Carleton UNIVERSITY

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2½D Finite Element Method for Electrical Impedance Tomography Considering the Complete Electrode Model

Navid Bahrani

Supervised by: Prof. Andy Adler

Carleton University

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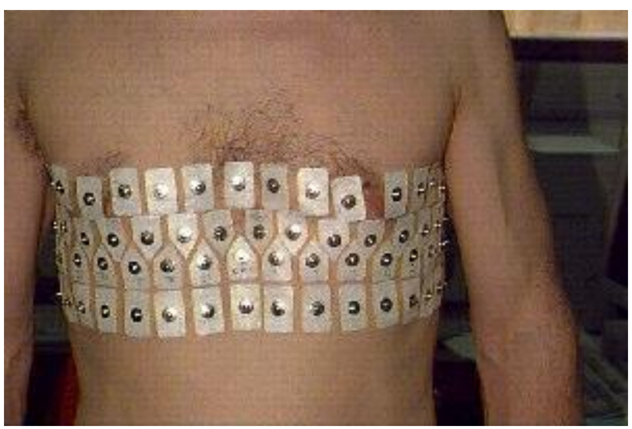
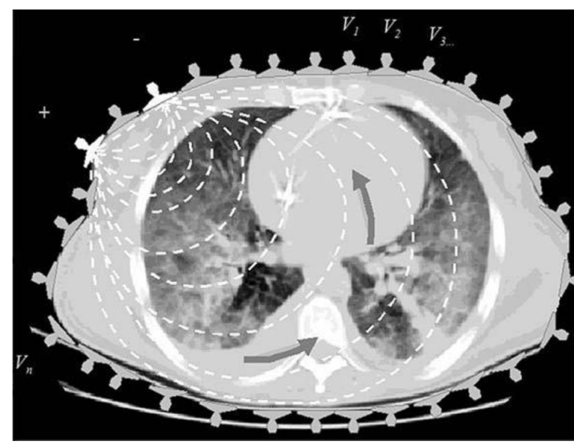
Electrical Impedance Tomography (EIT)

- EIT is used to generate images of the internal structure of sections of a body
- The EIT problem is
  - to reconstruct an unknown impedance distribution from boundary measurements.





Photos: (left) from Wikipedia/EIT, (right) from [4]

The EIT Problem

- Forward Model (2D & 3D)

$$\nabla_{2D} \cdot \left(\sigma(x,y)\nabla_{2D}\varphi(x,y)\right) = 0$$

$$\nabla_{3D} \cdot \left(\sigma(x,y,z)\nabla_{3D}\varphi(x,y,z)\right) = 0$$

$$\sigma\frac{\partial\phi}{\partial n} = \begin{cases} J & \text{on current electrodes} \\ 0 & \text{elsewhere on the surface} \end{cases}$$

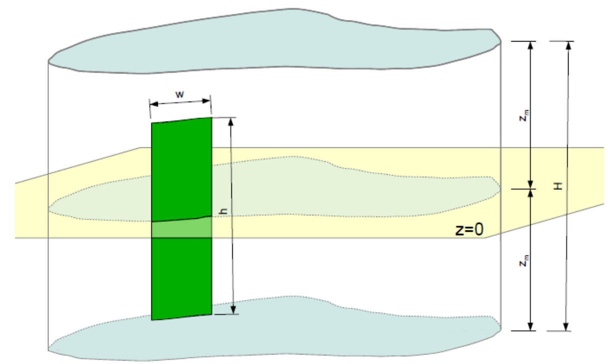
- Finite Element Method
- Current Patterns
- Electrode Models

2½D Motivation

- The 3D FE Model recruits too much elements.
  - => requires much more memory and Computational Complexity vs. 2D
  - Both Forward and Inverse Problem
  - Specially the inverse part
  - Requires more calculation time
    - ≠ Real time
  - Or a super-computer for fast imaging
    - ≠ Portability and Inexpensiveness

The 2½D Model

- Assumption
  - Translational Invariance along z
  - => Symmetric Voltages
- Equations



$$\nabla_{3D} \cdot \left(\sigma(x,y,z)\nabla_{3D}\varphi(x,y,z)\right) = 0$$

$$\varphi(x,y,z) = \sum_{k=0}^{\infty} V_k(x,y)\cos\left(\frac{k\pi}{a}z\right)$$

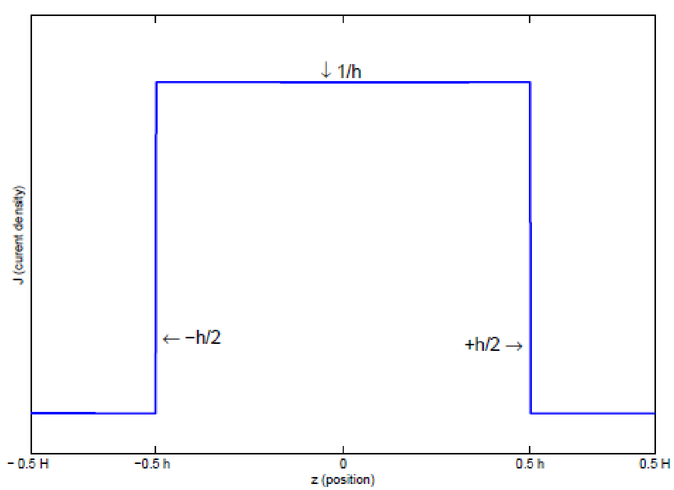
$$\begin{cases} \nabla_{2D} \cdot \left(\sigma(x,y)\nabla V_k(x,y)\right) - \sigma(x,y)\left(\frac{k\pi}{a}\right)^2 V_k(x,y) = 0 \\ \sigma(x,y)\frac{\partial}{\partial n}V_k(x,y) = J_k \end{cases}$$

Boundary Condition

$$\sigma(x, y)\frac{\partial V_k}{\partial n} = J_k$$

- for I = 1

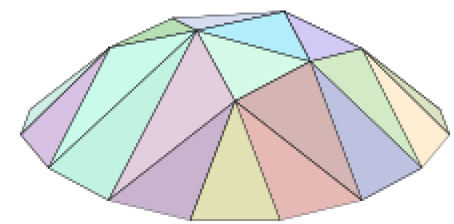
$$J_0 = \frac{1}{H} = \frac{1}{2a} \qquad J_k = \frac{2}{k\pi h}\sin\left(\frac{k\pi h}{2a}\right)$$



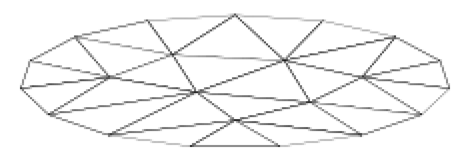
Finite Element Method

- Interpolation functions, i.e. basis

$$\tilde{u}_n(\vec{x}) = \sum_{i=1}^{M} u_i^n \phi_i(\vec{x})$$



- The Modified '*Stiffness Matrix*'



$$S'^k_{ij} = S^k_{ij} + \left(\frac{n\pi}{a}\right)^2 R^k_{ij} = \int_{\mathrm{E_k}} \nabla\phi_i \cdot \nabla\phi_j + \left(\frac{n\pi}{a}\right)^2 \phi_i\phi_j \, d\Omega$$

$$S'(n)U_n = I_n$$

Image from Wikipedia/Finite Element Method

Inverse Problem of EIT

- Static EIT, Difference EIT
- Jacobian (Sensitivity Matrix)

$$z = \Delta v = v_{\sigma_2} - v_{\sigma_1}$$

$$x = \mathbf{J}z + n \qquad x = \Delta\sigma = \sigma_2 - \sigma_1$$

$$\mathbf{J} = T\left[-\frac{\partial}{\partial\sigma}S^{-1}(\sigma)I\right] = T\left[S^{-1}(\sigma)\frac{\partial}{\partial\sigma}S(\sigma)S^{-1}(\sigma)I\right]$$

$$\hat{x} = (\mathbf{J}^T\mathbf{J})^{-1}\mathbf{J}^T z$$

$$\hat{x} = (\mathbf{J}^T\mathbf{J} + \lambda^2\mathbf{R}^T\mathbf{R})^{-1}\mathbf{J}^T z = Bz$$

