Continuous-Time signals $x(t)$

Discrete-Time signals $x[n]$

Quantized signals $x_q[n]$

$\Delta T_s$
A signal \( x(t) \) can be is represented as a sum of sinusoids

\[
x(t) = \sum_{k=1}^{N} A_k \cos(\Omega_k t + \phi_k)
\]

where each sinusoid can have

- its own frequency \( \Omega_k \)
- its own amplitude \( A_k \)
- its own phase \( \phi_k \)

To put the sinusoid in the spectrum representation, we need to expand each sinusoid into \textit{rotating phasors}. 
Frequency spectrum representation => Euler's formula.

For cosine:
\[ \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \]

For sine:
\[ \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \left\{ \frac{e^{j\theta} - e^{-j\theta}}{2} \right\} e^{-j\pi/2} \]

For our general sinusoid it is decomposed as:
\[ A \cos(\Omega t + \phi) = \frac{A}{2} \left\{ e^{j(\Omega t + \phi)} + e^{-j(\Omega t + \phi)} \right\} \]
We can represent the following sinusoid graphically in a frequency spectrum as depicted below.

\[ A \cos(5t) = \frac{A}{2} e^{j5t} + \frac{A}{2} e^{-j5t} \]

Amplitude, phase and frequency are represented.

Each bar represents an exponential.
Fourier analysis: represent signal as a spectrum of phasors

- DTFT (Discrete-time Fourier Transform)
  Also FFT (Fast Fourier Transform)
Compare two signals $x_1(t)$ and $x_2(t)$.
From the frequency spectrum of $x_1(t)$ and $x_2(t)$ we know they contain roughly the same frequency content. *When in time* did these frequencies occur?

**Possible answers:**
- Over all samples $n$ for all time
- Each frequency at a different point in time
- Each frequency at a various points in time, possibly overlapping, possible repeating

**Problem:** From spectrum alone, we do not know *when* in time a particular frequency component has occurred
$x_1(t)$: formed from a C-major scale.

$\begin{align*}
&\text{\includegraphics[width=\textwidth]{c_major_scale} }
\end{align*}$

$x_2(t)$: formed by playing notes simultaneously.

$\begin{align*}
&\text{\includegraphics[width=\textwidth]{notes_playing} }
\end{align*}$
How can we analyze a time-varying signal such where the frequency content changes over time?

We can calculate the DTFT over partitions of $x[n]$ and plot a 2D image where

- $x$-axis is time (specific partition)
- $y$-axis is frequency (index $k$ for $X[k]$)
- colour/height is magnitude-squared of $X[k]$

This is known as a spectrogram.
Each partition is a column $X[k]$. The columns are placed together corresponding to their place in time.
Spectrograms

Ideal C-Major Scale

After Mclellan, Schafer & Yoder, *DSP First*
Time-Frequency

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Spectrograms

C-Major Scale

C-Major Scale

GNU Octave/MATLAB:

fs=8192; t=0:1/fs:0.2; x=[];
for i=[3 5 7 8 10 12 14 15]
    w=2*pi*(220*2^(i/12));
    x=[x, cos(w*t)];
end;
specgram(x,256,fs);

Sinusoids ONLY

ARTIFACTS at Transitions

After Mclellan, Schafer & Yoder, DSP First
This effect is the *time-frequency* resolution trade-off. It is similar to the Heisenberg uncertainty principle.

GNU Octave/MATLAB:
```
fs=8192; t=0:1/fs:0.2; x=[];
for i=[3 5 7 8 10 12 14 15]
    w=2*pi*(220*2^(i/12));
    x=[x cos(w*t)];
end;
specgram(x,N,fs);
```
Spectrograms
Speech

Spectrogram of male speaker saying "fish".

"f"  "i"  "sh"

Time-Frequency
Slide 8C.14

0  0.2  0.4  0.6  0.8 

0 1000 2000 3000 4000 5000 

Time $t$ (seconds)

Frequency (Hz)

fricative start for "f"

high energy "white noise" for "sh"
vowel formant frequencies
Questions

- What kind of situations is time frequency analysis useful. Comment on analysis of:
  - Speech
  - Music
  - EMG signals

- What is the origin of the artefacts at the frequency transitions?
- If $F_s = 10$kHz and $N=1024$, what time frame does spectrogram window #9 represent?
- What about the phase in the phasor? Where is that shown in the spectrogram?