Continuous-Time signals $x(t)$

Discrete-Time signals $x[n]$

Quantized signals $x_q[n]$

Sampling Rate

Slides 8D

Slides 8B
Spectrum representation:

- Plot showing frequencies and phasors for a signal
- A frequency-domain representation

Ex:

\[ x(t) = 7 \cos(50\pi t) + 4 \cos(125\pi t) \]

\[ = 3.5 e^{j50\pi t} + 3.5 e^{-j50\pi t} \]

\[ + 2 e^{j125\pi t} + 2 e^{-j125\pi t} \]
How often/fast does \( x(t) \) need to be sampled for reconstruction?

**Shannon Sampling Theorem**
(Nyquist Sampling Theorem)

A continuous-time signal \( x(t) \) with frequencies no higher than \( F_{\text{max}} \) can be reconstructed **exactly** from its discrete-time samples \( x[n] = x(nT_s) \), if the samples are taken at a rate

\[
F_s = 1/T_s \quad \text{that is greater than} \quad 2F_{\text{max}} \quad \text{(the Nyquist rate)}.
\]

**Note:** We can also say that \( F_{\text{max}} = B \) which is the bandwidth of \( x(t) \).
When does \( y(t) = x(t) \)?

\[
x(t) \xrightarrow{\text{C-to-D}} x[n] \xrightarrow{\text{D-to-C}} y(t)
\]

\[
f_s = \frac{1}{T_s}
\]

\[
y(t) = x(t) \quad \text{if } f_s > 2F_{\text{max}}
\]

Note: \( 2.0 \times F_{\text{max}} \) is maximum frequency of \( x(t) \).

\( x(t) \) must be bandlimited (maximum frequency \( F_{\text{max}} \)) for \( x(t) \) to be perfectly reconstructed.
Shannon's Sampling Theorem can be justified by considering the rotating phasor (represented with a dot in a circle below).

**Experiment setup:** Disk spinning at unknown frequency.

**Possible result:** Stationary dot

If stationary dot occurs for $f_s = 10$ Hz, then what is the rotation frequency $f_0$ of the disk?

**Answer:** Any integer multiple of sampling period $f_s$.

i.e., $f_0 = 10$ Hz, 20 Hz, 30 Hz, 40 Hz, ...
Based on seeing the rotating wheel, we can't tell whether it is rotating one, twice or more between each flash of the strobe light.

We see this effect looking at wheels of nearby cars at night (neon lights flash at twice the power line frequency $\sim 2 \times 60\text{Hz} = 120\text{Hz}$)
If the rotational frequency is slightly different (slower or faster) than the strobe time (sampling frequency), then we see the wheel slowly rotating forward or backward.

This effect is called aliasing (a frequency shows up in a “disguise” as another frequency).
Example

- Strobe flashes at \((\text{Power Freq}) \times 2 \approx 100\text{Hz}\)
- Wheel 0 Rotates at 100 Hz \(\Rightarrow\) Period = \(1/100\text{Hz} \approx 10.0\text{ms}\)
- Wheel 1 Rotates at 101 Hz \(\Rightarrow\) Period = \(1/101\text{Hz} \approx 9.9\text{ms}\)
- Wheel 2 Rotates at 99 Hz \(\Rightarrow\) Period = \(1/99\text{Hz} \approx 10.1\text{ms}\)
Example

- Strobe flashes at (Power Freq)×2 ≈ 100Hz
- Wheel 0 Rotates at 100 Hz  ⇒ Period = 1/100Hz ≈ 10.0ms
- Wheel 1 Rotates at 101 Hz  ⇒ Period = 1/101Hz ≈ 9.9ms
- Wheel 2 Rotates at 99 Hz   ⇒ Period = 1/99Hz  ≈ 10.1ms

Time Axis

Strobe light flashes
Wheel 0:
- Rotates an extra 0s
- Fraction = 0s/10ms = 0 cycles
- Freq = 0 cyc/0.01s = 0 Hz

Wheel 1:
- Rotates an extra 0.1s
- Frac = 0.1s/9.9s = .01 cycles
- Freq = 0.01 cyc/0.01s = 1 Hz

Wheel 2:
- Rotates an extra -0.1s
- Frac = -0.1s/10.1s = -0.01 cycles
- Freq = -0.01 cyc/0.01s = -1 Hz
Imagine we can only measure the vertical position of the spot. This gives us a signal $x(t)$

**Curve 0:**
- $x_0(t) = A \cos (2\pi(100)t)$

**Curve 1:**
- $x_1(t) = A \cos (2\pi(101)t)$

**Curve 2:**
- $x_2(t) = A \cos (2\pi(99)t)$
Phasor Representation: Continuous time

\[ x_0(t) = A \cos(2\pi(100)t) = \frac{1}{2} A \left( e^{j2\pi(100)t} + e^{-j2\pi(100)t} \right) \]

\[ x_1(t) = A \cos(2\pi(101)t) = \frac{1}{2} A \left( e^{j2\pi(101)t} + e^{-j2\pi(101)t} \right) \]

\[ x_2(t) = A \cos(2\pi(99)t) = \frac{1}{2} A \left( e^{j2\pi(99)t} + e^{-j2\pi(99)t} \right) \]