Inverse Problem in 2½D

- Using Jacobian:
  - For each $n \rightarrow \Delta v_n$

$$n \rightarrow S_n \rightarrow \mathbf{J}_n \rightarrow \Delta\sigma_n$$

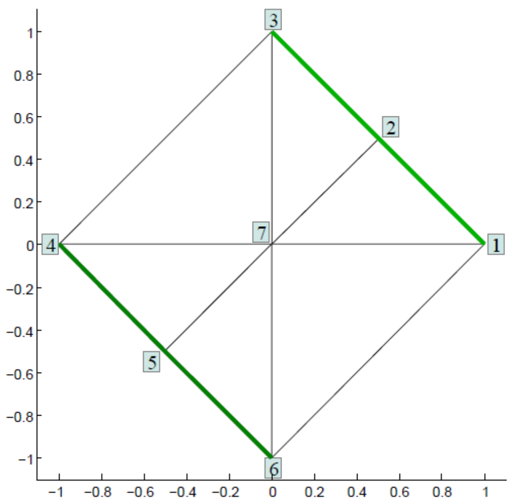
$$\Delta v_n = \mathbf{J}_n \Delta\sigma$$

$$\mathbf{J} = \sum_{n=0}^{\infty} \mathbf{J}_n$$

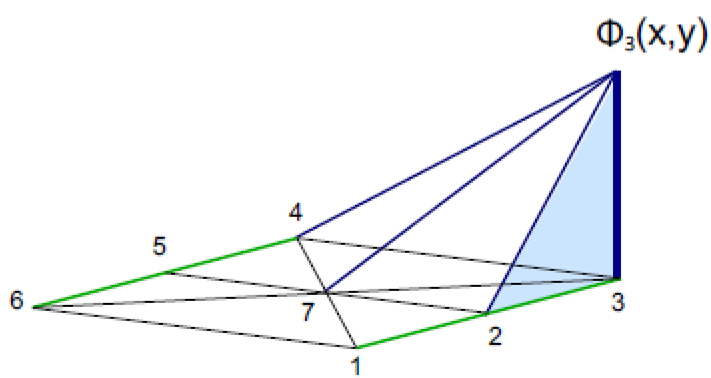
$$\Delta v = \sum_{k=0}^{\infty} \Delta v_k \cos\left(\frac{k\pi}{a} z\right)$$

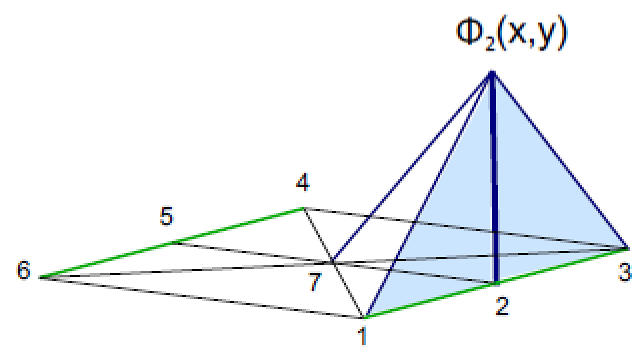
Complete Electrode Model

$$\begin{bmatrix} A_M + A_Z & A_W \\ A_W^T & A_D \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ I \end{bmatrix}$$

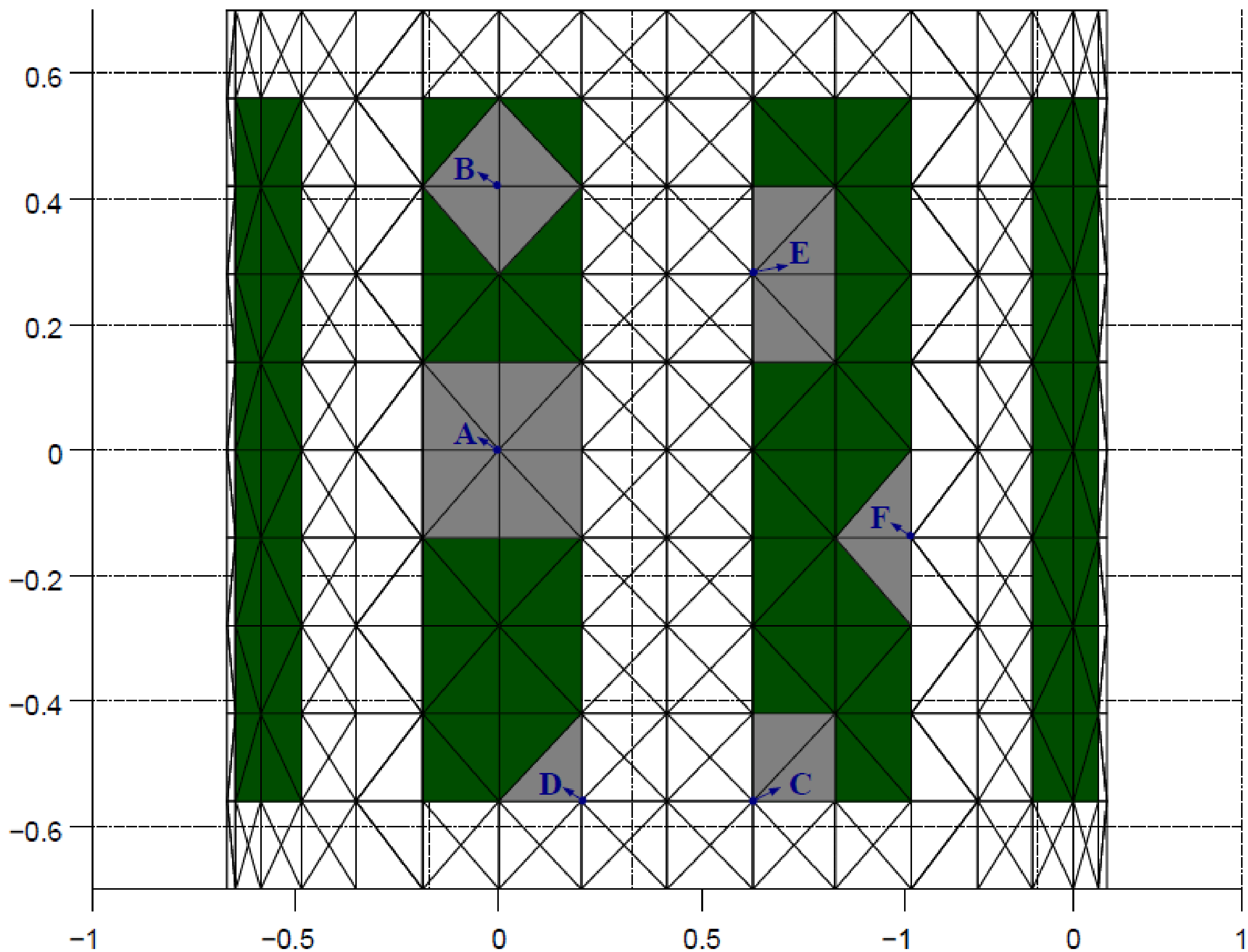


$$A_W = \begin{bmatrix} -\frac{\Delta}{2z_c} & -\frac{\Delta}{z_c} & -\frac{\Delta}{2z_c} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\Delta}{2z_c} & -\frac{\Delta}{z_c} & -\frac{\Delta}{2z_c} & 0 \end{bmatrix}^T$$

