Statistics of Normalized Data In Electrical Impedance Tomography

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Outline of the Presentation

- Formulation of the Problem.
- Motivation of the Study (2 slides).
  - The impact of the normalization (**dividing by reference measurement**) on the covariance matrix of the measured data.
  - **Presence** of spikes in the time series **after** normalization.
- Computing **estimates** of the **1st and 2nd order moments** of the **normalized data** by **Taylor** series expansion.
- Numerical Results and Discussions.
- Conclusion

- Underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- Ω denotes the sample space, $\mathcal{F}$ is a sigma field on Ω, and $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ is the probability measure.
- **Raw** Measurements represent a discrete time stochastic process $\mathbf{v} : \Omega \times \mathbb{N}^+ \rightarrow \mathbb{R}^{\mathrm{M}}$.
- Normalization

$$\delta v_{\mathrm{i}}[\mathrm{k}] = \frac{v_{\mathrm{i}}[\mathrm{k}]}{v_{\mathrm{i}}[\mathrm{k}_0]} - 1; \qquad 1 \leq \mathrm{i} \leq \mathrm{M} \tag{1}$$

- As a result of the division, the statistics of **Normalized Measurements differs** to that of the **Raw Measurements**.

- The **covariance matrix** of normalized **data** $\mathrm{C}_{\delta \mathrm{v}}[\mathrm{k}]$ features **explicitly** in the expressions for image reconstruction.
- One Step Gauss-Newton Reconstructor;

$$\delta\hat{\boldsymbol{\sigma}} = (\mathrm{J}^{\mathrm{T}}\mathrm{C}_{\delta \mathrm{v}}^{-1}\mathrm{J} + \mathrm{C}_{\delta\boldsymbol{\sigma}}^{-1})^{-1}\mathrm{J}^{\mathrm{T}}\mathrm{C}_{\delta \mathrm{v}}^{-1}\delta\mathbf{V} \quad (2)$$

- Bayesian Filtering (Kalman, Extended Kalman,...).
- Cramér Rao Bound Analysis.
- Spikes: The 1st and 2nd order **moments do NOT exist**, (i.e $\mathbb{E}(\delta v_i) = \pm\infty$, and $\mathrm{VAR}(\delta v_i) = \infty$).

Motivation of the Study (2/2)

If $(\mathbb{E}(\delta v_i) < \infty) \wedge (\mathrm{VAR}(\delta v_i) < \infty)$, then the **objective** is to compute $\mathrm{C}_{\delta \mathrm{v}}[\mathrm{k}]$ as defined below:

$$[\mathrm{C}_{\delta \mathrm{v}}[\mathrm{k}]]_{\mathrm{i,j}} = \mathbb{E}\left[\left(\frac{v_{\mathrm{i}}[k]}{v_{\mathrm{i}}[k_0]}\frac{v_{\mathrm{j}}[k]}{v_{\mathrm{j}}[k_0]}\right)\right] - \mathbb{E}\left[\frac{v_{\mathrm{i}}[k]}{v_{\mathrm{i}}[k_0]}\right]\mathbb{E}\left[\frac{v_{\mathrm{j}}[k]}{v_{\mathrm{j}}[k_0]}\right] \qquad (3)$$

- we assume that we know the statistics of our discretely sampled **RAW** data i.e (i) knowledge of the noise level in the hardware, (ii) problem under study.
- If we do not know how to measure and how to characterize the statistics of our **raw** signals, then we should attempt to solve that problem **as well**.

- Recalling, that random variables are **measurable** functions $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
- Consider a differentiable function $\mathrm{F} : \mathbb{R}^2 \to \mathbb{R}$, and the random variable pair (Z, W). The function is defined below by

$$\mathrm{F}(Z, W) := \frac{Z}{W}$$

- $Z = \mu_z + dZ$, and $W = \mu_w + dW$. Where, $dZ = Z - \mu_z$, $dW = W - \mu_w$.

$$\begin{aligned}\mathrm{F}(Z, W) = \quad & \mathrm{F}(\mu_z, \mu_w) + \mathrm{F_z}(\mu_z, \mu_w)dZ + \mathrm{F_w}(\mu_z, \mu_w)dW \\ & + \mathrm{F_{z,w}}dZ.dW + \mathrm{F_{w,w}}(\mu_z, \mu_w)\frac{(dW)^2}{2!} + \mathrm{H.O.T}\end{aligned}$$

Statistics of Normalized Data (1/3)

- After algebra and the expectation operation

$$\mathbb{E}(\delta v_i) = \mathbb{E}\left[\frac{Z}{W}\right] - 1 \simeq \frac{\mu_z}{\mu_w} - \frac{\sigma_z \sigma_w \rho_{z,w}}{\mu_w^2} + \frac{\mu_z}{\mu_w^3}\sigma_w^2 - 1 \qquad (4)$$

