Department of Systems and Computer Engineering
Carleton University
Ottawa K1S 5B6
Canada

SYSC3203 Bioelectronic Systems

Laboratory 0: Linear circuit analysis and measurement

SOLUTIONS

Prof. A. Adler

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1 Linear circuit analysis

For each of the circuits shown on the worksheet (Table 2), analyse the response you would expect to each of the three stimulus waveforms, given the following component values:

Table 1: Component values for the circuits

Circuit	R_1	R_2	C_1
(A)	20 kΩ	39 kΩ	-
(B)	39 kΩ	-	2.2 nF
(C)	20 kΩ	39 kΩ	2.2 nF

In each case:

- **Sketch the expected output waveforms in the boxes provided**
- **Write down the expected peak-to-peak voltage at the output**
- **Indicate whether you expect the output to *lead*, *lag*, or be *in phase* with the input signal**

SOLUTION

Circuit (A)

This is a simple voltage divider: the voltage appearing across the output terminals is a fraction $R_2/(R_1 + R_2)$ of that at the input. With $R_1 = 20k\Omega$ and $R_1 = 39k\Omega$,

$$\frac{V}{V_s} = \frac{39}{20 + 39} = 0.66$$

so with 2 V_{p-p} in, we should measure 1.32 V_{p-p} at the output. Neglecting parasitics (which should be insignificant at these frequencies) the response should be independent of frequency, so you should see the same peak-to-peak voltage for all three input signals, and all three should be *in phase with* the driving signal.

Circuit (B)

Sinusoidal inputs: This circuit is also a voltage divider, however its response depends on frequency as a result of the capacitor's frequency-dependent impedance $Z_C = 1/j\omega C$ i.e.

$$\frac{V}{V_s} = \frac{Z_C}{R_1 + Z_C} = \frac{1/j\omega C_1}{R_1 + 1/j\omega C_1}$$

$$= \frac{1}{1 + j\omega C_1 R_1}$$

The *magnitude* of the response will be given by

$$\left|\frac{V}{V_s}\right| = \left(\frac{1}{1 + \omega^2 C_1^2 R_1^2}\right)^{1/2}$$

At $f = 1$ kHz, $\omega = 2\pi f = 6283$ rad.s^{-1}, and taking $R_1 = 39k\Omega$ and $C_1 = 2.2nF$, $\omega C_1 R_1 = 0.539$ so

$$\left|\frac{V}{V_s}\right| = \left(\frac{1}{1 + 0.539^2}\right)^{1/2} = 0.880$$

so that the 2 V_{p-p} input becomes 1.76 V at the output; while at $f = 10$ kHz, $\omega = 2\pi f = 62831$ rad.s^{-1}, so that

$$\left|\frac{V}{V_s}\right| = \left(\frac{1}{1 + 5.39^2}\right)^{1/2} = 0.182$$

giving $V_{p-p} = 365$ mV at the output. In both cases the phase of the output *lags* the input: this is easiest to see if we re-write the response as

$$\frac{V}{V_s} = \frac{1}{1 + j\omega C R_1} = \frac{1 - j\omega C_1 R_1}{1 + \omega^2 C_1^2 R_1^2}$$

indicating a phase angle

$$\phi = \tan^{-1}\left(-\omega C_1 R_1\right)$$

The lag increases from 0° at DC towards −90° as $\omega \to \infty$. At $f = 1$ kHz, $\phi = -28.3°$, and at $f = 10$ kHz, $\phi = -79.5°$.