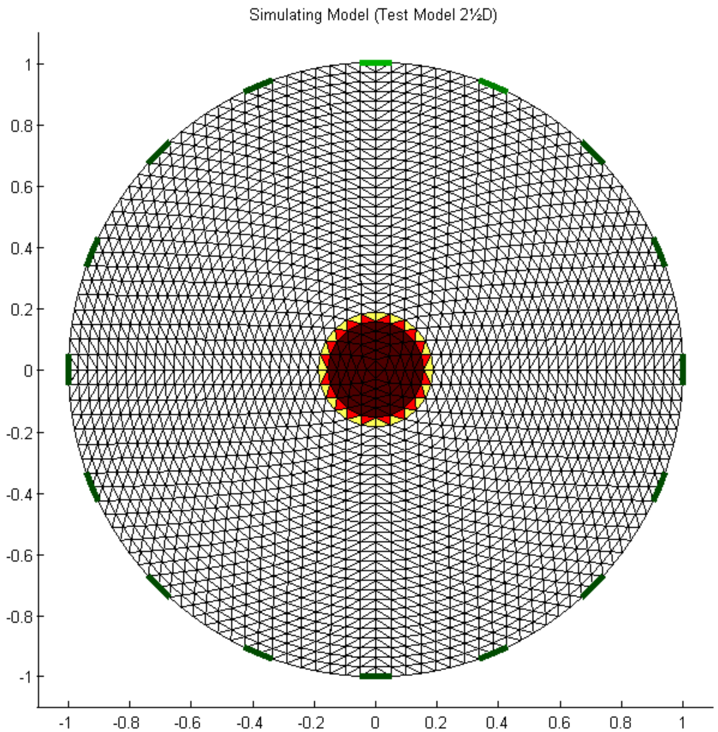


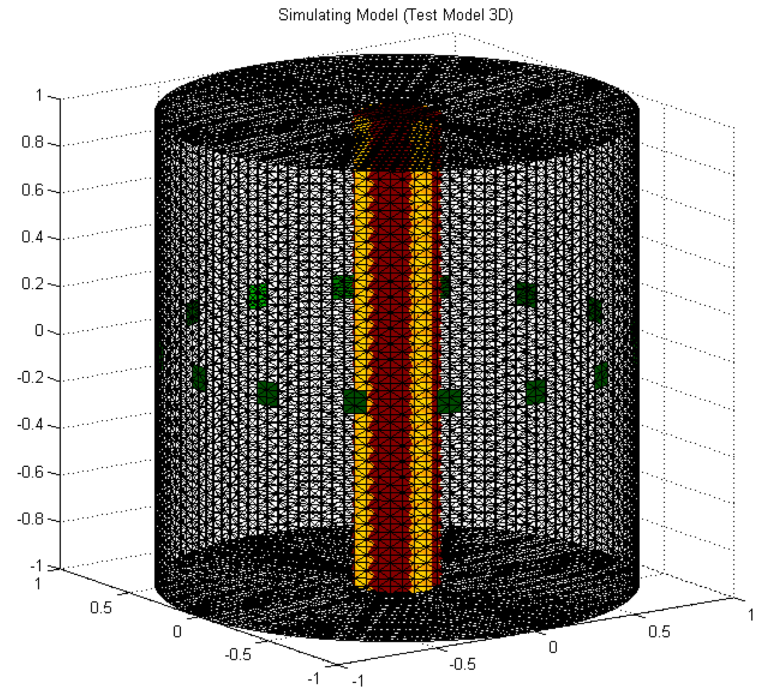


3D CEM



Mesh



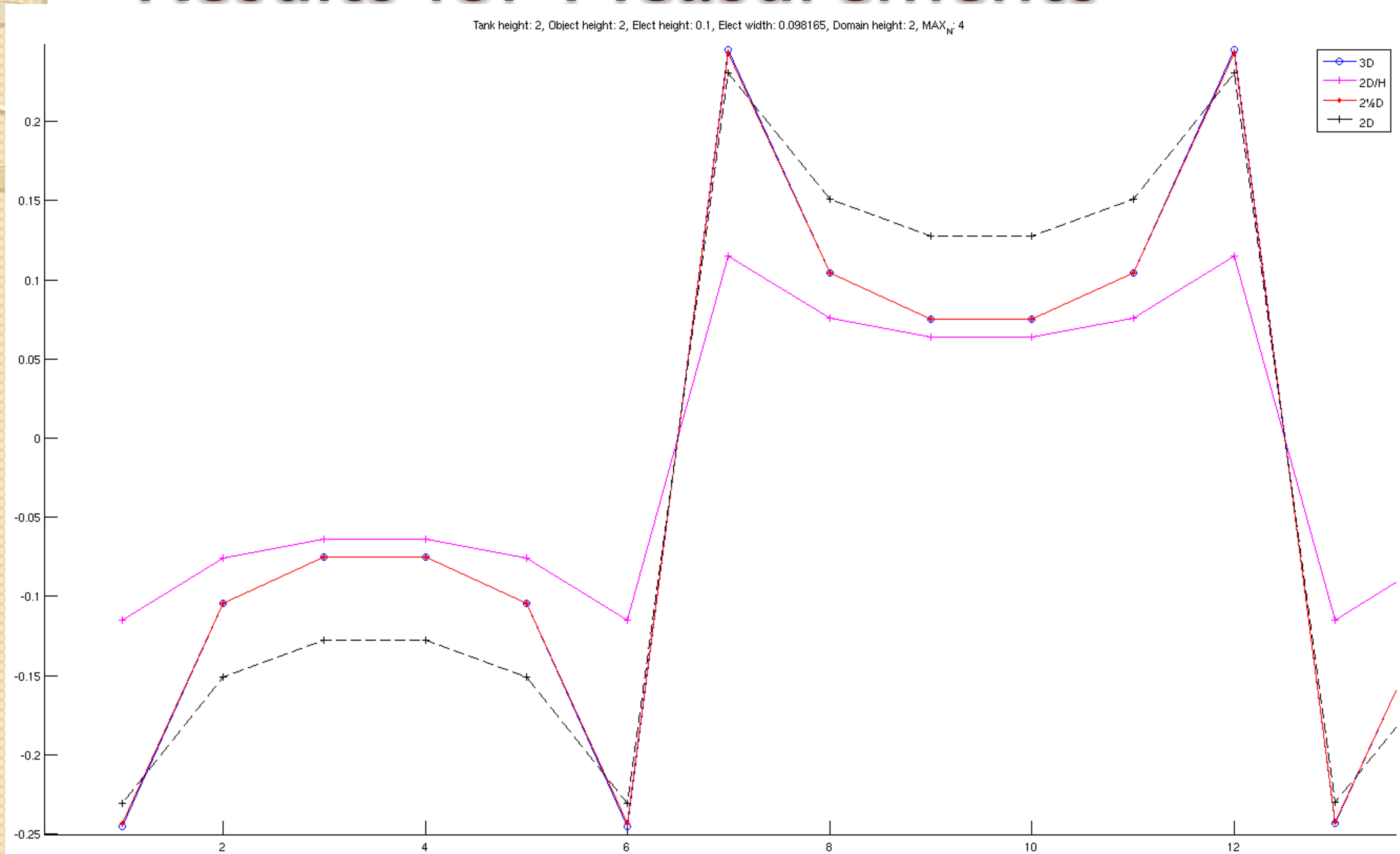


2D mesh with 4096 elements used for the 2½D method (32 layers in xy)
3D mesh with 737,280 elements (61 layers in z)

