Reconstruction of conductivities with jump changes

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Overview

- Methods for sharp changes
- Linear sampling/factorization
- Levelset
- Monotonicity
- Total variation
- Primal-Dual Interior Point method
- Implementation in EIDORS and results

Methods for Sharp changes

Tradition reconstruction methods for EIT include a regularization term that penalizes sharp changes. Put a different way they assume a prior distribution where smooth conductivity images are more likely. By contrast CT and MRI images show that material properties in the human body tend to have sharp changes between different anatomical structures.

Can EIT see sharp changes? Yes, in some circumstances sharp changes are easier to detect with EIT than smooth changes. However illconditioning and error in data mean that one might not be able to distinguish between sharp and smooth.

There are numerous approaches to recovering discontinuous conductivites in EIT, and we believe they merit the attention of the medical EIT community so will review some of them here. We go on to discuss the Primal-Dual Interior Point method applied to Total Variation (TV) regularization, which we believe is particularly important in medical EIT where we encounter both sharp changes and smooth gradations of conductivity.

Methods for finding jump discontinuities

While Total Variation is good for multi-component mixtures, and sharp changes while also allowing for gradients it is computationally expensive. Where only the boundary between regions of constant conductivity (two or more "phases") is needed there are several methods to detect the discontinuity.

- **Monotonicity** method of Tamburrino and Rubinacci. Used for two component mixtures where properties of components known. Requires measurements at driven electrodes not taken by all EIT. Fast and direct requiring only eigenvalues of trans-impedance matrices.
- **Shape reconstruction.** When the boundaries between phases are known to be smooth shape based methods such as level sets are useful. An iterative method but typically fewer parameters need than "imaging" methods.
- **Linear sampling** and **Factorization** methods are useful for detecting jump changes. Relatively large number of measurements needed and voltage on driven electrodes assumed known. Fast and direct.

Monotonicity

The monotonicity method of Tamburrino and Rubinacci[33] relies on the observation that if one conductivity is everywhere greater than another $\sigma_1 < \sigma_2$ then the trans-conductance matrices will satisfy $\Lambda_{\sigma_1} < \Lambda_{\sigma_2}$, where the inequality is defined by $\Lambda_{\sigma_2} - \Lambda_{\sigma_1}$ being positive definite.

The contra-positive of this is that a non-positive eigenvalue of $\Lambda_{\sigma_2} - \Lambda_{\sigma_1}$ means that $\sigma_1 \not< \sigma_2$.

This test is especially useful when the conductivity is assumed to take one of two values $\sigma_m < \sigma_M$ on each pixel (or voxel). One precomputes a test trans-conductance Λ_{Mk} for a conductivity of σ_M on the kth pixel and σ_m elsewhere. If the measured trans-conductance Λ_σ is such that $\Lambda_\sigma - \Lambda_{Mk}$ has a negative eigenvalue then we know the kth pixel cannot have conductivity σ_M so must be σ_m. Unfortunately the difference being positive definite does not imply the pixel has conductivity σ_M, in that case the test is inconclusive.

The procedure is repeated using test trans-conductance matrices Λ_{mk} for a conductivity of σ_m on the kth pixel and σ_M elsewhere. This results in two disjoint subsets of pixels Ω_M that definitely have conductivity σ_M and Ω_m that definitely have conductivity σ_m. There may be a set of pixels that are not determined depending on the number and size of electrodes and the number of pixels.

Monotonicity continued

Complexity

For an N electrode and K pixel system the algorithm requires 2K $(N-1) \times (N-1)$ matrices to be tested for a negative eigenvalue. So far implementations have calculated all the eigenvalues each of which is an operation with complexity $O(N-1)^3$. This can be reduced if the aim is only an estimate of the smallest eigenvalue sufficiently accurate to determine if it is negative. The algorithms has the advantage that it can be efficiently parallelized on up to $2K$ processors each with modest memory requirements and without a communications overhead.

Measurement accuracy

With practical measurements one needs to test that the smallest eigenvalue is sufficiently negative to be conclusive. See[33, 34]. An example is given the eigenvalues λ_j of the difference between trans-conductance matrices to consider the sign index $s = \sum_j \lambda_j / \sum_j |\lambda_j|$ and the test is then $s < 1 - \epsilon$ for some small ϵ. Of course we will still have more pixels unknown as accuracy decreases.

Complex admittivity

The method is known to work for purely real or purely imaginary admittivity but has so far been extended to the complex admittivity or multifrequency data.

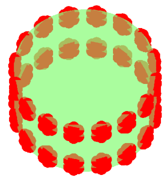
MCMC Recent work uses Markov-Chain Monte Carlo method to resolve the conductivity of the unknown pixels [3].

Shape Reconstruction

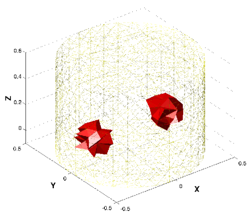
The shape reconstruction approach parameterizes the curves (or in 3D surfaces) of discontinuity in a piece-wise constant conductivity.
Typically parametrization of smooth curves (surfaces) requires fewer parameters, consequently the non-linear fitting problem is less costly computationally and can be better conditioned. Two approaches have emerged to parametrization these internal boundaries. The level set approach, first suggested in EIT by Santosa [28], and recently developed by Dorn et al[16][31]. Another approach taken by Kolehmainen et al parameterizes internal boundaries using smooth functions such as Fourier series [21][22].

Level set reconstruction

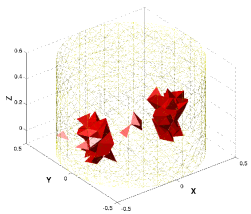
3D Level set reconstruction, simulated data from Soleimani Lionheart and Dorn[31]



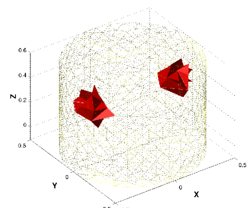
Electrodes for 3D simulation



True image



level set solution



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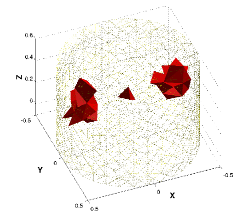

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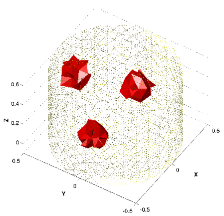