Squarewave input (direct method): Let the voltage across C_1 be $v(t)$. Then the current through R_1 is

$$i(t) = \frac{v_s - v}{R_1}$$

Meanwhile, the charge on C_1 is $q = C_1 v$, so that

$$i = \frac{dq}{dt} = C_1 \frac{dv}{dt}$$

Equating, we have

$$C_1 \frac{dv}{dt} = \frac{v_s - v}{R_1}$$

$$\frac{dv}{dt} + \frac{1}{R_1 C_1} v = \frac{1}{R_1 C_1} v_s(t)$$

This type of first-order inhomogeneous ODE is easily solved using an *integrating factor*

$$\phi(t) = \exp\left(\frac{t}{R_1 C_1}\right) = \exp\left(t/\tau\right)$$

such that

$$\frac{d}{dt}\phi v = \phi \frac{dv}{dt} + \frac{d\phi}{dt} v = \phi \frac{dv}{dt} + \frac{1}{\tau}\phi v$$

allowing us to write the LHS as a *total derivative*, giving

$$\frac{d}{dt}\phi v = \frac{1}{\tau}\phi v_s(t)$$

with solution

$$\phi v = \frac{1}{\tau}\int \phi v_s dt + \text{const.}$$

Now consider the *unit step response* of the circuit; that is, if we start with v_s in steady state at 0 V and the capacitor uncharged, then instantaneously apply a voltage $v_s = 1\ V$ at time $t = 0$

$$v_s(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

so that

$$e^{t/\tau} v - e^{0/\tau}(0\ V) = \frac{1}{\tau}\int_0^t e^{t'/\tau} dt' = e^{t/\tau} - 1$$

Dividing through by $e^{t/\tau}$, we get

$$v(t) = 1 - \exp\left(-t/\tau\right)\ \text{V} \qquad t \geq 0$$

which starts from 0 V and rises asymptotically towards 1 V with time constant $\tau = R_1 C_1$.

In our case, we have a square wave that switches (near) instantaneously from -1 V to +1 V rather than from 0 V to +1 V, so the response is scaled by a factor 2 and shifted down by 1 V i.e.

$$v(t) = 1 - 2\exp\left(-t/\tau\right)\ \text{V}$$

By symmetry, the turn-off response is

$$v(t) = -1 + 2\exp\left(-t/\tau\right)\ \text{V}$$

where $t = 0$ now corresponds to the falling edge of the input square wave.

The peak-to-peak output swing will be determined by how closely the response approaches the asymptote in each half-cycle of the square wave i.e.

$$v_{p-p} = 2\left[1 - \exp\left(-\frac{T/2}{R_1 C_1}\right)\right]\ \text{V}$$

With $\tau = R_1 C_1 = (39\ k\Omega)(2.2\ nF) = 85.8\ \mu s$ and $T/2 = 500\mu s$, we should observe $v_{p-p} = 1.99\ V$ i.e. it should essentially reach its final value.

Squarewave input (transform method): As an alternative to direct integration in the time domain, we could analyse the response using Laplace transform methods, starting from the response we wrote down for the sinusoidal inputs

$$H(s) = \frac{V}{V_s} = \frac{1}{1 + j\omega C_1 R_1} = \frac{1}{1 + s\tau}$$

Now consider the unit step response. Since $u(t)$ has Laplace transform

$$u(t) \leftrightharpoons \frac{1}{s}$$

we can write the output as

$$V(s) = H(s).\frac{1}{s} = \frac{1}{s(1 + s\tau)}$$

Using partial fractions

$$\frac{1}{s(1 + s\tau)} = \frac{A}{s} + \frac{B}{1 + s\tau}$$

$$1 = A(1 + s\tau) + Bs$$

i.e. $A = 1$, $B = -\tau$ we get

$$V(s) = \frac{1}{s} - \frac{\tau}{1 + s\tau}$$

Inverting each term using tables,

$$v(t) = 1 - \exp(-t/\tau) \qquad t \geq 0$$

Since our actual input is $\pm 1\ V$, the output will be scaled and shifted i.e.

$$v(t) = 2\left[1 - \exp(-t/\tau)\right] - 1 \qquad t \geq 0$$

$$= 1 - 2\exp(-t/\tau)\ \text{V} \qquad t \geq 0$$

Circuit (C)

Sinusoidal inputs: A slightly more complicated frequency-dependent voltage divider: just like (B), we can write down the response in terms of the component impedances as

$$\frac{V}{V_s} = \frac{R_2}{R_1 + R_2 + Z_C}$$
$$= \frac{R_2}{R_1 + R_2 + 1/j\omega C_1}$$