H=2; h=0.1, w ≈ 0.1

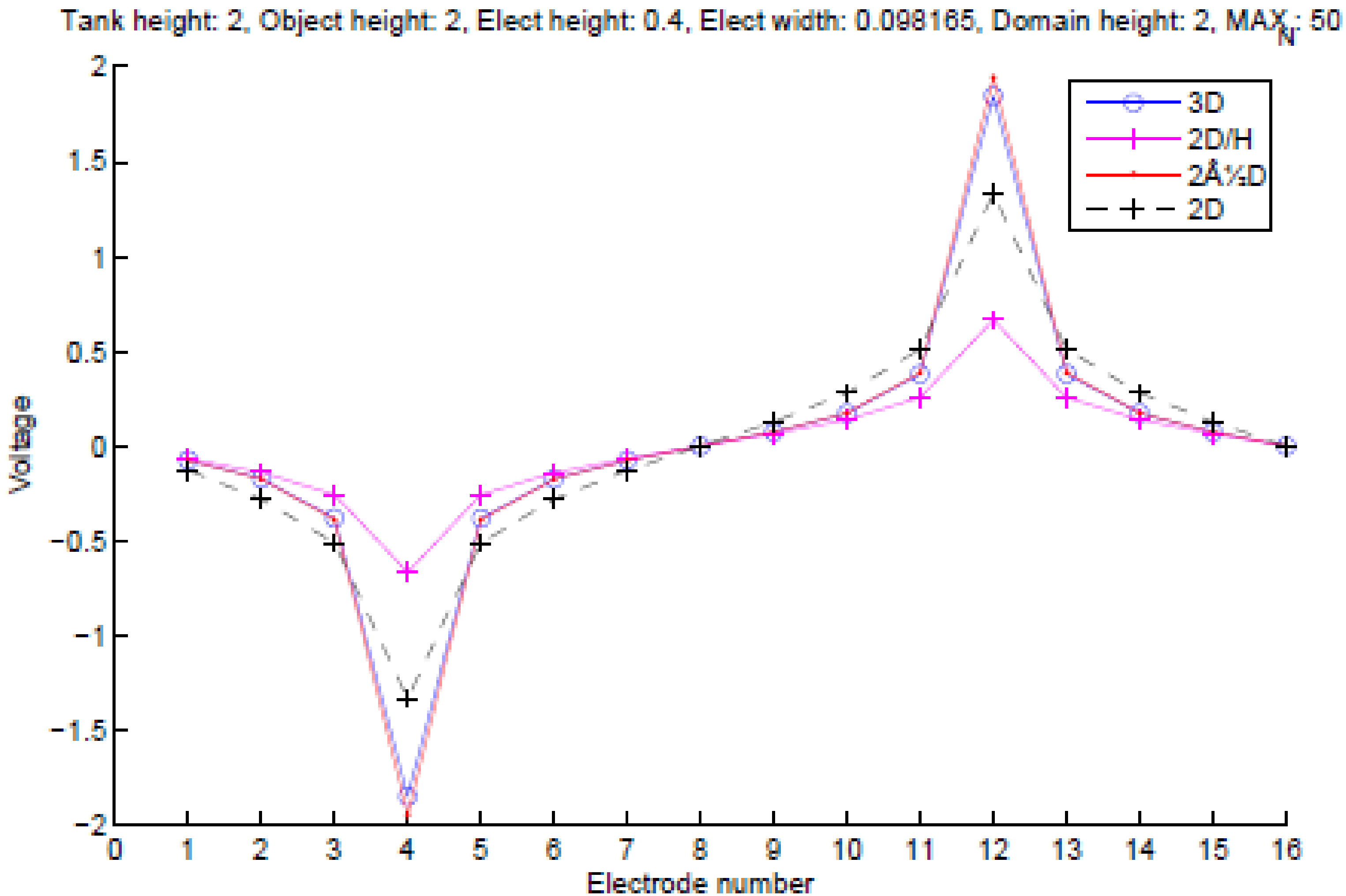
The Images are produced by EIDORS

Results for Measurements

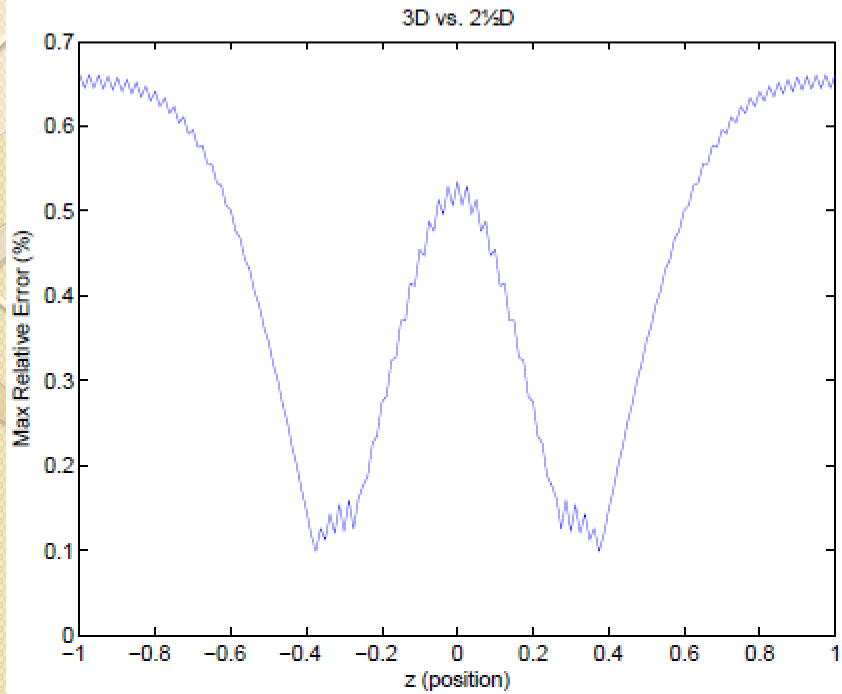


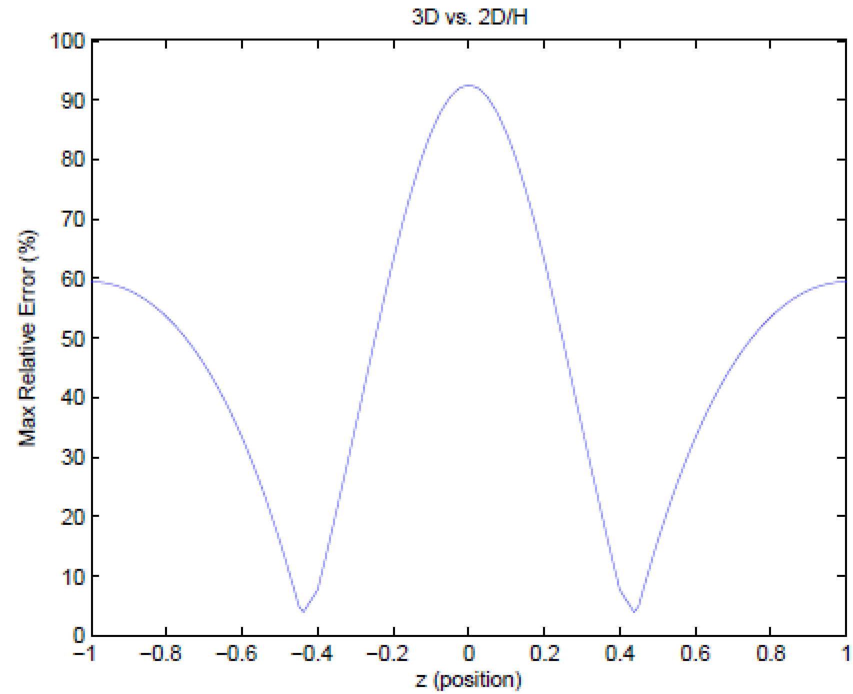
Measurements (Difference Voltage of Electrodes) – Opposite Pattern - Only first 5 terms

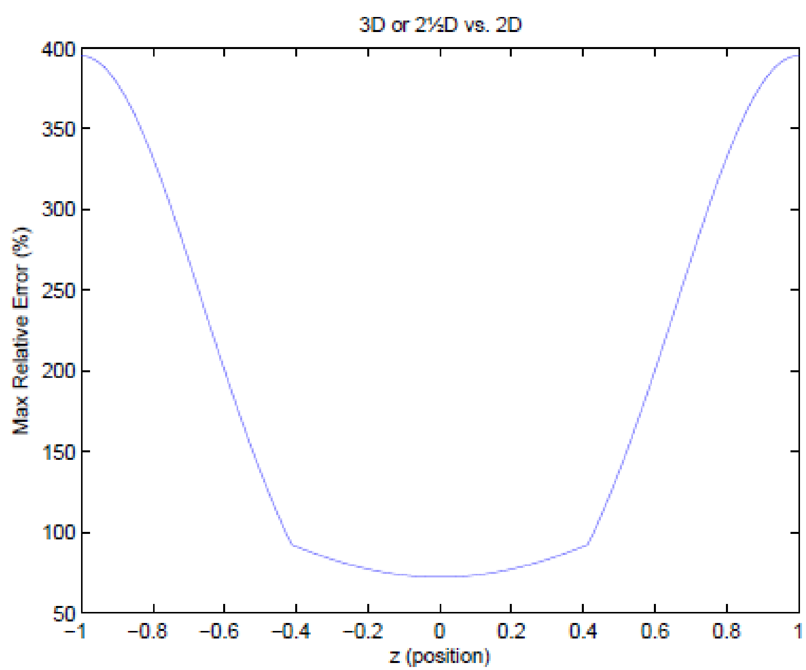
Maximum error: 0.82% (0.002)