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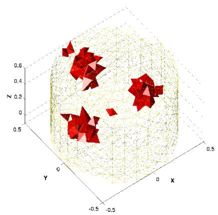
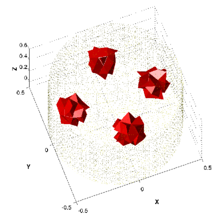


level set solution



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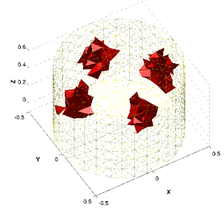
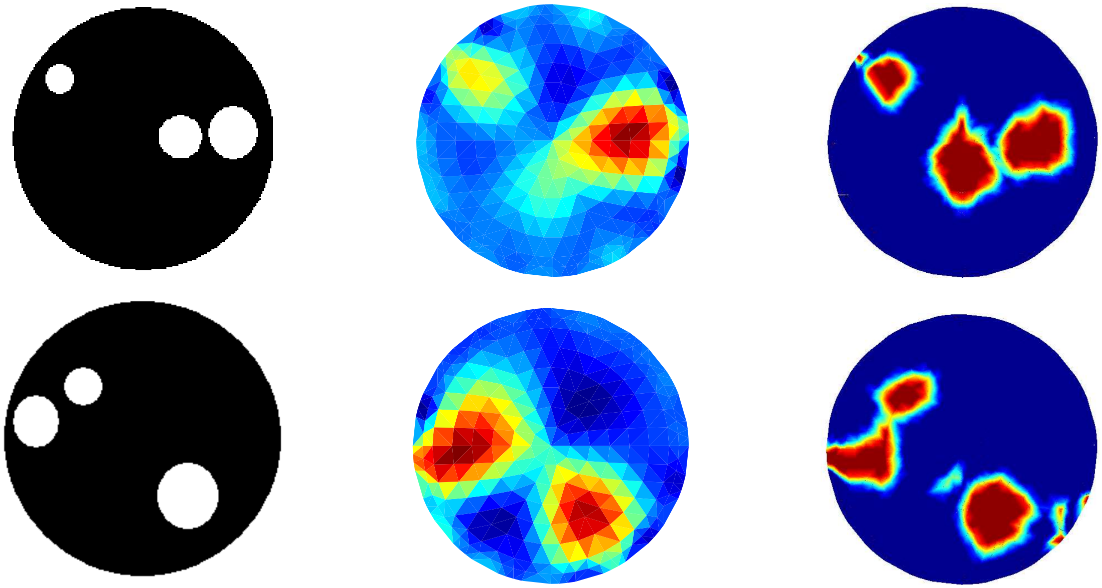


level set solution

Level set 2D tank data

2D Level set reconstruction compared with Tikhonov Regularized Gauss-Newton from Soleimani Lionheart and Dorn[31]



True | Gen Tik Gauss-Newton | Level set

Linear Sampling or Factorization

“Linear Sampling" or “Factorization” methods were developed originally for inverse scattering problems. They provide a direct method of finding the jump discontinuities of in a piecewise constant conductivity from boundary measurements and have been applied to EIT by Hanke and Bruhl [9, 17]

In common with the monotonicity points in the domain are tested to see if they are inside “inclusions”, and each test involves calculation of eigenvalues of an operator on the boundary. More specifically for each point z in the domain h_z is the potential on the boundary due to a dipole source at z with zero Neumann condition. In the factorization method one tests if h_z at the boundary is in the range of the operator $(R - R_1)^{1/2}$ where R is the trans-resistance measured at the boundary (Neumann-Dirichlet map) and R_1 is the trans-resistance for the homogeneous case. It is in the range if and only if z is in the inclusion.

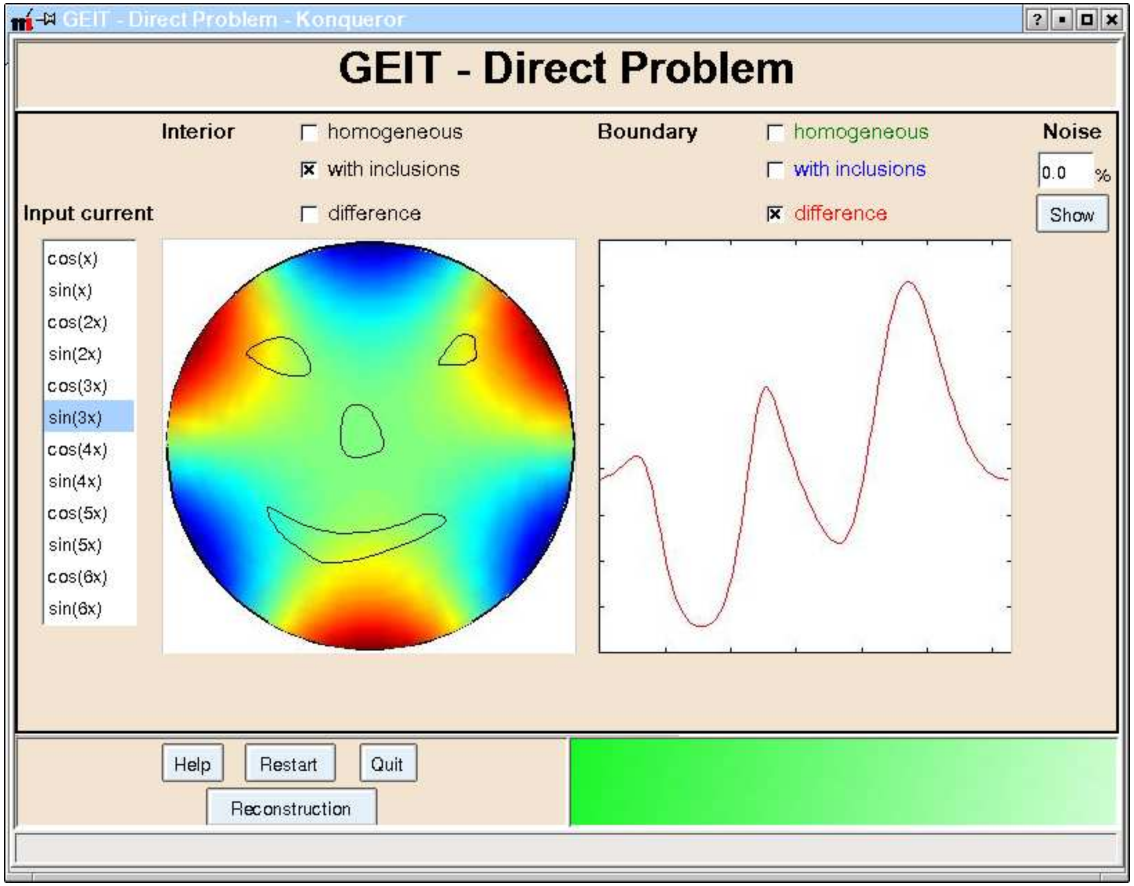
The test to see if h_z is in the range is essentially the *discrete Picard criterion*. Let v_k be the eigenfunctions of $R - R_1$ and λ_k the eigenvalues then h is in the range if and only if