It's instructive to split this as

$$\frac{V}{V_s} = \left(\frac{R_2}{R_1 + R_2}\right) \cdot \left(\frac{R_1 + R_2}{R_1 + R_2 + 1/j\omega C_1}\right)$$
$$= \left(\frac{R_2}{R_1 + R_2}\right) \cdot \left(\frac{j\omega RC_1}{1 + j\omega RC_1}\right)$$

from which we can see that the overall response will consist of a first-order high pass filter (HPF) with cutoff frequency $f_c = [2\pi(R_1 + R_2)C_1]^{-1}$ multiplied by a simple frequency-independent voltage divider.

The *magnitude* of the response will be given by

$$\left|\frac{V}{V_s}\right| = \left(\frac{R_2}{R_1 + R_2}\right) \cdot \left(\frac{\omega^2 R^2 C_1^2}{1 + \omega^2 R^2 C_1^2}\right)^{1/2}$$

At $f = 1$ kHz, $\omega = 2\pi f = 6283$ rad.s^{-1}, and taking $R_1 = 20k\Omega$, $R_2 = 39k\Omega$ and $C_1 = 2.2nF$, $\omega C_1(R_1 + R_2) = 0.815$ so

$$\left|\frac{V}{V_s}\right| = \left(\frac{0.815^2}{1 + 0.815^2}\right)^{1/2} = 0.632$$

Using the result from (A), the net response at $f = 1$ kHz is then

$$\left|\frac{V}{V_s}\right| = (0.66)(0.632) = 0.418$$

so that the 2 V_{p-p} input becomes approximately 0.84 V_{p-p}.

At $f = 10$ kHz, $\omega C_1(R_1 + R_2) = 17.4$, and

$$\left(\frac{8.15^2}{1 + 8.15^2}\right)^{1/2} = 0.993$$

i.e. we are far enough above the cutoff frequency that the signal is passed without significant attenuation: the response is dominated by the resistive divider and the peak-to-peak output voltage will be the same as for circuit (A), i.e. 1.32 V_{p-p}

By writing

$$\frac{j\omega RC_1}{1 + j\omega RC_1} \times \frac{1 - j\omega RC_1}{1 - j\omega RC_1}$$
$$= \frac{\omega^2 R^2 C_1^2 + j\omega RC_1}{1 + \omega^2 R^2 C_1^2}$$

we can see that the output will *lead* the input, with a positive phase angle

$$\phi = \tan^{-1}\left[\frac{1}{\omega C_1(R_1 + R_2)}\right]$$

The lead decreases from 90° at DC towards 0° as $\omega \to \infty$. At $f = 1$ kHz, $\phi = 50.8°$, and at $f = 10$ kHz, $\phi = 7.0°$ - almost in phase.

Squarewave input (direct method): The voltage v_C across C_1 can be written using Kirchhoff's Laws and Ohm's Law as

$$v_C = v_s - iR_1 - iR_2$$

where as for circuit (B),

$$i = \frac{dq}{dt} = C_1 \frac{dv_C}{dt}$$

Hence

$$\frac{dv_C}{dt} + \frac{v_C}{(R_1+R_2)C_1} = \frac{1}{(R_1+R_2)C_1} v_s$$

This is the same equation that you solved in (B) for $v(t)$ i.e. its unit step response is given by

$$v_C(t) = \begin{cases} 1 - \exp(-t/\tau) \text{ V} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

this time with $\tau = (R_1 + R_2)C_1$. From KVL, the sum of the voltages across R_1 and R_2 is whatever remains i.e. $1 - v_C$, and the output voltage $v(t)$ is the fraction of that across R_2.