- It is worth recalling $\mathrm{VAR}(\delta v_i) = \mathbb{E}[(\delta v_i - \mathbb{E}[\delta v_i])^2]$

$$\mathbb{E}[(\delta v_i - \mathbb{E}[\delta v_i])^2] = \mathbb{E}[(\delta v_i)^2] - (\mathbb{E}[(\delta v_i)])^2$$

$$\mathrm{VAR}(\delta v_i) = \mathrm{VAR}\left(\frac{Z}{W} - 1\right) \simeq \frac{\sigma_z^2}{\mu_w^2} - 2\left(\frac{\mu_z \sigma_z \sigma_w \rho_{z,w}}{\mu_w^3}\right) + \frac{\mu_z^2 \sigma_w^2}{\mu_w^4} \quad (5)$$

If $(\mathbb{E}(\delta v_i) < \infty) \wedge (\mathrm{VAR}(\delta v_i) < \infty)$, then recall:

$$[\mathrm{C}_{\delta \mathrm{v}}[\mathrm{k}]]_{\mathrm{i,j}} = \mathbb{E}\left[\left(\frac{v_{\mathrm{i}}[k]}{v_{\mathrm{i}}[k_0]}\frac{v_{\mathrm{j}}[k]}{v_{\mathrm{j}}[k_0]}\right)\right] - \mathbb{E}\left[\frac{v_{\mathrm{i}}[k]}{v_{\mathrm{i}}[k_0]}\right]\mathbb{E}\left[\frac{v_{\mathrm{j}}[k]}{v_{\mathrm{j}}[k_0]}\right] \quad (6)$$

- We have the following product pair $(v_i[k]v_j[k], v_i[k_0]v_j[k_0])$. We must, first compute the mean, variance and the correlation coefficient of this product pair.

Let $X_1 = v_i[k], X_2 = v_j[k]$, $X_3 = v_i[k_0]$, $X_4 = v_j[k_0]$, $Z = X_1 X_2$, and $W = X_3 X_4$.

The expression for $\mathrm{Var}(Z)$, is given below by

$$\sigma_z^2 = (\mu_1\sigma_2)^2 + 2\mu_1\mu_2\sigma_{1,2} + (\mu_2\sigma_1)^2 + (\sigma_1\sigma_2)^2 + \sigma_{1,2}^2 \qquad (7)$$

By inspection, $\mathrm{Var}(W)$ reads as

$$\sigma_w^2 = (\mu_3\sigma_4)^2 + 2\mu_3\mu_4\sigma_{3,4} + (\mu_4\sigma_3)^2 + (\sigma_3\sigma_4)^2 + \sigma_{3,4}^2 \qquad (8)$$

The expression for $\mathrm{Cov}(Z, W)$, is given below by

$$\begin{aligned} \sigma_{z,w} = \quad & \mu_1\mu_3\sigma_{2,4} + \mu_1\mu_4\sigma_{2,3} + \mu_2\mu_3\sigma_{1,4} \\ & + \mu_2\mu_4\sigma_{1,3} + \sigma_{1,3}\sigma_{2,4} + \sigma_{1,4}\sigma_{2,3} \end{aligned} \qquad (9)$$

Results and Discussions

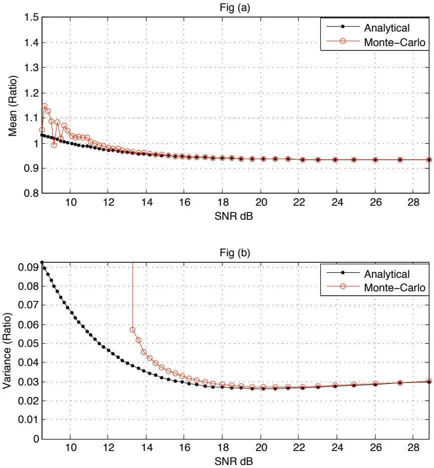

Figure: This figure shows the first and second order moments of the ratio Z/W as a function of $20 \log \frac{\mu_{\boldsymbol{w}}}{\sigma_{\boldsymbol{w}}}$ on the x-axis.

Conclusions

- We derived closed form expressions for computing the first and second order statistics of the normalized data, and validated it using a simple Monte-Carlo procedure.
- These expressions are accurate provided that the signal to noise ratio of the reference measurement is above 16 dB.
- In order to guarantee accuracy in the reconstructed images, the minimum SNR of the reference measurement should be above 13 dB($|\mu_w| > 4\sigma_w$.).