$$\sum_{k=1}^{\infty} \lambda_k^{-1} \langle h, v_k \rangle < \infty$$

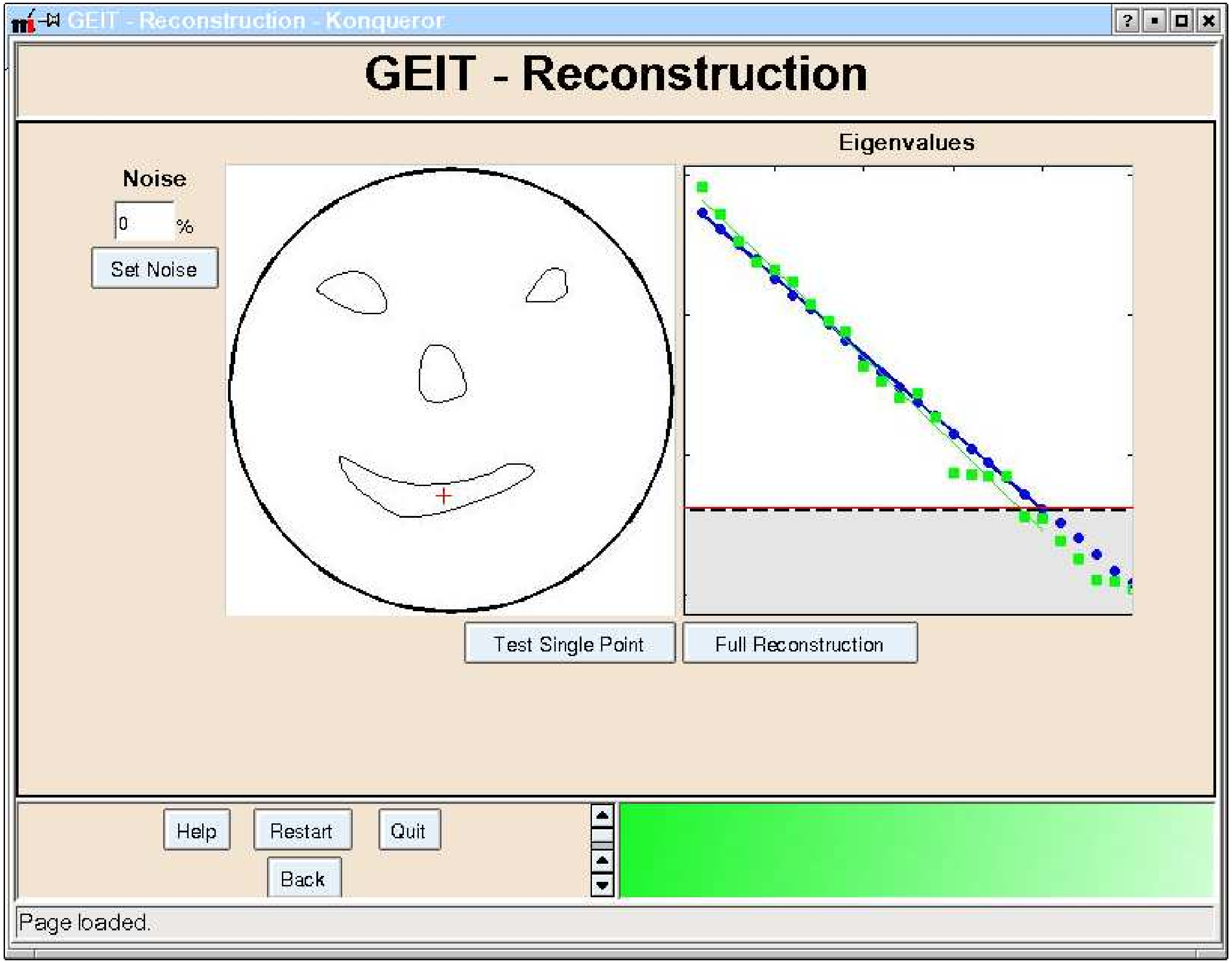
“Linear sampling” means the same test without the power $1/2$ of the operator, in this

Factorization:Numerical studies

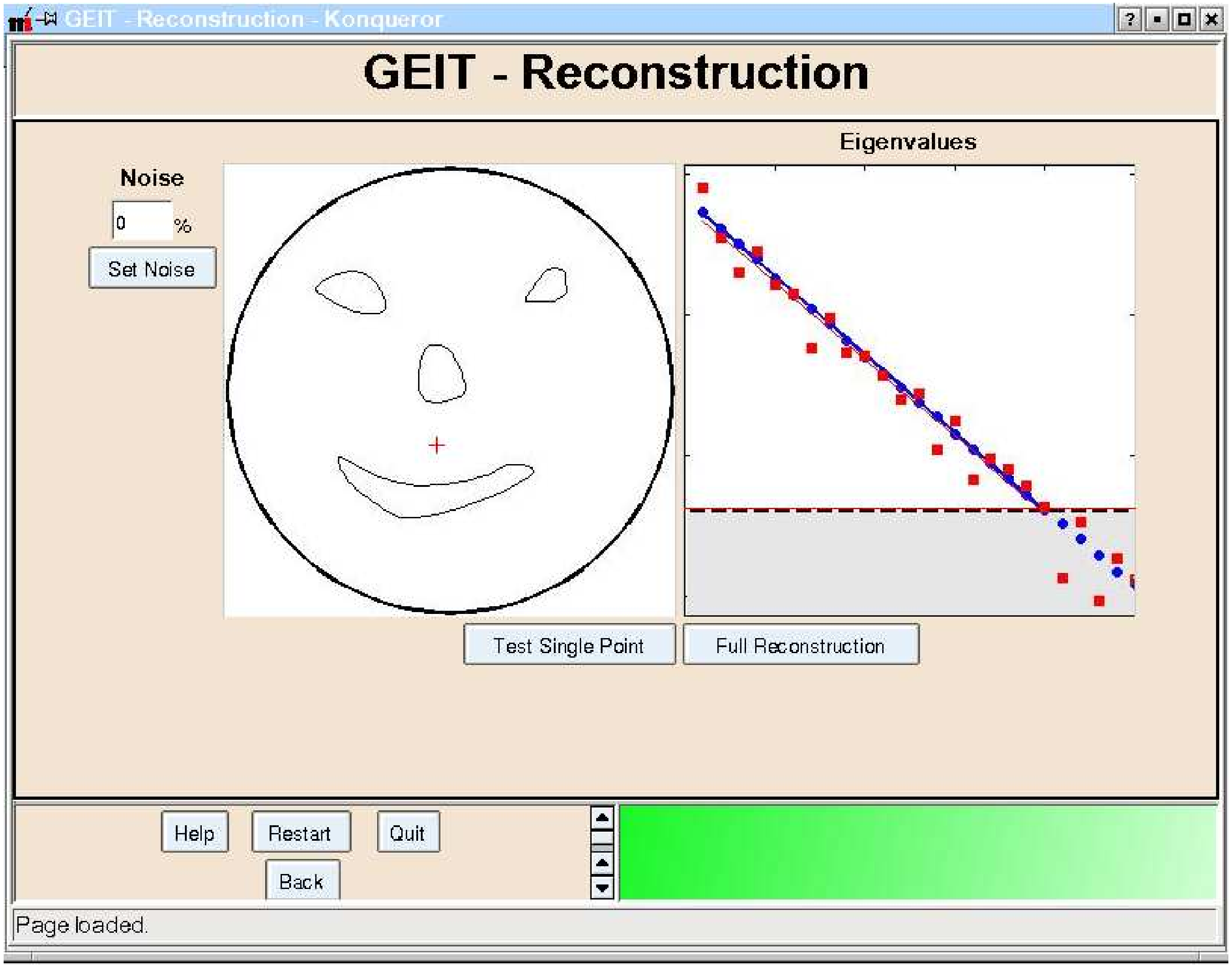
Bruhl and Hanke[9] tested factorization method on 2D data from RPI's ACT3 system, Schappel[30] also reported tests on half-plane experiemntal data. The reconstruction code of Bruhl is available as an on-line Java (adapted for the web by B. Gebauer) as GEIT [18]. Here are some sample results.



