due to the relatively small inter-path delays. The performance of the optimal receiver for unresolved multipath Rayleigh channels has been considered by Aiken [2] for signals that are orthogonal to all time shifts (widely orthogonal signals or uniformly orthogonal signals) and thus equivalent to transmission over a resolved multipath channel. Mazo [3] evaluated the matched filter bound on performance for a two paths Rayleigh fading channel assuming the channel to be known exactly. He showed that the matched filter provides diversity-like improvement at high SNR, even without path resolvability.

In this paper we consider the performance of commonly used binary modulation schemes such as Frequency Shift Keying (FSK) and Differential Phase Shift Keying (DPSK) over two and three paths Rayleigh fading channels. Following the same approach as in [3] we determine single pulse performance (equivalent to matched filter bounds) but without full tracking the channel. Instead, only a second order statistic of the channel path gains is assumed to be known. For small inter-path delays, it is to be expected that the effect of inter-symbol interference is small, making legitimate the evaluation of single pulse performance. The paper is organized as follows. Section II considers receiver structures, optimum as well as sub-optimum. Section III presents performances of binary FSK and DPSK over two and three paths Rayleigh fading channels. Section IV presents the conclusions.

2. Receiver structures

Assume transmission of one of $M$ possible bandpass signals of finite energy over a fading multipath channel. For convenience the $M$ possible transmitted signals will be represented by their complex envelope $\tilde{s}_m(t)$. Under the hypothesis $H_m$ (i.e. that $\tilde{s}_m(t)$ was transmitted), the complex envelope of the received signal $r(t)$ is given by

$$\tilde{r}(t) = \sum_{i=0}^{L-1} a_i \tilde{s}_m(t - \tau_i) + \tilde{n}(t) \quad m = 1, 2, ..., M$$

where $a_i$ are independent circularly complex Gaussian random variables with zero mean and variance $E[a_i a_i^*] = \sigma_i^2$ [4, 5]. The multipath delays $\tau_i$ are assumed to be known and $\tau_i \neq \tau_j$ if $i \neq j$. The effect of the channel noise is modeled by an additive zero mean circularly complex Gaussian process $\tilde{n}(t)$ satisfying $E[\tilde{n}(t)\tilde{n}^*(u) / 2] = N_0 \delta(t - u)$ and statistically independent of the signal process $\tilde{v}_m(t) = \sum_{i=0}^{L-1} a_i \tilde{s}_m(t - \tau_i)$.

We assume that the observation interval $T_0$ is much longer than the multipath delays such that the delayed signals $\tilde{s}_m(t - \tau_0), \tilde{s}_m(t - \tau_1), ..., \tilde{s}_m(t - \tau_{L-1})$ are contained in the observation interval. Furthermore we assume that these delayed signals are linearly independent. Notice that linear independence is a much weaker constraint compared to the orthogonality constraint $(\int_0^{T_0} \tilde{s}_m(t - \tau_i) \tilde{s}_m(t - \tau_j) dt = 0, i \neq j)$ required by the resolvability condition [1]. It has been shown in [6] that any square integrable waveform with standard Fourier transform will satisfy the linear independence condition.

We use the following notation: Capital letters denote matrices whereas lowercase letters denote vectors or scalars. The symbols $\tilde{r}, \tilde{s}$ and $\dagger$ denote respectively the...
transposition, complex conjugation and Hermitian conjugation of a matrix or vector. The $ij$th entry of a matrix $B$ is denoted as $[B]_{ij}$ and the $i$th entry of a vector $v$ is denoted as $[v]_i$. Since $T_0 \gg \tau_i$ for all $i$, the energies of the signals $\bar{s}_m(t-\tau_i)$ over the observation interval will be identical and denoted as $E_m$. We define the correlation matrix of the signal under $H_m$, $\Gamma_m$ as $[\Gamma_m]_{ij} = \frac{1}{T_0} \int_0^{T_0} \bar{s}_m(t-\tau_i) \bar{s}_m(t-\tau_j)dt$. The covariance matrix of the channel $C$, defined as $[C]_{ij} = E[a_i a_j^+]$, is a diagonal positive definite matrix with $i$th diagonal entry $2\sigma_i^2$.