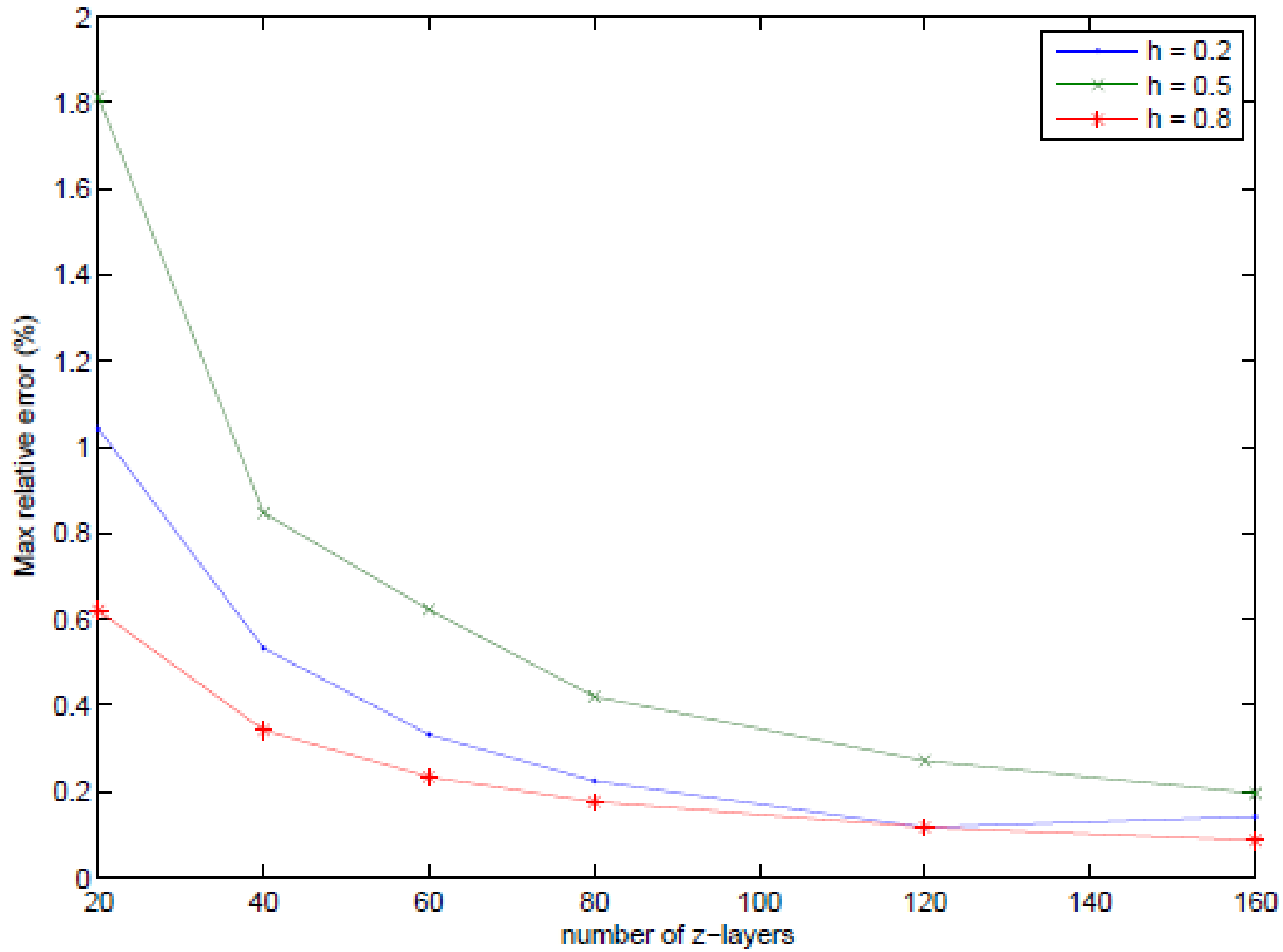
Comparing 3D, 2D, 2D/H (first term of 2½D) and 2½D CEM solutions for electrode voltages - CEM (W = 0.1, H = 2,h = 0.4)


3D vs. 2½D
Max Relative Error (%)
z (position)


3D vs. 2D/H
Max Relative Error (%)
z (position)

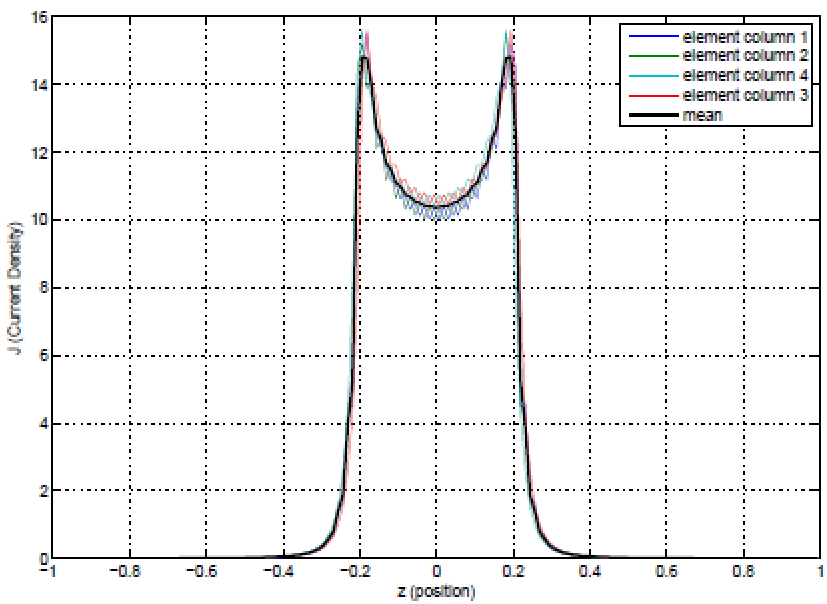

3D or 2½D vs. 2D
Max Relative Error (%)
z (position)

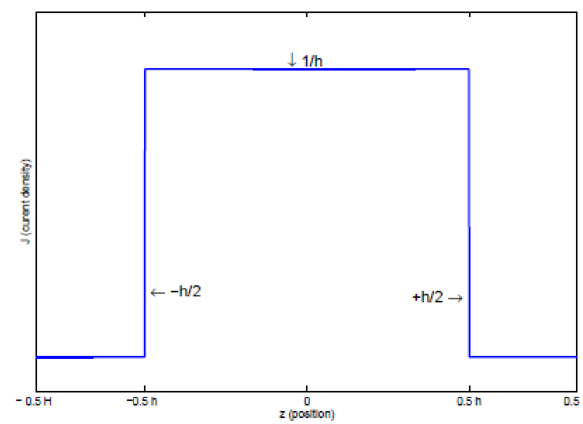
Decrement of the Error by Decrement of the Element size