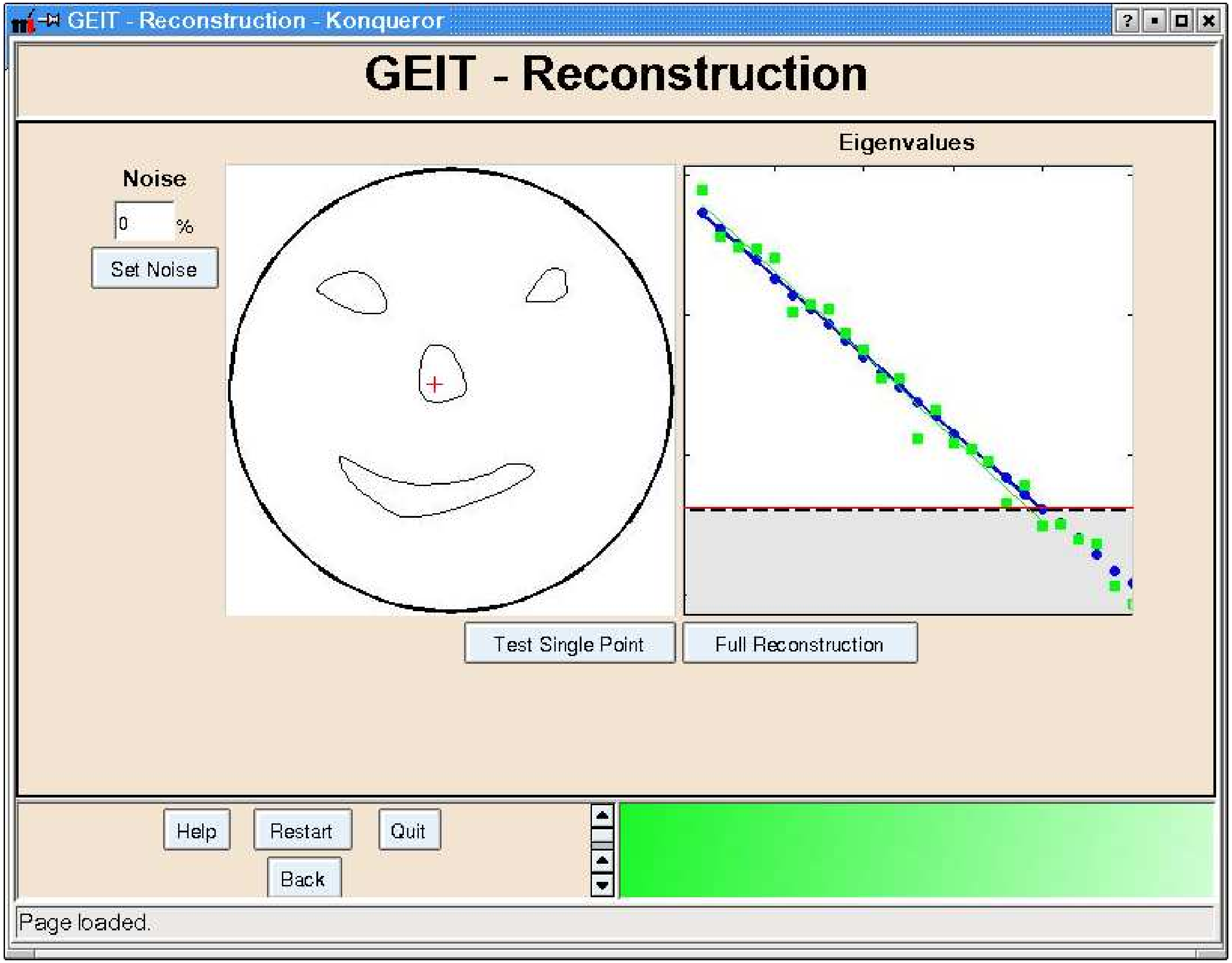
Factorization:GEIT numerical studies cont