Nyquist rate analysis: \( f_s > 2f_{\text{max}} \)

- \( f_s = 100\text{Hz}, \) so \( f_{\text{max}} < \frac{1}{2} (100\text{Hz}) = 50\text{Hz} \)
We sample the vertical position at $t=nT_s$. This gives us a signal $x[n]$

Curve 0:
\[- x_0[n] = A \cos (2\pi(100)n(0.01)) = A \cos (2\pi(1.00)n) \]

Curve 1:
\[- x_1[n] = A \cos (2\pi(101)n(0.01)) = A \cos (2\pi(1.01)n) \]

Curve 2:
\[- x_2[n] = A \cos (2\pi(99)n(0.01)) = A \cos (2\pi(0.99)n) \]
Curve 0:
- \( x_0[n] = A \cos (2\pi(1.00)n) \)
  = \( A \cos (2\pi + 0)n \)
  = \( A \cos (0) = A \)

Curve 1:
- \( x_1[n] = A \cos (2\pi(1.01)n) \)
  = \( A \cos (2\pi + 0.02\pi)n \)
  = \( A \cos (2\pi(0.01)n) \)

Curve 2:
- \( x_2[n] = A \cos (2\pi(0.99)n) \)
  = \( A \cos (2\pi - 0.02\pi)n \)
  = \( A \cos (-2\pi(0.01)n) \)
  = \( A \cos (2\pi(0.01)n) \)
- So \( x_1[n] = x_2[n] \) in this case
Phasor Representation: Discrete time

\[ x_0[n] = A \cos(2\pi(0.00)n) = \frac{1}{2} A \left( e^{j2\pi(0.00)t} + e^{-j2\pi(0.00)n} \right) \]

\[ x_1[n] = A \cos(2\pi(0.01)n) = \frac{1}{2} A \left( e^{j2\pi(0.01)t} + e^{-j2\pi(0.01)n} \right) \]

\[ x_2[n] = A \cos(-2\pi(0.01)n) = \frac{1}{2} A \left( e^{j2\pi(0.01)t} + e^{-j2\pi(0.01)n} \right) \]

Given the sampled \( x[n] \), the signals could have come from any of these original signals.

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**Units**

- \( f \) (cycles/sample)
- \( \omega \) (radians/sample)
- \( F \) (cycles/second)
Origin of aliased signals

Step 1: (C-T)
Continuous time signal (here, most signal at one freq)

Step 2: (D-T)
Signal is sampled. $x[n]$ could originate from any replica of the spectrum

Step 3: Reconstruction
We interpret (reconstruct) the signal from the baseband spectral replica

Perfect Reconstruction because $f_{\text{max}} < \frac{1}{2} f_s$
Origin of aliased signals: non-folding case

Step 1: (C-T)
Continuous time signal
(here, most signal at one freq)

Step 2: (D-T)
Signal is sampled.
$x[n]$ could originate from any replica of the spectrum

Step 3: Reconstruction
We interpret (reconstruct) the signal from the baseband spectral replica

Aliased Reconstruction because $f_{\text{max}} > \frac{1}{2} f_s$
Step 1: (C-T)
Continuous time signal
(here, most signal at one freq)

Step 2: (D-T)
Signal is sampled.
$x[n]$ could originate
from any replica of the
spectrum

Step 3: Reconstruction
We interpret (reconstruct) the
signal from the baseband
spectral replica

Aliased Reconstruction because $f_{\text{max}} > \frac{1}{2} f_s$
Non-Folding case:
- \( x_1[n] = A \cos \left( 2\pi(101)t + \theta \right) \)
  - \( = A \cos \left( 2\pi(101)(0.01)n + \theta \right) \)
  - \( = A \cos \left( 2\pi(1.01)n + \theta \right) \)
  - \( = A \cos \left( (2\pi + 0.02\pi)n + \theta \right) \)
  - \( = A \cos \left( 2\pi(0.01)n + \theta \right) \)

Folding case:
- \( x_2[n] = A \cos \left( 2\pi(99)t + \theta \right) \)
  - \( = A \cos \left( 2\pi(99)(0.01)n + \theta \right) \)
  - \( = A \cos \left( 2\pi(0.99)n + \theta \right) \)
  - \( = A \cos \left( (2\pi - 0.02\pi)n + \theta \right) \)
  - \( = A \cos \left( -2\pi(0.01)n - \theta \right) \)
  - \( = A \cos \left( 2\pi(0.01)n - \theta \right) \)

Non-folding:
- Frequency is wrong
- Direction is right

Folding:
- Frequency is wrong
- Direction is wrong
Questions

- EMG signal has maximum content at a $f=5\text{kHz}$. What sampling frequency required? What $T_s$?
- Assume uniform frequency content in EMG. Filter with a 4\text{th} order LPF with $f_c$ of 1kHz. Sample: $f_s=20\ \text{kSamples/s}$. Is there aliasing? Estimate maximum amplitude of aliasing contribution?
- Sample signal with ADC. What resolution required so the quantization error is less than the aliasing contribution?
- Range of ADC is –2 V to +2V. EMG signal is –100mV to +100mV. Specify the ADC in bits (B) and $f_s$. 