With multipath Rayleigh channels, we have the classical problem of detecting a continuous time Gaussian random signal $\bar{v}_m(t) = \sum_{i=1}^{L} a_i \bar{s}_m(t-\tau_i)$ in additive white Gaussian noise [7, pp. 419-421]. A minimum probability of error receiver forms the likelihood ratio between each one of the hypotheses $H_m : \bar{r}(t) = \bar{v}_m(t) + \bar{n}(t), m=1,2,...,M$ and a null hypothesis $H_0 : \bar{r}(t) = \bar{n}(t)$. The decision is made in favor of the largest likelihood ratio [7, p. 11]. The likelihood ratios $\Lambda_m(\bar{r}(t))$ between $H_m$ and $H_0$ are given by [7, pp. 419-423]

$$\Lambda_m(\bar{r}(t)) = \lim_{N \to \infty} \frac{P_{r_m|H_m}[r_m|H_m]}{P_{r_m|H_0}[r_m|H_0]} \quad m = 1, \ldots, M \quad (1)$$

where $r_m$ is a $N$-dimensional vector whose components are the projections of $\bar{r}(t)$ on the eigenvectors $\{\phi_m(t)\}_i$ associated with the covariance function of $\bar{v}_m(t)$ (Karhunen-Loève expansion), and the limit is in probability. The covariance of the signal process $\bar{v}_m(t)$, $K_m(t,u) = \sum_{i=1}^{L} 2\sigma_i^2 \bar{s}_m(t-\tau_i) \bar{s}_m(u-\tau_i)$, is a Pincherle-Goursat (degenerate or finite dimensional) kernel with well known eigenvalues and eigenvectors [8, p. 55]. The eigenfunctions are obtained by a linear transformation of $\{\bar{s}_m(t-\tau_i), i = 0, 1, \ldots, L - 1\}$

$$\phi_m(t) = \frac{1}{\sqrt{E_m}} \sum_{i=0}^{L-1} x_m^i \bar{s}_m(t-\tau_i) \quad (2)$$

Under these circumstances, the problem of finding the eigenvalues and eigenvectors of $K_m(t,u)$ reduces to the following eigenvalue problem $\lambda_m x_m^i = E_m C T_m^* x_m^i$, where $x_m^i = [x_m^0, x_m^1, \ldots, x_m^{L-1}]^T$. Since $C$ and $\Gamma_m$ are both Hermitian positive definite matrices, this algebraic system has at most $L$ solutions (i.e. $L$ real eigenvalues and $L$ corresponding linearly independent eigenvectors). Therefore our problem is in fact finite dimensional and we do not need to take the limit in the likelihood ratios (1).

Let $D_m^*$ denotes the diagonal matrix whose diagonal entries are the eigenvalues of the matrix $C T_m^*$. The eigenvalues of the signal process are given by $E_m [D_m^*]_{ii}$. The eigenfunctions are given by (2), where the matrix $X_m$ defined as $[X_m]_{ij} = x_m^i$ satisfies the equations $X_m C T_m^* X_m = I$ and $C T_m^* X_m^2 = X_m^2 D_m^*$. Performing some mathematical manipulations yields the log-likelihood ratios $\ln [\Lambda_m(\bar{r}(t))]$ between $H_m$ and $H_0$ [6]

$$\ln [\Lambda_m(\bar{r}(t))] = \gamma_m^2 r_m^i \left[ (D_m^*)^{-1} + \gamma_m I \right]^{-1} r_m^i - \ln \left[ \det(I + \gamma_m D_m^*) \right] \quad (3)$$