Hence the unit step response of the circuit as a whole for $t \geq 0$ is

$$v(t) = \frac{R_2}{R_1+R_2} \exp(-t/\tau) \text{ V} \qquad t \geq 0$$

Our actual square wave input is a sequence of 2 V positive going steps and -2 V negative going steps, so in each complete cycle of input, the output will spike up to

$$v^+ = 2\left(\frac{R_2}{R_1+R_2}\right) \text{ V}$$

and then decay exponentially down towards 0 V with time constant $\tau = (R_1+R_2)C_1$, and then spike down to

$$v^- = -2\left(\frac{R_2}{R_1+R_2}\right) \text{ V}$$

before decaying back towards 0 V with the same time constant. The peak-to-peak output swing will therefore be

$$v_{p-p} = 4\left(\frac{R_2}{R_1+R_2}\right) \text{ V}$$

or approximately 2.64 V. *You will probably observe a value significantly less than this, likely due to the finite rise and fall times of the Picoscope source.*

Squarewave input (transform method): From the preceding frequency domain analysis, we have that

$$H(s) = \frac{V}{V_s} = \left(\frac{R_2}{R_1+R_2}\right) \cdot \left(\frac{j\omega RC_1}{1 + j\omega RC_1}\right)$$

$$= \left(\frac{R_2}{R_1+R_2}\right) . \frac{s\tau}{1+s\tau}$$

Now consider the unit step response. Since $u(t)$ has Laplace transform

$$u(t) \leftrightharpoons \frac{1}{s}$$

we can write the output as

$$V(s) = H(s) . \frac{1}{s}$$

$$= \left(\frac{R_2}{R_1+R_2}\right) . \frac{\tau}{1+s\tau}$$

which (from elementary tables) gives the time domain response

$$v(t) = \left(\frac{R_2}{R_1+R_2}\right) . \exp(-t/\tau)$$

With a 2 V peak-to-peak square wave input, the output will be twice this, as before.

2 Circuit measurements

Construct each of the circuits using one of the solderless breadboards provided. Using the Picoscope at your workstation, apply each of the input signals in turn and carefully measure the circuit's output.

Compare the measured results with those from your analysis. Are the discrepancies significant? **Try to explain any differences**.

3 Rise and fall times

Return to circuit (B) with the square-wave input. If you did not already do so, calculate the circuit's *time constant*. Use this result to calculate the expected 20-80 % rise and fall times for this circuit.

Solution: Using the unit step response calculated previously

$$v(t) = 1 - \exp(-t/\tau) \text{ V} \qquad t \geq 0$$

we need to solve

$$0.2 = 1 - \exp(-t_1/\tau)$$

$$0.8 = 1 - \exp(-t_2/\tau)$$

for $t_2 - t_1$.

Rearranging

$$0.8 = \exp(-t_1/\tau)$$

$$0.2 = \exp(-t_2/\tau)$$

and dividing top by bottom

$$\frac{0.8}{0.2} = \exp((t_2 - t_1)/\tau)$$

i.e.

$$t_2 - t_1 = \tau \ln 4$$

Note: if you want the 10%-90% times, just replace $\ln 4$ by $\ln 9$.

Time constant τ (ms)	20-80 % risetime t_r	80-20 % falltime t_f
0.0858	119 μs	119 μs

