Quantization: approximating a continuous range of values by a small set of discrete values.
Quantization function

Quantizer represented by:

\[ x_q[n] = Q(x[n]) \]

where

\[ x_q[n] = \begin{cases} 
X_L & \text{if } T_L < x < +\infty \\
X_{L-1} & \text{if } T_{L-1} < x < T_L \\
X_1 & \text{if } T_1 < x < T_2 \\
X_0 & \text{if } -\infty < x < T_1 
\end{cases} \]
Our quantizer has
- $L$ levels
- covers the range from $X_{\text{min}}$ to $X_{\text{max}}$
- Level $L_k$ has thresholds
  - Upper Threshold $T_k$
  - Lower Threshold $T_{k-1}$

Quantizers can be
- Uniform
  - $T_k - T_{k-1} = \Delta = \text{constant}$
  - $\Delta$ is the quantization step
- Non-uniform
  - $T_{k-1} - T_k$ varies
- Adaptive
  - $\Delta$ is changes with time
Given $X_{\text{min}}$ and $X_{\text{max}}$, we calculate the quantization step, $\Delta$, for a uniform quantizer:

$$\Delta = \frac{X_{\text{max}} - X_{\text{min}}}{L - 1} = \frac{\text{Range}}{L}$$
Resolution = Quantization step = $\Delta$

$\Delta = \frac{(X_{\text{max}} - X_{\text{min}})}{(L-1)}$

$X_{\text{max}}$ is $\Delta$ below top of range

$\Delta = \frac{\text{Range}}{L}$

**Examples:**

- **8-bit converter:** Range is 0 to 4V
  
  $\Delta = \frac{\text{Range}}{L} = \frac{(4 - 0)}{2^8} = 15.6\text{mV}$
  
  $X_{\text{max}} = 4V - 15.6\text{mV}$

- **12-bit converter:** Range is –1 to 1V
  
  $\Delta = \frac{\text{Range}}{L} = \frac{(1 - -1)}{2^{12}} = 0.49\text{mV}$
  
  $X_{\text{max}} = 1V - 0.49\text{mV}$
The quantized signal is encoded into $B$ bits where

$$L \leq 2^B$$

Normally the maximum number of levels are chosen for the number of bits. Common values are 8, 12, 16.
Questions

- What is the role of
  - Sample and Hold?
  - Quantizer?
  - Encoder?

- An 8-bit ADC has a range of 0V – 1V
  - What is the resolution
  - What is the sensitivity?
  - To what voltages do the maximum and minimum output values correspond?

- What happens if the signal \( x(t) \) is outside the range \( X_{\text{min}} \) to \( X_{\text{max}} \)?
Question

- You are processing data, and you see values of $X_{\text{min}}$ or $X_{\text{max}}$ in the recorded data.
- What does that mean?
- How should you process it?

![Graph showing $X_{\text{max}}$ and $X_{\text{min}}$ over time.](chart.png)
Encoding: 8, 12, 16 bit ADCs exist (also called bit depth)

Technologies
- Direct conversion (Flash conversion) – fastest, least bits
- Successive-approximation
- Sigma-Delta ADC
- Integrating ADC – slowest, largest bits

Terminology
- Range: \( X_{\text{max}} - X_{\text{min}} \)
  - Single Supply: \( X_{\text{min}} = 0 \)
  - Differential: \( X_{\text{min}} = -X_{\text{max}} \)
- Resolution: Number of discrete values (= \( L \))
  - Resolution in Volts is \( (X_{\text{max}} - X_{\text{min}})/(L-1) \)
- Sampling Frequency
Quantization
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ADC Example
(Unipolar / Single Supply)

- 3-bit ADC  =>  $2^3 = 8$ Levels
- Input Range: Example $0 – 2V$
- $\Delta = (\text{Input Range})/\text{Levels} = 2V / 8 = 0.250V$
- $X_{\text{max}} = \text{Max input range} – \Delta = 1.750$

<table>
<thead>
<tr>
<th>Output Level</th>
<th>Code</th>
<th>Min</th>
<th>Max</th>
<th>Level (V)</th>
<th>Min (V)</th>
<th>Max (V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>000</td>
<td>$-\infty$</td>
<td>$T_1$</td>
<td>0.000</td>
<td>$-\infty$</td>
<td>0.125</td>
</tr>
<tr>
<td>1</td>
<td>001</td>
<td>$T_1$</td>
<td>$T_2$</td>
<td>0.250</td>
<td>0.125</td>
<td>0.375</td>
</tr>
<tr>
<td>2</td>
<td>010</td>
<td>$T_2$</td>
<td>$T_3$</td>
<td>0.500</td>
<td>0.375</td>
<td>0.625</td>
</tr>
<tr>
<td>3</td>
<td>011</td>
<td>$T_3$</td>
<td>$T_4$</td>
<td>0.750</td>
<td>0.625</td>
<td>0.875</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>$T_4$</td>
<td>$T_5$</td>
<td>1.000</td>
<td>0.875</td>
<td>1.125</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
<td>$T_5$</td>
<td>$T_6$</td>
<td>1.250</td>
<td>1.125</td>
<td>1.375</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
<td>$T_6$</td>
<td>$T_7$</td>
<td>1.500</td>
<td>1.375</td>
<td>1.625</td>
</tr>
<tr>
<td>7</td>
<td>111</td>
<td>$T_7$</td>
<td>$+$∞</td>
<td>1.750</td>
<td>1.625</td>
<td>$+$∞</td>
</tr>
</tbody>
</table>
Single Supply / Unipolar ADC