Sources of Mismatch

- 3D Interpolation Function
- Injected Current Pattern



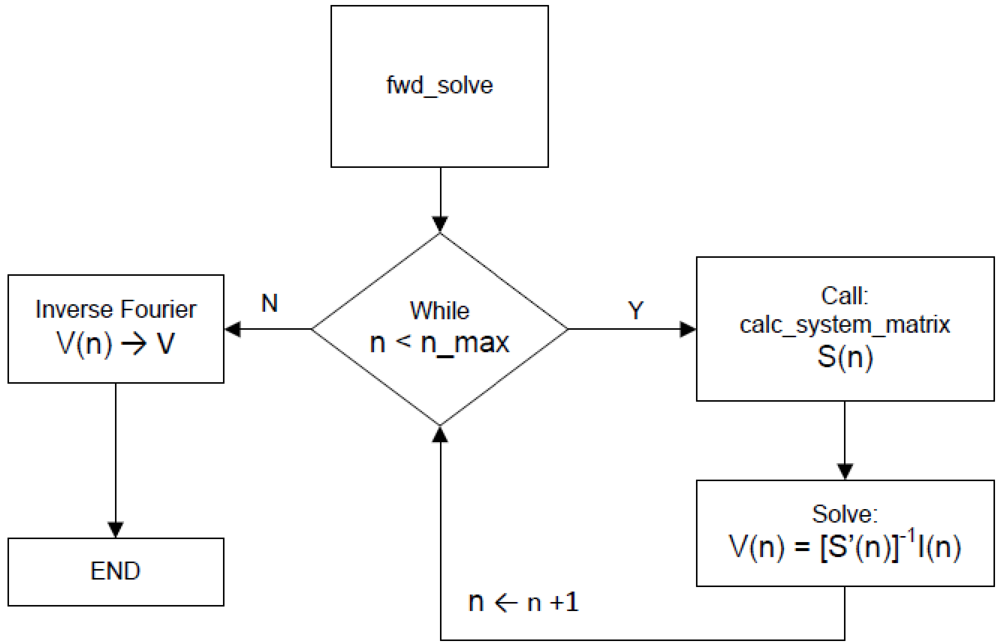


- 2D-based Complete Electrode Modelling

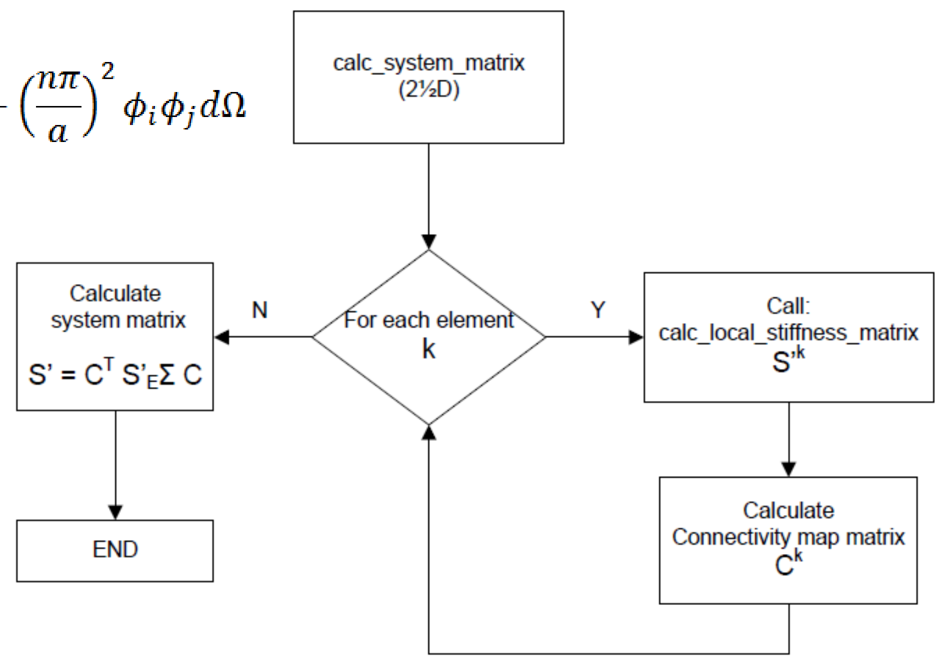
Truncation Point

$$V_{2\frac{1}{2}D} = \frac{1}{H} S^{-1} I + \sum_{n=1}^{\infty} \frac{2}{n\pi h} \sin\left(\frac{n\pi h}{2z_m}\right) \cos\left(\frac{n\pi}{z_m} z\right)\left(S + \left(\frac{n\pi}{z_m}\right)^2 R\right)^{-1} I$$

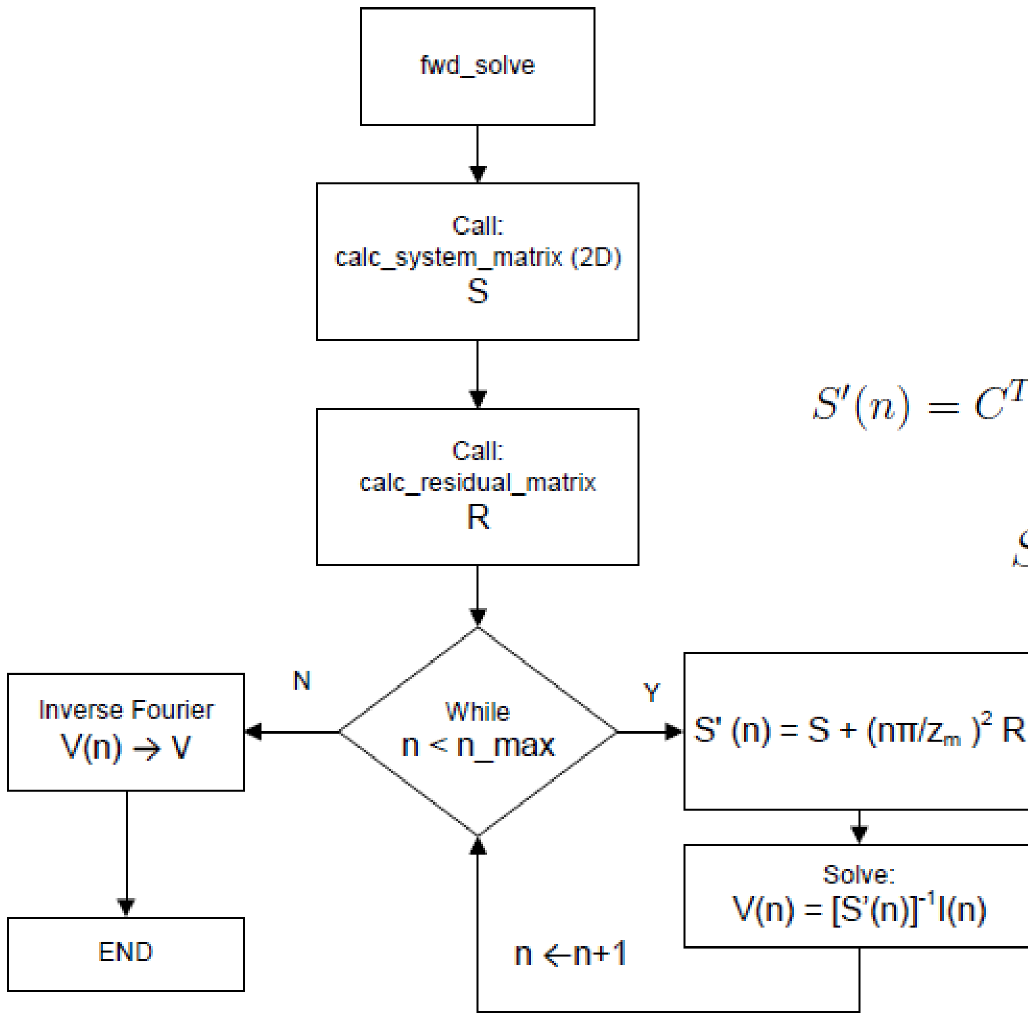
$\frac{H}{r}$	$\frac{h}{H}$	n_max
2	$(0.025,\ 0.05,\ 0.1)$	3
	(> 0.1)	2
4	(0.025)	7
	(0.05)	6
	(0.1)	5
	(≥ 0.2)	3
10	(0.025)	13
	(0.05)	11
	(≥ 0.1)	7



$$S'^{k}_{ij} = S^{k}_{ij} + \left(\frac{n\pi}{a}\right)^2 R^{k}_{ij} = \int_{E_k} \nabla\phi_i \cdot \nabla\phi_j + \left(\frac{n\pi}{a}\right)^2 \phi_i\phi_j d\Omega$$



Speed/Computation Improvement



$$R = C^T \; R_E \Sigma \; C$$

$$R_{ij}^k = \int_{E_k} \phi_i \phi_j \, d\Omega$$

$$S'(n) = C^T \; S_E \Sigma \; C + (\frac{n\pi}{z_m})^2 (C^T \; R_E \Sigma \; C)$$

$$S'(n) = S + (\frac{n\pi}{z_m})^2 R$$

Achievements

$$t_{2\frac{1}{2}D} = t_{2D} + n\_max \times t_{\text{2D-size Forward\_Solve}} + t_R + t_{IFT}$$

Mesh Structure:

$$[2\tfrac{1}{2}D : Mesh.Nodes]_{N\times 2} \quad \text{vs.} \quad [3D : Mesh.Nodes]_{MN\times 3}$$

$$[2\tfrac{1}{2}D : Mesh.Elements]_{K\times 3} \quad \text{vs.} \quad [3D : Mesh.Nodes]_{3(M-1)K\times 4}$$

2D: 2,113 nodes and 4,094 elements
3D: 128,893 nodes and 736,920 elements

if M = 61 slices

System Matrix:

$$[S_{2\frac{1}{2}D}]_{N\times N} \quad \text{vs.} \quad [S_{3D}]_{MN\times MN}$$

M^2 = 61 61 = 3,681

Matrix Inversion at the Forward Problem:

$$[S^{-1}_{2\frac{1}{2}D}]_{N\times N} \quad \text{vs.} \quad [S^{-1}_{3D}]_{MN\times MN}$$

M^2 = 61 61 = 3,681

The EIDORS Project

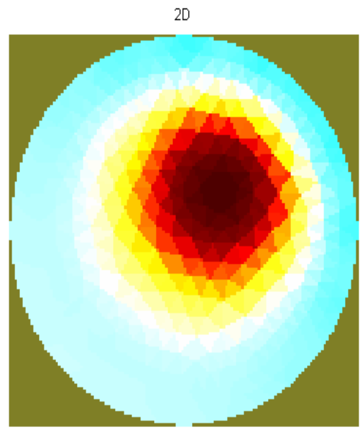
- http://eidors3d.sourceforge.net/
- ***Electrical Impedance and Diffuse Optical Tomography Reconstruction Software***
- A collaborative project where many groups working on EIT are involved around the world
- Modular-Based structure
- Medical & Industrial Applications

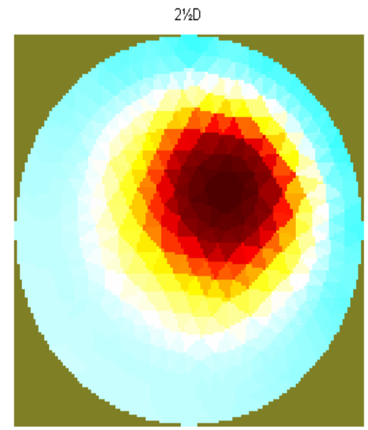
Questions

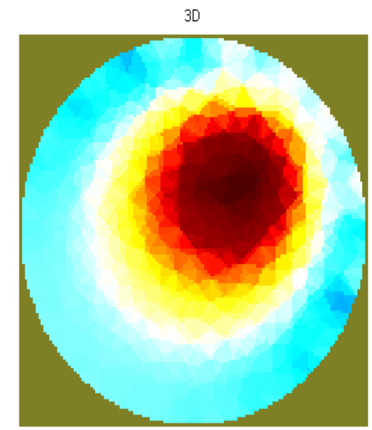
?