Factorization:GEIT numerical studies cont

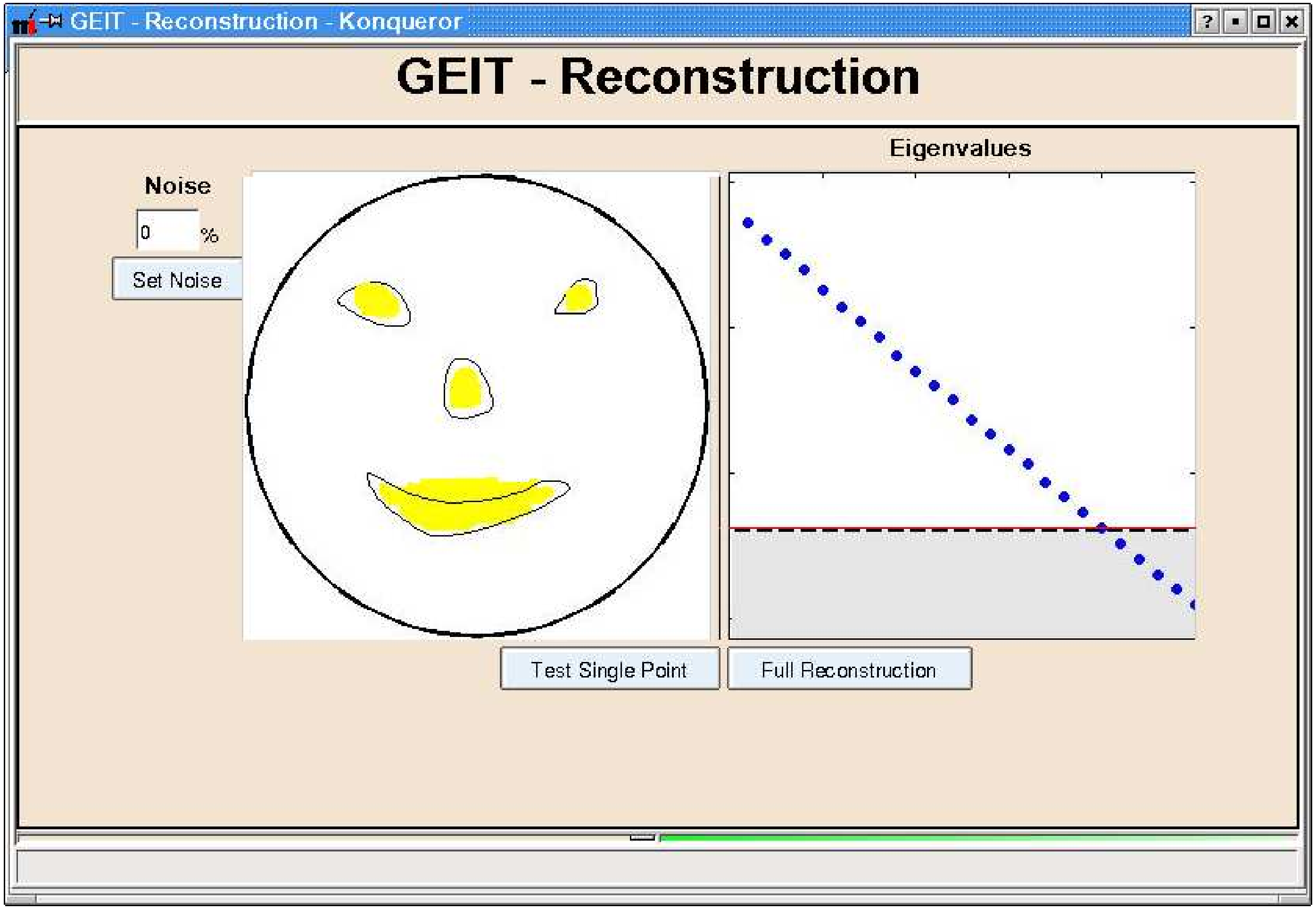


Factorization:GEIT numerical studies cont



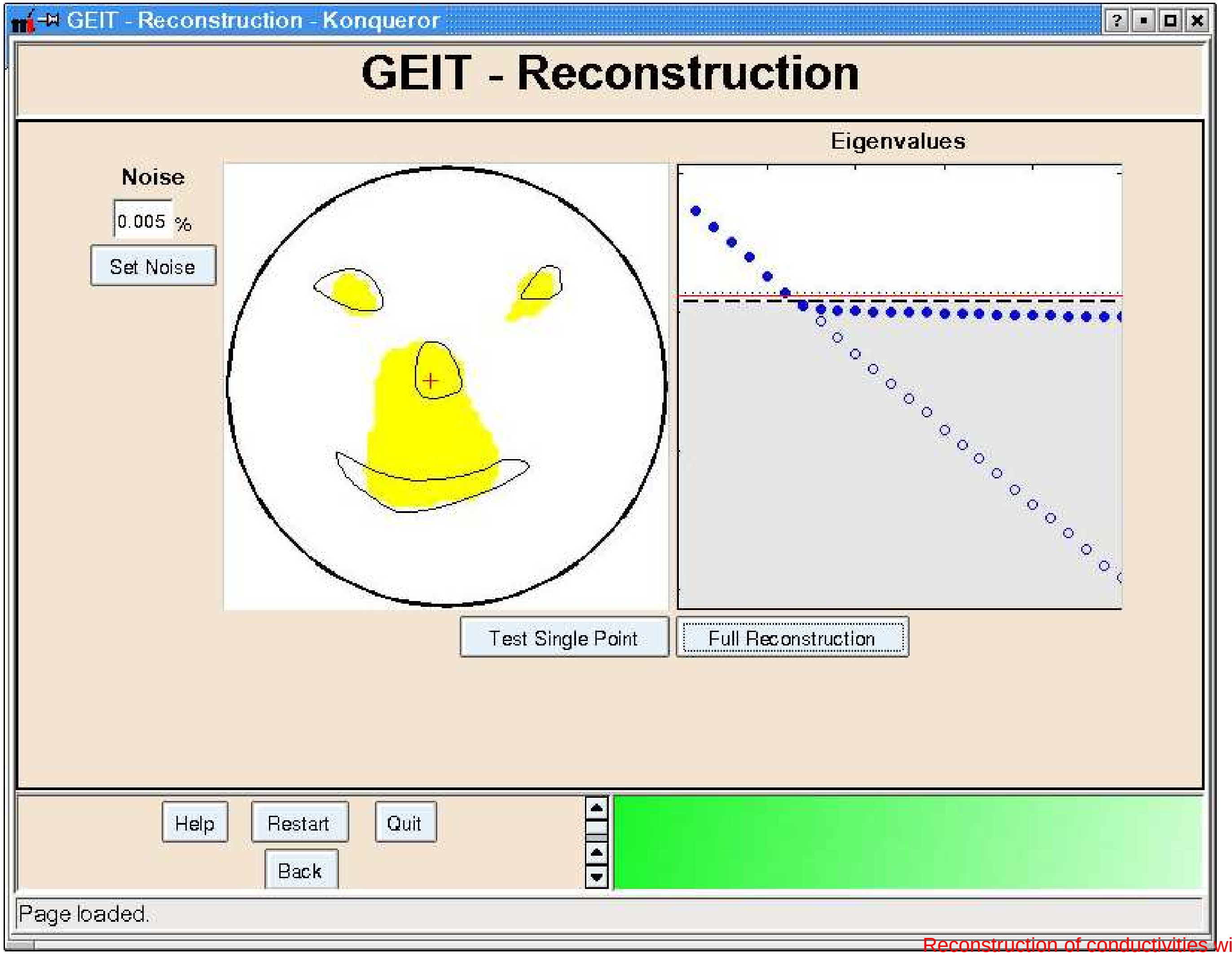
Factorization:GEIT numerical studies cont

Full reconstruction with no noise



Factorization:GEIT numerical studies cont

Full reconstruction with noise



Factorization method: comments

- Factorization method should work as well in 3D, a student of Hanke is working on it for experimental data.
- The method appears to need the diagonal of R, that is voltages on current carying electrdodes. It can still work with other data missing.
- Like monotonicity it is a fast direct method, full 3D data cost of calculating eigensystem increases but this can be alleviated by using multiple processors.
- Systematic approaches to regularization need to be considered.