where $I$ is the identity matrix and $\gamma_m$ is the signal-to-noise ratio, $\gamma_m = \frac{E_m}{2\sigma_m^2}$. The components of the $L$ dimensional vector $r_m^i$ are the projections of the received signal on the eigenvectors $\phi_m(t)$ normalized by the signals energy, $r_m^i = \frac{1}{\sqrt{E_m}} \int_0^{T_0} \bar{r}(t) \phi_m(t)dt$. The receiver suggested by (3) is the Rayleigh fading channel version of the Quadratic Decorrelation Receiver (QDR) considered in [6]. The QDR block diagram for the $m$th hypothesis is illustrated in Fig. 1. The complete receiver is composed of $M$ blocks such as Fig. 1 followed by a decision device that chooses the largest output. It is shown in [6, 9] that this receiver first performs an orthogonalization (or decorrelation) operation ($r_m^i = X_m^* u_m^i$) and then implements a resolved multipath optimum decision rule for the transformed signals and channel. An equivalent form of the log-likelihood ratio is given by

$$\gamma_m^2 u_m^i (C^{-1} + \gamma_m \Gamma_m)^{-1} u_m - \ln \left[ \det(I + \gamma_m C T_m^*) \right]$$

where $u_m^i = \frac{1}{E_m} \int_0^{T_0} \bar{r}(t) \bar{s}_m^i (t-\tau_i)dt$ and is essentially the receiver found by Aiken in [10], assuming zero Doppler shift.

In order to assess the performance improvement due to the decorrelation operation, we consider also a receiver very similar to QDR except that it does not employ decorrelation, the Quadratic Receiver (QR). The QR is obtained from the QDR (see Fig. 1) by replacing the matrices $X_m$ and $D_m^*$ with the matrices $I$ and $C$. In other words, this receiver has the following decision variable $\gamma_m^2 u_m^i (C^{-1} + \gamma_m I)^{-1} u_m^i - \ln \left[ \det(I + \gamma_m C) \right]$. Notice that the variables $u_m^i$ could have been also obtained by using a tapped-delay line. Then they could be combined and sent to the decision circuit. Thus the QR is a typical example of a RAKE receiver[11, 12] and is to be considered as the conventional receiver over multipath fading channels.

3. Performance

Following [3] we consider single-pulse performance of the QDR scheme when used with binary Frequency Shift Keying (FSK) and Differential Phase Shift Keying (DPSK). We will also investigate the performance of the QR scheme and compare with the QDR. This will show the improvement due to decorrelation. Both receivers makes their decision rule based on a Hermitian quadratic form in jointly Gaussian random variables, $\Delta$ (for instance $\Delta = r_1^T Q_i r_1 - r_2^T Q r_2$ for the QDR scheme). It is well known that the probability of error in these cases can be evaluated analytically by inverting the characteristic function $\varphi_{\Delta}(J, \delta)$, of $\Delta$ [1]. For a Rayleigh fading
channel, the characteristic function reduces to
\[ \varphi_{\Delta|H_k}(jt) \triangleq E[e^{jt\Delta}] = [\det(I - jtM_k Q)]^{-1} \]
where \( I \) is the identity matrix, \( Q \) is the matrix associated with \( \Delta \), \( M_k = E \left[ (r' - r) (r' - r)^\dagger \right] \)
and \( r' = E[r'] \). The probability density function (PDF) of \( \Delta \) is given by the Fourier transform of \( \varphi_{\Delta}(jt) \), and the two pairwise probabilities of error are obtained by integrating the PDF of \( \Delta \) \[1\]. We are left with line integrals,
\[
\int_{-\infty}^{\infty} e^{-\frac{s}{\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{-iz} dz.
\]
which can be evaluated using the residue theorem \[13, p. 89\].