- **Calculate the 20 % and 80 % voltage levels, given a 2 V p-p input signal**

Solution: 20% of 2 V is 0.4 V and 80% is 1.6 V, **measured up from -1 V**

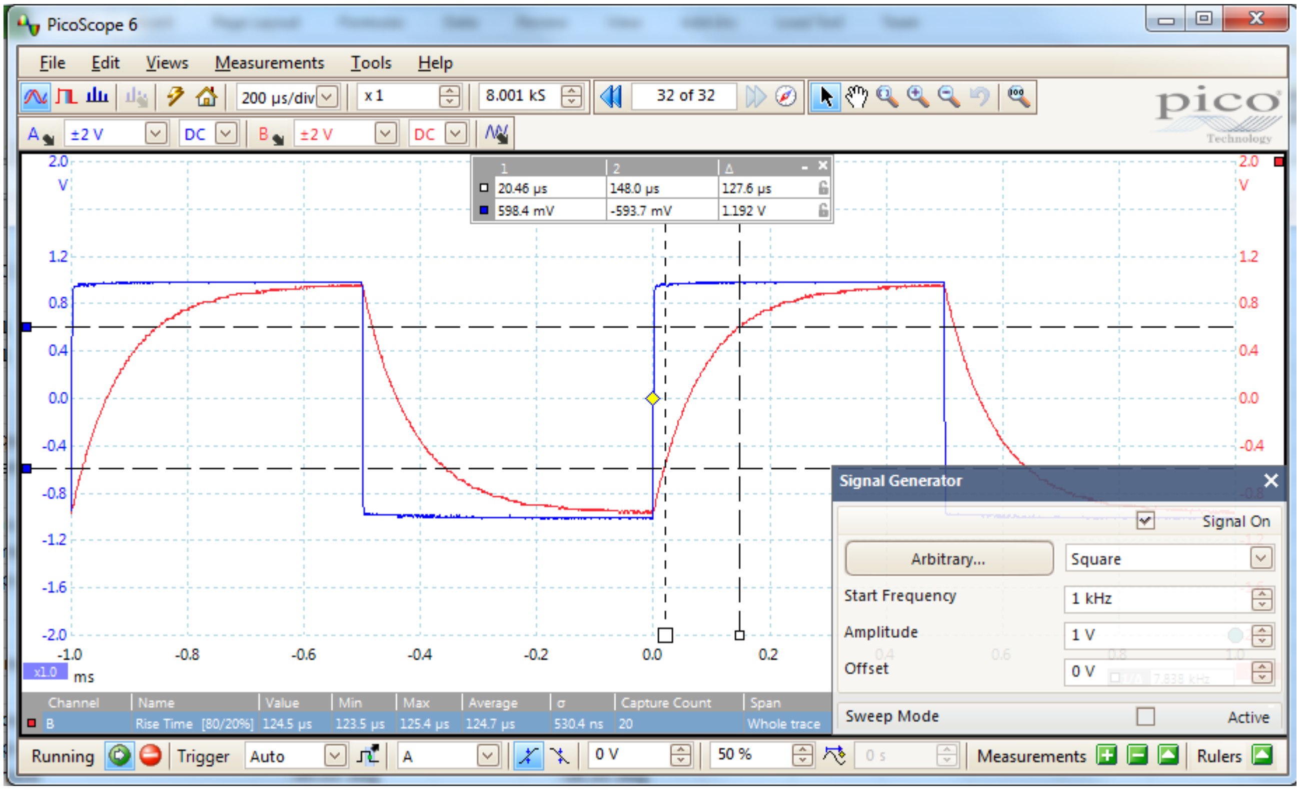
20 % level (mV)	80 % level (mV)
-0.6 V	+0.6 V

- **Use the Picoscope's moveable vertical and horizontal markers to estimate the rise time (20-80%) and fall time (80-20%) of the output signal (Figure 1)**

- **Repeat your measurement using the Picoscope's automated measurement feature. Does the result agree with your manual measurement?**