TV – the idea

Let $F(\sigma) = V$ be the forward problem the a typical regularization method is to solve

$$\min_{\sigma} \left(\|V - F(\sigma)\|^2 + G(\sigma) \right)$$

for a penalty function G. In generalized Tikhonov regularization $G(\sigma) = \alpha^2 \|L(\sigma - \sigma_0)\|^2$ for a differential operator L. The penalty term is smooth so standard (eg Gauss-Newton) optimization will work fine. This regularization incorporates the a priori information that the conductivity is smooth.
The Total Variation functional $G(\sigma) = \alpha \|\nabla(\sigma - \sigma_0)\|$ still prevents wild fluctuations in σ but allows step changes. The optimization is now of a non-smooth function as G is *not differentiable* at $\sigma = \sigma_0$. An efficient method for solving this is the **Primal Dual Interior Point Method.** This method tracks a solution between a primal and dual problem avoiding the singularity. It is still more computationally costly than Gauss Newton for a smooth penalty.

What TV measures

Total variation measures the total amplitude of the oscillations of a function. For a differentiable function on a domain Ω the total variation is [15]

$$TV(f) = \int_{\Omega} |\nabla f| \tag{1}$$

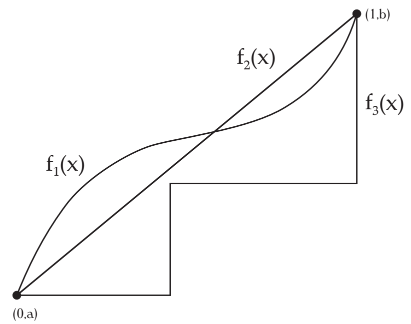
The definition can be extended to non–differentiable functions as:

$$TV(f) = \sup_{\mathbf{v} \in \mathcal{V}} \int_{\Omega} f \ \mathrm{div}\, \mathbf{v} \tag{2}$$

where $\mathcal{V}$ is the space of continuously differentiable vector-valued functions that vanish on $\partial\Omega$ and $\|\mathbf{v}\|_{\Omega} \leq 1$.
As the TV functional measures the variations of a function over its domain, it can be understood to be effective at reducing oscillations in the inverted profile, if used as a penalty term. The same properties apply however to ℓ_2 regularisation functionals. The important difference is that the class of functions with bounded total variation also includes discontinuous functions, which makes the TV particularly attractive for the regularisation of non–smooth profiles.

Simple TV example

The following one-dimensional example illustrates the advantage of using the TV against a quadratic functional in non-smooth contexts. Let $F = \{f : [0,1] \to \mathbb{R}, | \, f(0) = a, f(1) = b\}$, we have $\min\limits_{f \in F} \int_0^1 |f'(x)| \mathrm{d}x$ is achieved by any monotonic function, including discontinuous ones. $\min\limits_{f \in F} \int_0^1 (f'(x))^2 \mathrm{d}x$ is achieved only by the straight line connecting the points $(0, a)$ $(1, b)$.



The figure shows three possible functions f_1, f_2, f_3 in F. All of them have the same total variation, including f_3 which is discontinuous. Only f_2 however minimises the H^1 semi–norm $|f|_{H_1} = \left(\int_0^1 (f'(x))^2 \, \mathrm{d}x \right)^{1/2}$. The quadratic functional, if used as penalty, would therefore bias the inversion toward the linear solution and the function f_3 would not be admitted in the solution set as its H^1 semi-norm is infinite.

TV details cont

- Dobson and Santosa [**?**] suitable for the linearised problem and suffers from poor numerical efficiency.
- Somersalo *et. al.* [32] and Kolehmainen *et. al.* [23]. MCMC methods to solve the TV regularised inverse problem. Do not suffer from problems resulting from non-differentiability of the TV functional.
- Borsic [6, 5] applied Primal Dual Interior Point Method (PD-IPM) to TV regularized 2D EIT. Generalized to 3D in[24], and now partially incorporated in EIDORS.

Primal and Dual problems

In inverse problems, with linear forward operators, the discretised TV regularised inverse problem, can be formulated as

$$(P) \quad \min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 + \alpha\|\mathbf{L}\,\mathbf{x}\| \tag{3}$$

where $\mathbf{L}$ is a discretization of the gradient operator. We will label it as the primal problem. A *Dual* problem to (P), which can be shown to be equivalent [5] is

$$(D) \quad \max_{\mathbf{y}:\|\mathbf{y}\|\leq 1} \;\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 + \alpha\,\mathbf{y}^T\mathbf{L}\mathbf{x} \tag{4}$$

PD cont

The optimisation problem

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{b}\|^2 + \alpha\,\mathbf{y}^T\mathbf{L}\mathbf{x} \tag{5}$$

has an optimal point defined by the first order conditions

$$\mathbf{A}^T(\mathbf{A}\mathbf{x}-\mathbf{b}) + \alpha\,\mathbf{L}^T\mathbf{x} = 0 \tag{6}$$

the dual problem can be written therefore as

$$(D) \qquad \max_{\substack{\mathbf{y}\,:\,\|\mathbf{y}\|\leq 1 \\ A^T(A\mathbf{x}-\mathbf{b})+\alpha\,L^T\mathbf{y}=0}} \frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{b}\|^2 + \alpha\,\mathbf{y}^T\mathbf{L}\mathbf{x} \tag{7}$$

PD cont

The complementarity condition for **??** and **??** is set by nulling the primal dual gap

$$\frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{b}\|^2+\alpha\,\|L\,\mathbf{x}\|-\frac{1}{2}\|A\mathbf{x}-\mathbf{b}\|^2-\alpha\mathbf{y}^T\mathbf{L}\mathbf{x}=0 \tag{8}$$

which with the dual feasibility $\|\mathbf{y}\|\leq 1$ is equivalent to requiring that

$$\mathbf{L}\mathbf{x}-\|\mathbf{L}\mathbf{x}\|\mathbf{y}=0$$

PD cont

The PD-IPM framework for the TV regularised inverse problem can thus be written as