The bit error probabilities for the QDR and QR schemes are presented as functions of the received signal-to-noise-ratio per bit \( \frac{E_b}{N_0} \). For a two paths channel, the received SNR is given by \( \frac{E_b}{N_0} = \frac{2\sigma^2}{E_b} (1 + s) \frac{E_b}{N_0} \) where \( s = \frac{\gamma^2}{\sigma^2} \) and \( E_b \) is the energy per bit of the real signal (i.e., half of the energy per bit of its complex envelope). For FSK we have \( E_b = \frac{E_m}{2} \), \( m = 1, 2 \) since the observation interval is equal to the symbol duration plus the maximum of the channel inter-path delays. FSK modulation with frequency separation \( f_1 - f_2 \) will be denoted as FSK\((v)\), where \( v = (f_1 - f_2)T \). Both conventional Differential Phase Shift Keying (DPSK) and symmetrical Differential Phase Shift Keying \[14\] (SDPSK) are considered in this paper. Since with DPSK, the transition between the carrier phase of consecutive bits carries the information, the observation interval needs to be twice the symbol duration \( T \). Therefore for DPSK, \( E_b = \frac{E_m}{4} \), \( m = 1, 2 \) which takes into account the energy gain due to the longer observation interval. The relative delay between the first path and the second path, \( \tau = \tau_1 - \tau_0 \) is expressed as a percentage of the symbol duration \( T \). The two paths Rayleigh fading channel is characterized by a parameter \( s \), the relative Rayleigh path's strength between the first and the second path.

The numerical results are presented in Fig. 2-5. Several parameter index pointing to the same curve show that corresponding curves overlap or are very close to each other. For clarity only one of those curves is plotted. Fig. 2 presents the probability of error for four modulation schemes considered in this paper: FSK\((1)\), FSK\((1/2)\), DPSK and SDPSK, with equal path's strength and a very-small inter-path delay \( \tau = 0.1T \). Such delay corresponds to a completely unresolved channel. It is seen that the QDR scheme provides the diversity-like gains typical of classical channel diversity techniques even when the multipath is unresolved. It is also seen that DPSK detected with the QDR gives the best performance. More precisely at high SNR DPSK gives 4 dB improvement compared to FSK\((1/2)\) and between 10 and 13 dB compared to FSK\((1)\). However 3dB are gained because the observation interval used with DPSK is twice the one used with FSK. We can see that the QDR yields nearly the same performance with DPSK and SDPSK. The improved per-
formance of DPSK and SDPSK over FSK can be explained by the fact that the required SNR to achieve the diversity gain of order two is lower for DPSK and SDPSK than for FSK. As far as FSK(1) is concerned, with QDR, FSK(1) is better for $E_b/N_0 < 20 dB$, and FSK(1/2) is better for $E_b/N_0 > 20 dB$. Same type of behavior has been observed on a two paths mixed mode Ricean/Rayleigh channel [6]. Moreover it is seen that for FSK(1/2) and DPSK the QR scheme has a marked error floor, while the QDR provides a two fold diversity curve slope.

![Figure 2: Performance of FSK and DPSK over 2 paths Rayleigh fading channels](image)

The results for a two paths channel show that the performance of the QDR scheme is always better than the performance of the QR scheme. It is to be expected that the gains and the superior performance of the QDR scheme over QR are even larger over a three paths channel as can be inferred from Fig. 3-5. For a three paths channel, the received SNR is given by $\frac{E_b}{N_0} = 2\tau_0^{2}(1 + s + s')\frac{E_b}{N_0}$ where $s' = \tau_0^{2}$ We use the same notation as for the two paths channel. The relative delay between the first path and the third path, $\tau' = \tau_2 - \tau_0$, is expressed as a percentage of the symbol duration $T$. The three paths Rayleigh fading channel is characterized by the value of the parameters $s$ (defined as for the two paths channel) and $s'$ which represents the Rayleigh path's strength of the third path relative to the first.