	Calculated	Measured
20-80 % risetime t_r	119 μs	
80-20 % falltime t_f	119 μs	

Figure 1: Rise and fall time measurement



Show your results to your instructor

Table 2: SYSC3203 Lab 0 worksheet

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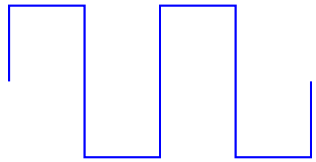
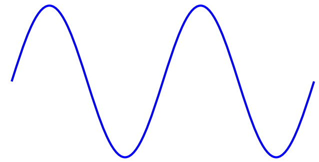
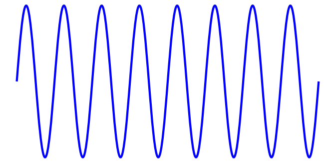
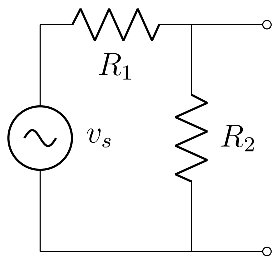
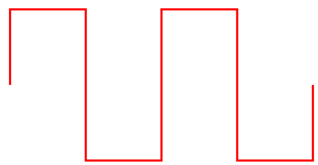
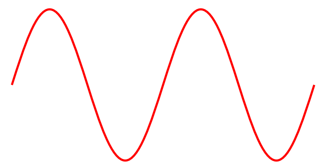
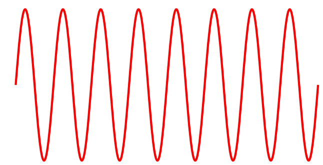
<tr><th>Circuit</th><th colspan="3">Input</th></tr>
<tr><td></td><td>Square wave, 1 kHz 2 Vp-p
</td><td>Sine wave, 1 kHz 2 Vp-p
</td><td>Sine wave, 10 kHz 2 Vp-p
</td></tr>
<tr><td>(A)

</td><td>
$Vpp = 1.32$</td><td>
$Vpp = 1.32$ In phase</td><td>
$Vpp = 1.32$ In Phase</td></tr>
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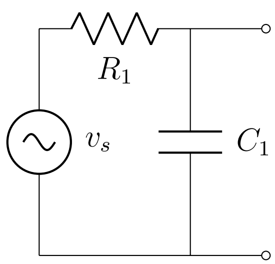
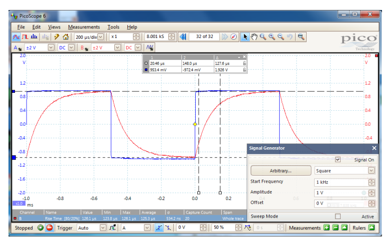
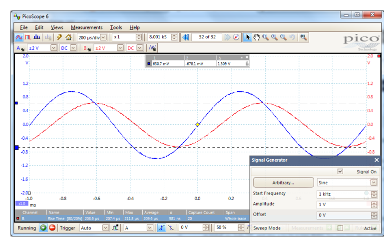
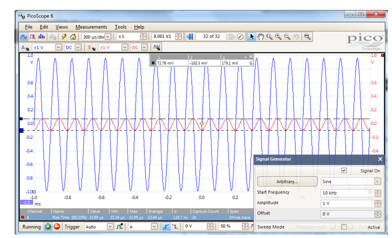

</td><td>
$Vpp = 1.99V$</td><td>
$Vpp = 1.76V$ Phase lags</td><td>
$Vpp = 0.365V$ Phase lags</td></tr>
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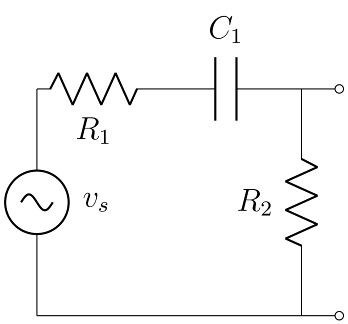
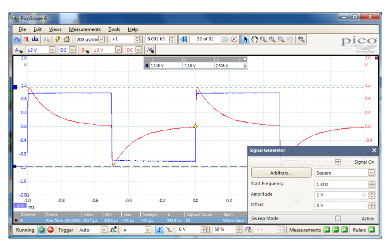
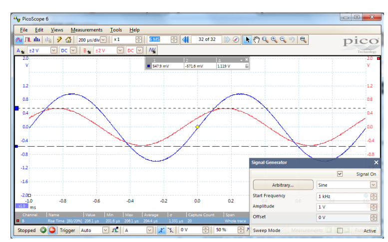
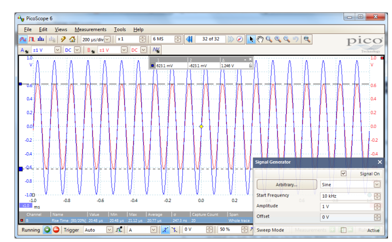

</td><td>
$Vpp = 2.64V$</td><td>
$Vpp = 0.84V$ Phase leads</td><td>
$Vpp = 1.32$ Phase leads</td></tr>
</table>