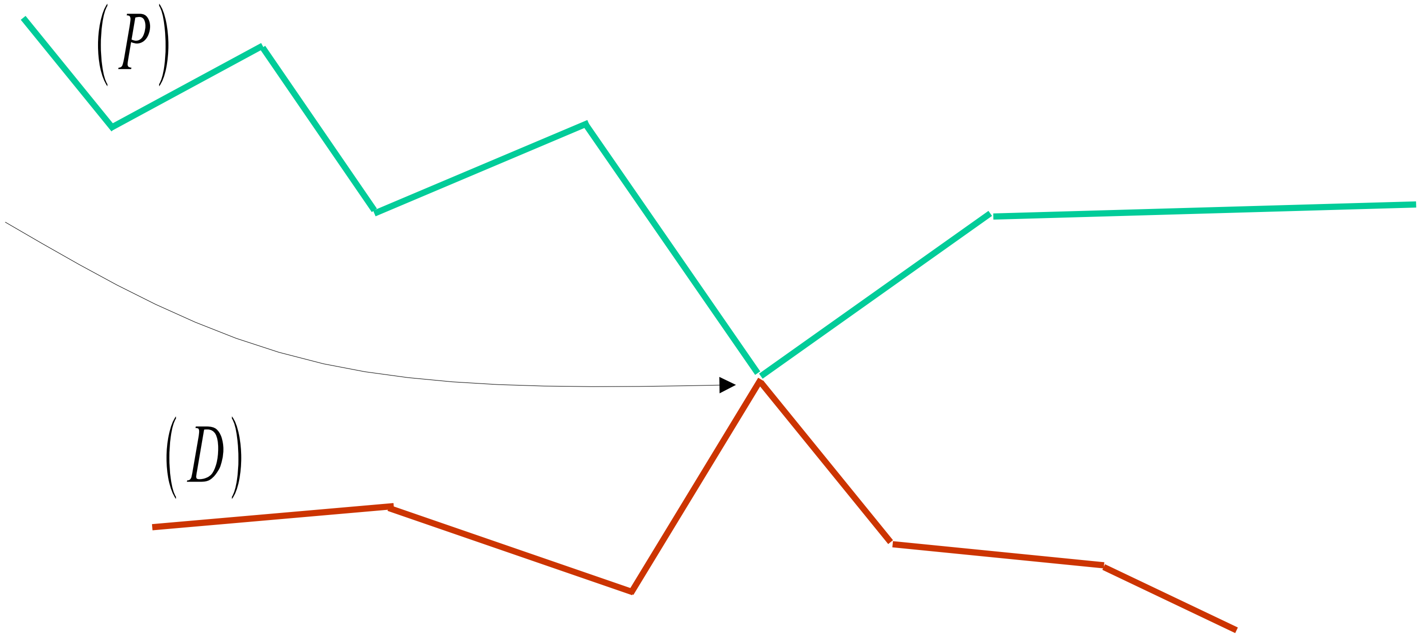
$$\|\mathbf{y}\| \leq 1 \tag{9a}$$

$$\mathbf{A}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) + \alpha\, L^T \mathbf{y} = 0 \tag{9b}$$

$$\mathbf{L}\mathbf{x} - \|\mathbf{L}\mathbf{x}\|\mathbf{y} = 0 \tag{9c}$$

PD cont

It is not possible to apply the Newton Method directly to (**??**) as (**??**) is not differentiable for $\mathbf{Lx} = 0$. A centering condition has to be applied, obtaining a smooth pair of optimisation problems (P$_\beta$) and (D$_\beta$) and a central path parameterised by β. This is done by replacing $\|\mathbf{Lx}\|$ by $(\|\mathbf{Lx}\|^2 + \beta)^{\frac{1}{2}}$ in (**??**).



Application to EIT

The PD-IPM algorithm in its original form [10] was developed for inverse problems with linear forward operators. The following section (based on [5]) describes the numerical implementation for EIT reconstruction. The implementation is based on the results of the duality theory for inverse problems with linear forward operators. Nevertheless it was possible to apply the original algorithm to the EIT inverse problem with minor modifications, and to obtain successful reconstructions. The formulation for the EIT inverse problem is

$$\begin{aligned} \mathbf{s}_{rec} &= \text{arg min}_{\mathbf{s}} f(\mathbf{s}) \\ f(\mathbf{s}) &= \frac{1}{2}\|F(\mathbf{s}) - \mathbf{V}_{\mathsf{m}}\|^2 + \alpha\, TV(\mathbf{s}) \end{aligned} \tag{10}$$

PD cont

The system of non–linear equations that defines the PD–IPM method for (**??**) can be written as

$$\begin{gathered} \|\mathbf{y}\| \leq 1 \\ J^T(F(\mathbf{s}) - \mathbf{V}_{\mathsf{m}}) + \alpha\, L^T \mathbf{y} = 0 \\ L\mathbf{s} - E\mathbf{y} = 0 \end{gathered} \tag{11}$$

with $E = \sqrt{\|\mathbf{L}\,\mathbf{s}\|^2 + \beta}$, and J the Jacobian of the forward operator $F(\mathbf{s})$. Newton's method can be applied to solve (**??**) obtaining the following system for the updates $\delta\mathbf{s}$ and $\delta\mathbf{y}$ of the primal and dual variables

$$\begin{bmatrix} J^T J & \alpha\, L^T \\ EL & -E \end{bmatrix} \begin{bmatrix} \delta\mathbf{s} \\ \delta\mathbf{y} \end{bmatrix} = - \begin{bmatrix} J^T(F(\mathbf{s}) - \mathbf{b}) + \alpha\, L^T \mathbf{y} \\ L\mathbf{s} - E\mathbf{y} \end{bmatrix} \tag{12}$$

with

$$h = 1 - \frac{\mathbf{y}\mathbf{L}\,\mathbf{s}}{E} \tag{13}$$

PD cont

which in turn can be solved as follows

$$[J^T J + \alpha\, L^T E^{-1} h L]\, \delta\mathbf{s} \quad = \quad -[J^T (F(\mathbf{s}) - \mathbf{b}) + \alpha\, L^T E^{-1} L\, \mathbf{s}] \tag{14a}$$

$$\delta\mathbf{y} \quad = \quad -\mathbf{y} + E^{-1} L\mathbf{s} + E^{-1} h L\, \delta\mathbf{s} \tag{14b}$$

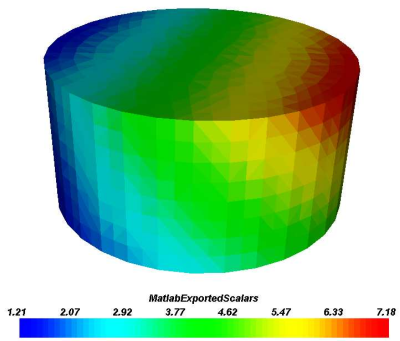
Equations (**??**) can therefore be applied iteratively to solve the non–linear inversion (**??**). Some care must be taken on the dual variable update, to maintain dual feasibility. A traditional line search procedure with feasibility checks is not suitable as the dual update direction is not guaranteed to be an ascent direction for the penalised dual objective function (D_β). The simplest way to compute the update is called the *scaling rule* [2] which is defined to work as follows

$$\mathbf{y}_{k+1} = \lambda(\mathbf{y}_k + \delta\mathbf{y}_k) \tag{15}$$

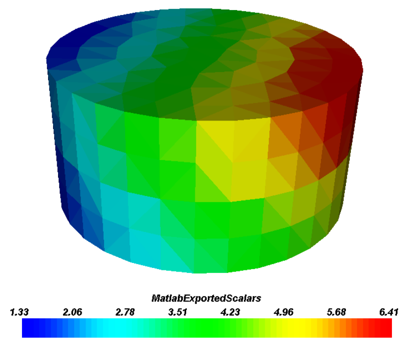
where

$$\lambda = \max\{\lambda : \ \lambda\|\mathbf{y}_k + \delta\mathbf{y}_k\| \leq 1\} \tag{16}$$