- Range: 0 to 1 V
- B = 2
- $L = 2^2 = 4$
- $\Delta = (1 - 0) / 4 = 0.25$
- $X_{\text{min}} = 0$
- $X_{\text{max}} = 1V - \Delta = 0.75V$
- Thresholds, $T_1 \ldots T_3$

\[
T_1 = L_0 + (1 - \frac{1}{2}) \times \Delta = 0.125 \\
T_2 = L_0 + (2 - \frac{1}{2}) \times \Delta = 0.375 \\
T_3 = L_0 + (3 - \frac{1}{2}) \times \Delta = 0.625
\]
Single Supply / Unipolar ADC

- Range: –1 to +1 V
- \( B = 2 \)
- \( L = 2^2 = 4 \)
- \( \Delta = (1 - (-1)) / 4 = 0.5 \)
- \( X_{\text{min}} = -1 \text{V} \)
- \( X_{\text{max}} = 1 \text{V} - \Delta = 0.5 \text{V} \)
- Thresholds, \( T_1 \ldots T_3 \)
  \[ T_1 = L_0 + (1 - \frac{1}{2}) \times \Delta = -0.75 \]
  \[ T_2 = L_0 + (2 - \frac{1}{2}) \times \Delta = -0.25 \]
  \[ T_3 = L_0 + (3 - \frac{1}{2}) \times \Delta = +0.25 \]
### ADC Example (Bipolar / Dual Supply)

- **3-bit ADC** => \(2^3 = 8\) Levels
- **Input Range**: Example –2V to +2V
- \(\Delta = (\text{Input Range})/\text{Levels} = 4V / 8 = 0.50V\)
- \(X_{\text{max}} = \text{Max input range} – \Delta = 1.50\)

<table>
<thead>
<tr>
<th>Output Level</th>
<th>Code</th>
<th>Min</th>
<th>Max</th>
<th>Level (V)</th>
<th>Min (V)</th>
<th>Max (V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
<td>-(\infty)</td>
<td>(T_1)</td>
<td>-2.00</td>
<td>-(\infty)</td>
<td>-1.75</td>
</tr>
<tr>
<td>1</td>
<td>101</td>
<td>(T_1)</td>
<td>(T_2)</td>
<td>-1.50</td>
<td>-1.75</td>
<td>-1.25</td>
</tr>
<tr>
<td>2</td>
<td>110</td>
<td>(T_2)</td>
<td>(T_3)</td>
<td>-1.00</td>
<td>-1.25</td>
<td>-0.75</td>
</tr>
<tr>
<td>3</td>
<td>111</td>
<td>(T_3)</td>
<td>(T_4)</td>
<td>-0.50</td>
<td>-0.75</td>
<td>-0.25</td>
</tr>
<tr>
<td>4</td>
<td>000</td>
<td>(T_4)</td>
<td>(T_5)</td>
<td>0.00</td>
<td>-0.25</td>
<td>+0.25</td>
</tr>
<tr>
<td>5</td>
<td>001</td>
<td>(T_5)</td>
<td>(T_6)</td>
<td>+0.50</td>
<td>+0.25</td>
<td>+0.75</td>
</tr>
<tr>
<td>6</td>
<td>010</td>
<td>(T_6)</td>
<td>(T_7)</td>
<td>+1.00</td>
<td>+0.75</td>
<td>+1.25</td>
</tr>
<tr>
<td>7</td>
<td>011</td>
<td>(T_7)</td>
<td>+(\infty)</td>
<td>+1.50</td>
<td>+1.25</td>
<td>+(\infty)</td>
</tr>
</tbody>
</table>
Quantization

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Quantization Error

The difference $x_q[n] - x[n] = e[n]$ is called the Quantization Error.
Quantization Error Limits

Input
\[ x[n] \]

Output
\[ x_q[n] \] +

\[ e[n] \] is always less than \( \frac{1}{2} \Delta \) if

\[ (X_{\text{min}} - \frac{1}{2}\Delta) < x[n] < (X_{\text{max}} + \frac{1}{2}\Delta) \]

We can consider \( e[n] \) to be random noise, if

- Statistics of \( e[n] \) do not change with time
- \( e[n] \) is uncorrelated with input \( x[n] \)
Questions

- What happens to the quantization error if $X_{\text{min}} > x[n]$?
- If $L=100$, how many bits $B$ are required for the encoder?
- If $L$ is even and one level is 0, how do we assign levels for a double sided ADC?
- If the measurement noise at the sensor results in a SNR of 40 dB, and the input range of the ADC is 0-100°C. We want the quantization noise to be lower than the measurement noise. How many quantization bits, $B$, should the ADC have?

(You may assume the temperature is uniformly distributed between 10-50°C, and the ADC input voltage is linearly related to temperature)
An 8-bit ADC has an input range of +1 V to −1 V (corresponding to output codes of −128 to 127). You use this ADC to design a circuit to measure EMG signals with a range of ±50 mV.

- What is the range of digital output values from the ADC we expect from these EMG signals?
- What is the resolution (in Volts)?
- What is the maximum quantization error?
- When you look at the data, there are a number of recorded values of −128 in the data stream. What does this mean?