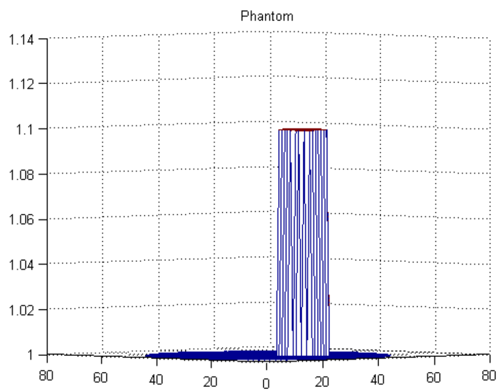
Reference

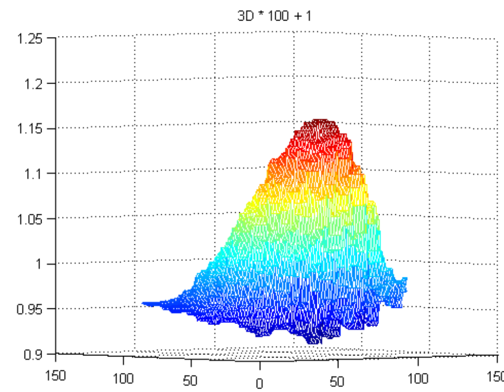
- [0] Ider *et al*, *Electrical impedance tomography of translationally uniform cylindrical objects with general cross-sectional boundaries. IEEE Trans. Med. Imaging 9 49–59*, 1990.
- [1] Lionheart W R B, *Uniqueness, shape and dimension in EIT, Ann. NY Acad. Sci. 873 466–71*, 1999
- [2] K Jerbi, W R B Lionheart, *et al sensitivity matrix and reconstruction algorithm for EIT assuming axial uniformity,* Physiol. Meas. 21 (2000) 61–66
- [3] David Holder, *Electrical impedance tomography: methods, history, and applications*, 2004
- [4] Costa E.L.V., Lima R. Gonzalez, Amato M.B.P., "Electrical Impedance Tomography", *Yearbook of Intensive Care and Emergency Medicine, 2009.*
- *[6] …*