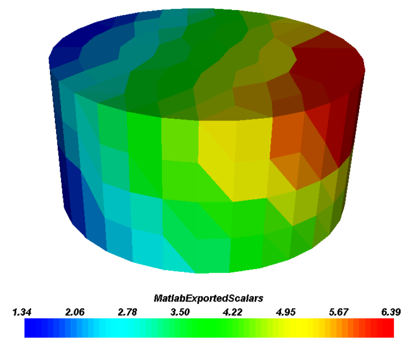
3D EIT smooth example



Original smooth conductivity projected on courser



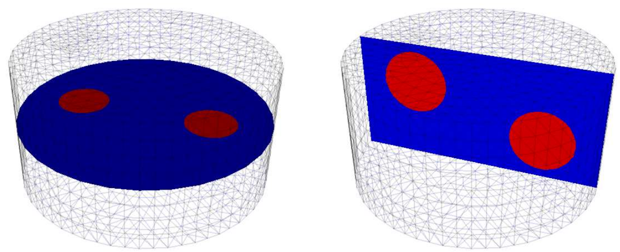
Smoothly regularized Gauss-Newton reconstruction



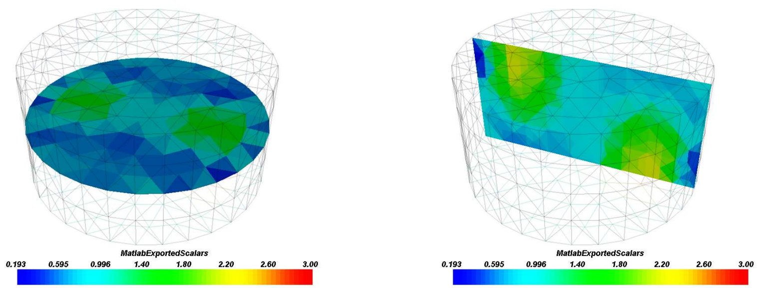
TV regularized PD-IPM reconstruction

Reconstruction of a smooth conductivity using a smooth regularization penalty function using Smooth and TV regularization.

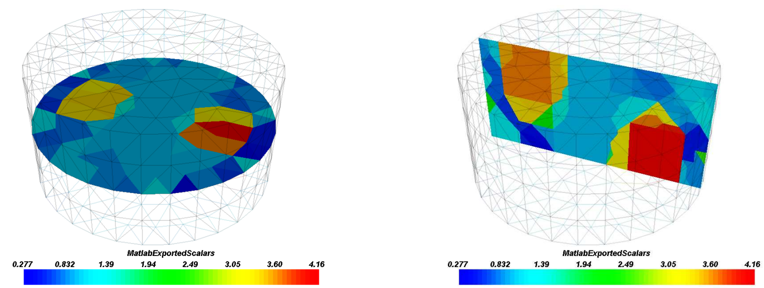
3D EIT discontinuous example



Two spheres test object



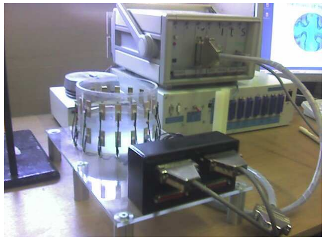
Tikhonov reg. GN reconstruction



TV reconstruction.

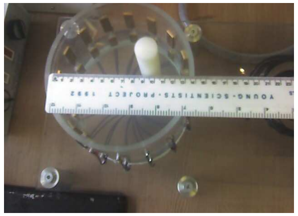
2D TV reconstruction - tank data

As a preliminary test of the PDIP-TV implementation in EIDORS David Stephenson (University of Manchester) compared a single iteration of Generalized Tikhonov regularized Gauss-Newton with a PDIP TV applied to the linearized problem. The data was collected on the ITS ERT system. Although the data collection was 2D – a limitation of the system, a 3D reconstruction algorithm was used.

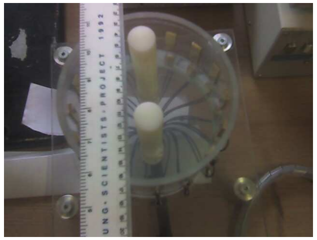


(a)

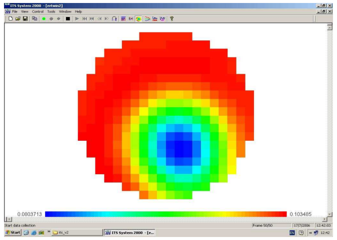
(b)



(c)

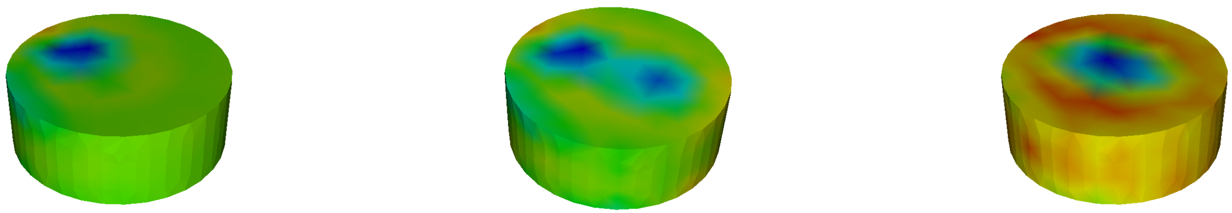


(d)

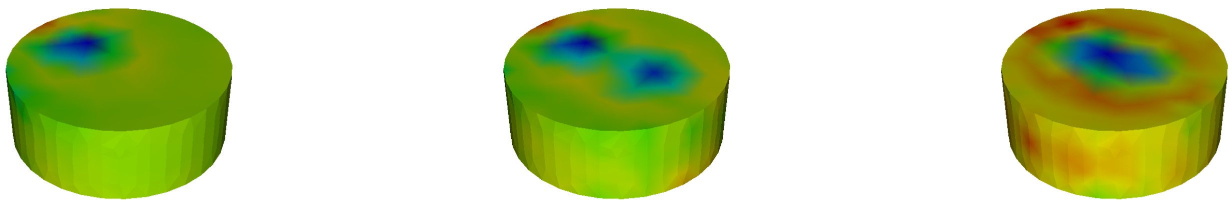


(e)

2D TV reconstruction - results



Tikhonov reg. GN reconstruction



TV reconstruction.

References

[1] Andersen KD and Christiansen E, 1995, A Newton barrier method for minimizing a sum of Euclidean norms subject to linear equality constraints. Technical Report, Department of Mathematics and Computer Science, Odense University, Denmark.

[2] Andersen KD, Christiansen E, Conn A, and Overton ML, 2000, An efficient primal-dual interior-point method for minimizing a sum of Euclidean norms. SIAM J. on Scientific Computing, 22:243-262, 2000.

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