Fig. 3 presents the probability of error for FSK(1) over two and three paths Rayleigh fading channels respectively curves C1 and C2) when the SNR is between 20 and 70dB. From Fig. 3 we see that the QDR scheme gives higher diversity-like gains over the three paths channel. For sufficiently high SNR, the probability of error of FSK(1) decreases by an order of magnitude for an increase of 3.5dB in $E_b/N_0$, and thus yields diversity gains of order three. However for FSK(1) the required SNR to achieve order three diversity gains is relatively high since the QDR scheme yields only two-fold diversity gains for a SNR of 40dB. In fact from Fig. 4 and Fig. 5, it seems that the required SNR is highly dependent on modulation schemes. Fig. 3 illustrates the error floor phenomenon in the performance of the QR scheme. It appears that over the three paths channel there is also an error floor with FSK(1) unlike over the two paths channel. The QDR scheme however eliminates this error floor.

![Figure 3: Performance of FSK(1) signaling over 2 and 3 paths Rayleigh fading channels](image)

Fig. 4 presents the probability of error for FSK(1/2) over two and three paths Rayleigh fading channels, and Fig. 5 presents similar curves for DPSK and SDPSK. From Fig. 4 and Fig. 5 we see that the QDR scheme gives also higher diversity-like gains over the three paths channel for FSK(1/2), DPSK and SDPSK. However three fold diversity gains are obtained at a much lower SNR with these modifications than with FSK(1). Fig. 5 shows that the QDR scheme yields nearly the same performance with DPSK and SDPSK for two as well as three paths Rayleigh fading channels. From Fig. 3, Fig. 4 and Fig. 5 we see that all the effects observed over the two paths channel(diversity-like gains and error floor phenomenon) are enhanced over the three paths channel. It is to be expected that the same trend exists as the number of paths increases showing that the improvement of the QDR scheme with respect to QR might be even higher.

All the results confirm that the performance of the QR (dashed lines) for small delay depends heavily on modulation schemes. For FSK(1) or SDPSK the QR scheme performs reasonably well giving diversity gains. With FSK(1/2) or DPSK however the QR scheme does not give any diversity gains. The improved performance of the QDR scheme for DPSK modulations can be explained as follows. Over an equal path strength channel, the performance of the QR scheme depends on two factors, 1) the two correlations matrices $\Gamma_1$ and $\Gamma_2$, 2) the cross correlation matrix $\Gamma_{12}$, defined as $\Gamma_{12} = \frac{1}{\sqrt{E_b\\E_b}} \int_0^T \delta_1(t - \tau) \delta_2'(t - \tau) dt$. These matrices are closely related to the shape of the signals and the inter-path delay of the channel. For
SDPSK and FSK, the signals corresponding to alternate hypotheses are conjugate to each other; thus the correlation matrices are also conjugate to each other having identical eigenvalues. On the other hand, for DPSK the two correlation matrices have different eigenvalues. Let's recall that the QDR scheme uses the eigenvalues of the matrices \( \Gamma_1, \Gamma_2, \Gamma_3 \), which are equal for SDPSK and FSK and the QR scheme uses the eigenvalues of the matrix C. The relatively good performance of the QR scheme for SDPSK can then be partly explained by the fact that though the QR scheme does not use the appropriate eigenvalues, it uses an identical set of eigenvalues for both hypotheses similar to QDR. The form of the cross correlation matrix \( \Gamma_1 \) has also an important role since though the QR scheme uses a similar set of eigenvalues for FSK(1/2) as QDR, it yields some error floor.

4. Conclusions

In this paper we studied single-pulse performance of optimum (the QDR scheme) and sub-optimum (the QR scheme) structures for commonly used binary modulation schemes such as FSK and DPSK over two and three paths Rayleigh fading channels. The multipath delays were assumed to be known but unresolved. It was shown that it is possible to get diversity-like gains at high SNR even if the channel is not fully tracked. However this requires in general the use of receivers especially designed to handle the non-resolvability condition. Indeed all the results show that the QDR reduces significantly the error floors and provides SNR gains with respect to the QR scheme. However it was shown that the required SNR to achieve the diversity-like gains is highly dependent on the modulation formats. Consequently when the multipath is unresolved DPSK and SDPSK have the best performance at high SNR, better than FSK(1/2) and FSK(1).

References