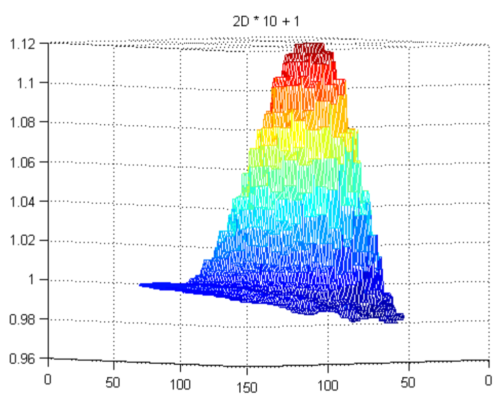

2D

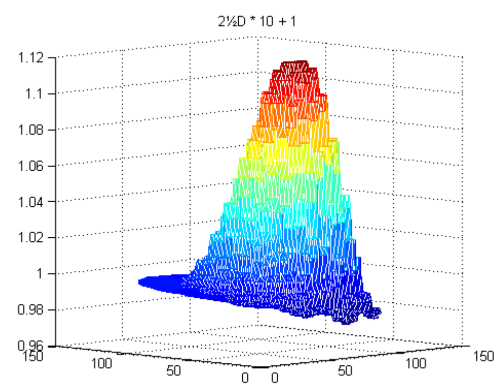

2½D


3D

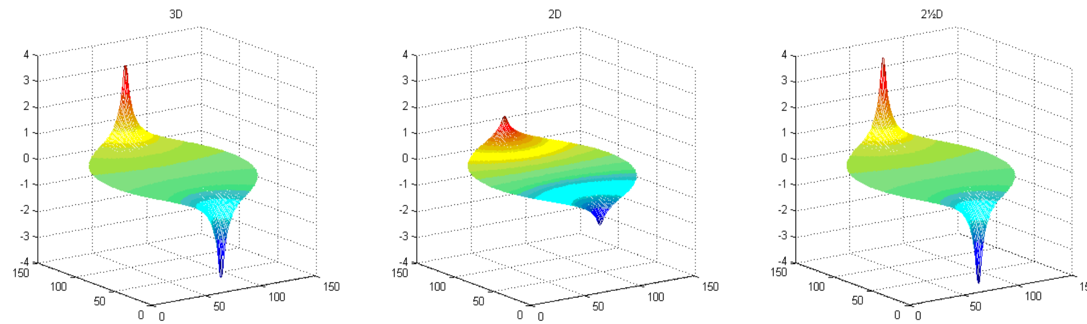

Phantom
1.14
1.12
1.1
1.08
1.06
1.04
1.02
1
80 60 40 20 0 20 40 60 80

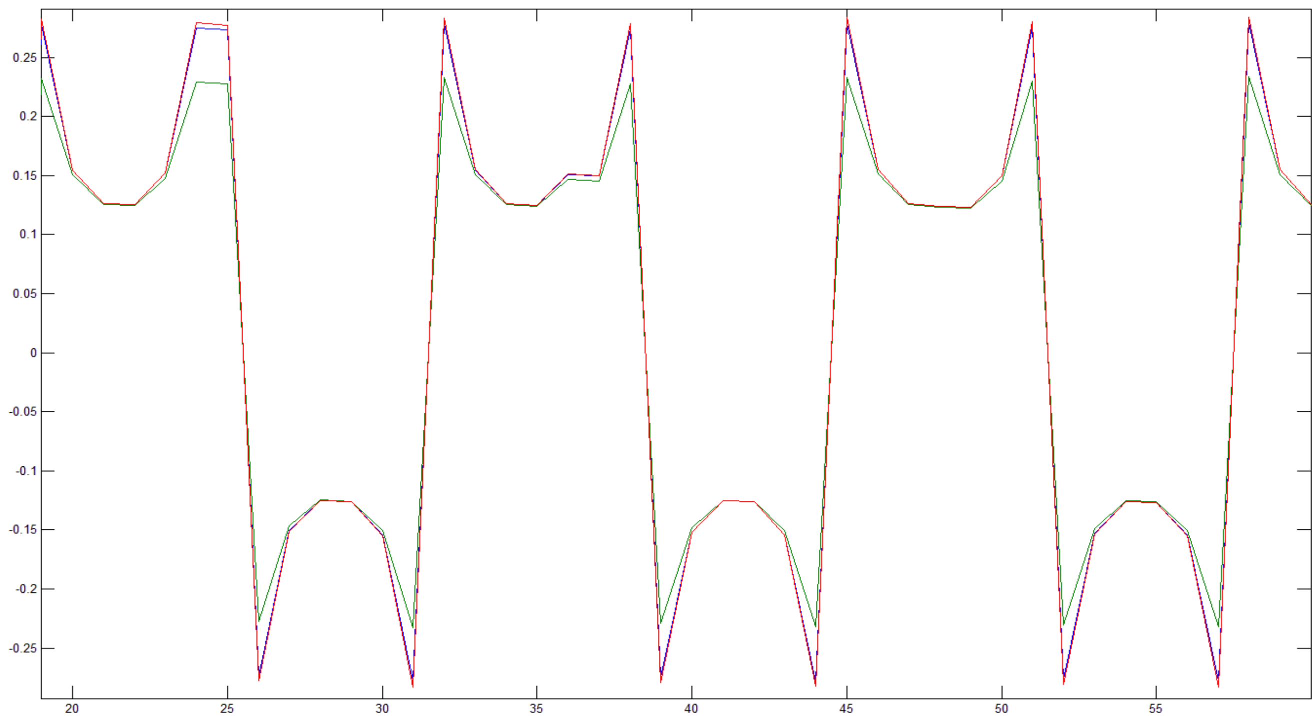

3D * 100 + 1
1.25
1.2
1.15
1.1
1.05
1
0.95
0.9
150 100 50 0 50 100 150

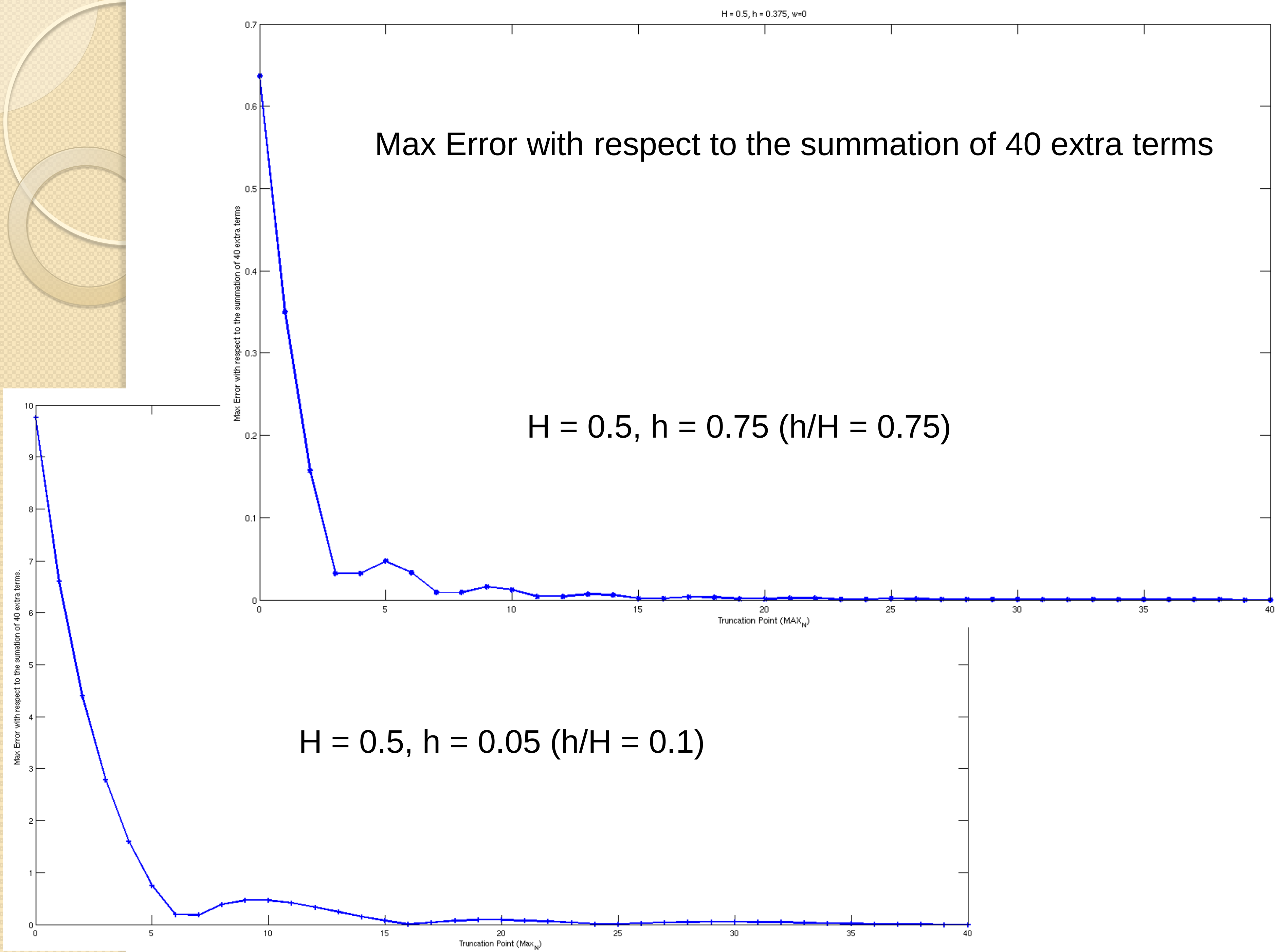

2D * 10 + 1
1.12
1.1
1.08
1.06
1.04
1.02
1
0.98
0.96
0 50 100 150 100 50 0


2½D * 10 + 1
1.12
1.1
1.08
1.06
1.04
1.02
1
0.98
0.96
150 100 50 0 0 50 100 150

Results of the Forward Model



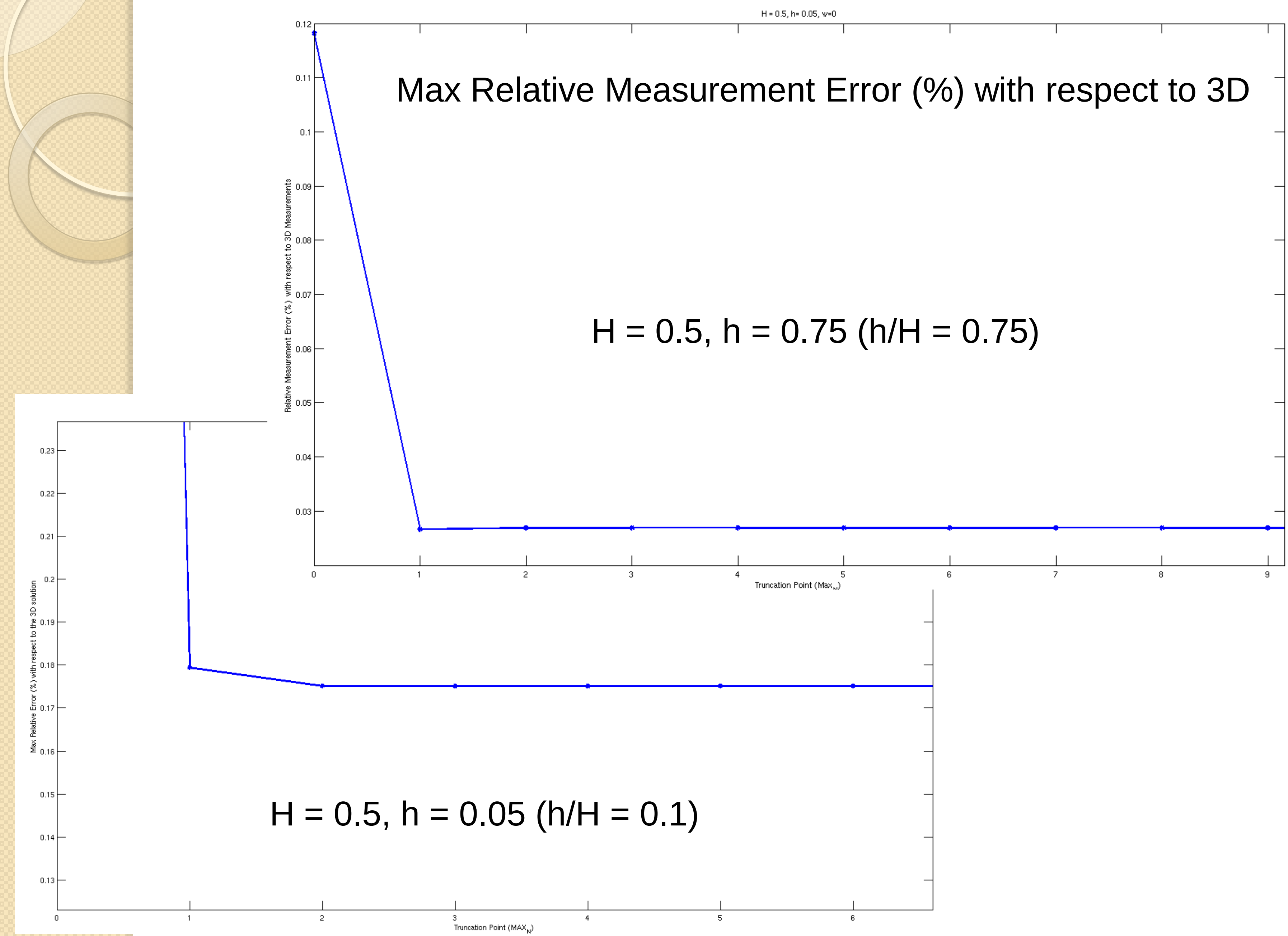




Max Error with respect to the summation of 40 extra terms

H = 0.5, h = 0.75 (h/H = 0.75)

H = 0.5, h = 0.05 (h/H = 0.1)



Max Relative Measurement Error (%) with respect to 3D
H = 0.5, h = 0.75 (h/H = 0.75)
H = 0.5, h = 0.05 (h/H = 0.1)
H = 0.5, h= 0.05, w=0
Relative Measurement Error (%) with respect to 3D Measurements
Truncation Point (Max$_N$)
Max Relative Error (%) with respect to the 3D solution
Truncation Point (MAX$_N$)

Decrement of the Error by Decrement of the Element size