Reciprocity in electromagnetic systems

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Abstract: We review reciprocity for electromagnetic (E&M) systems, including EIT. We note that reciprocity is valid for all linear E&M systems and that it is clostly related to reciprocity in circuit theory. The correct reference for earliest work is Lorentz (1895).

1 Introduction

"Reciprocity" refers to a mutual and equivalent exchange. The term has been used in electrical engineering and phyics to refer to the equivalences when exchanging electromagnetic sources and fields. In Fig 1, $V_{ab} = V_{cd}$ if $I_{cd} = I_{ab}$, since the electrode positions, electrical networks and roles are reciprocal. This example illustrates the two reciprocity results, for electrical networks and for electromagentic systems generally.

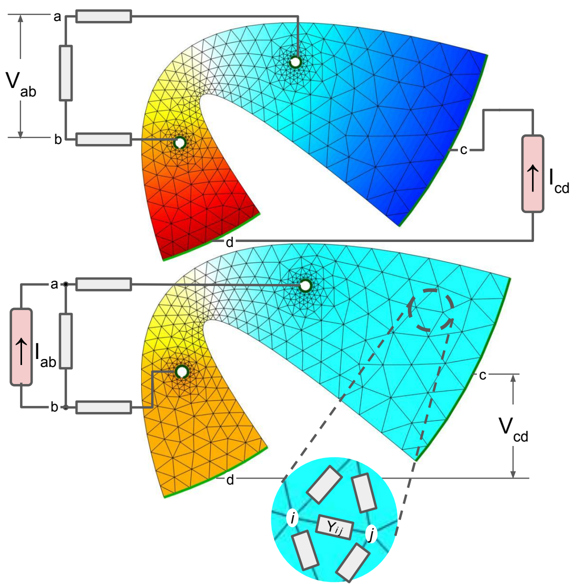

Figure 1: Illustration of reciprocity. In the upper figure, current (I_{cd}) is applied between outer electrodes and voltage (V_{ab}) meausured through a resistor network. In the lower figure these are reversed. Colour corresponds to voltage in the medium. The lower inset illustrates the discretization of the medium onto an electrical network, where nodes i and j are connected by admittance Y_{ij}.

2 Reciprocity in Electromagnetic Systems

Reciprocity is valid for linear, passive electromagnetic media, with sinusoidal electric field ($\vec{E}$) and current density ($\vec{J}$) and complex conductivity $\sigma^* = \sigma + i\omega\epsilon$.

$$\vec{J} = \left[\sigma^* - (i\omega\mu)^{-1}(\nabla\times\nabla\times)\right]\vec{E} \qquad (1)$$

where $\times$ represents the cross product.

Consider a current density $\vec{J_1}$ with angular frequency ω which produces electric and magentic fields $\vec{E_1}$ and $\vec{H_1}$, as well as another current $\vec{J_2}$ and fields $\vec{E_2}, \vec{H_2}$. If we have an isolated system with no energy from the outside, the reciprocity theorem [2] says:

$$\int \vec{J_1}\cdot\vec{E_2}dV = \int \vec{J_2}\cdot\vec{E_1}dV \qquad (2)$$

This is the Lorentz reciprocity, published in 1895 following analogous results on sound and light (for a history, see [3]).

Within the EIT community, it has been common to cite Geselowitz [1] who rediscovered the analogous theorem in electrostatics (Green's reciprocity). The Lorentz reciprocity is more general and applies to all linear electromagnetic networks. It is not valid for non-linear elements (e.g. diodes).

3 Reciprocity in Electrical Networks

Linear, passive electrical networks are reciprocal for currents and voltages measured at any two ports (such as A and B in Fig 1). Such networks can be characterized by an admittance matrix $\mathbf{Y}$, which relates the voltage on all nodes, $\mathbf{V}$, to the current flowing into each note, $\mathbf{I}$, via $\mathbf{I} = \mathbf{YV}$. Setting $V = 0$ on a ground node we have reduced matrices (not including the ground node) $\tilde{\mathbf{I}}$, $\tilde{\mathbf{V}}$, and $\tilde{\mathbf{Y}}$, and calculate an impedance matrix $\tilde{\mathbf{Z}} = \tilde{\mathbf{Y}}^{-1}$, such that $\tilde{\mathbf{V}} = \tilde{\mathbf{Z}}\tilde{\mathbf{I}}$.

Any voltage measured on the network can be described by $v = \mathbf{t}^T\mathbf{ZI}$, where *test* vector $\mathbf{t}$ represents the gain on each network node (and specifically those corresepondning to electrodes). For pair-drive EIT where current flows between nodes a and b, $\mathbf{I}_a = -\mathbf{I}_b = I_{\text{applied}}$, with other elements in $\mathbf{I}_{ab}$ zero. Using a gain, G, between electrodes c and d, $\tilde{\mathbf{t}}_{cd} = 0$ except for $\tilde{\mathbf{t}}_c = -\tilde{\mathbf{t}}_d = G$, Reciprocity follows from $\mathbf{Y}$ being symmetric, and thus $\mathbf{t}^t_{ab}\tilde{\mathbf{Z}}\tilde{\mathbf{I}}_{cd} = \mathbf{t}^t_{cd}\tilde{\mathbf{Z}}\tilde{\mathbf{I}}_{ad}$.

The symmetrical nature of $\mathbf{Y}$ may be motivated as follows: the body is discretized into points characterized by a voltage and current source. We conceptually build up the space from empty (no connections between points). When adding admittance Y_{jk} then current $Y_{jk}(V_j - V_k)$ flows into j and $Y_{jk}(V_k - V_j)$ flows into k. The admittance matrix, $\mathbf{Y}$, will increase by Y_{jk} at (j,k) and (k,j) and decrease by Y_{jk} at (j,j) and (k,k). $\mathbf{Y}$ will remain symmetrical unless external power drives more current into j than leaves k.

4 Discussion

Within the EIT community, it has sometimes not been clear that reciprocity has a longstanding mathematical consideration and that it applies to all electromagnetic problems. The goal of this abstract is to illustrate the relationship between reciprocity in electromagnetic systems and electrical networks, and to describe its origin in less mathematical terms. We recommend that for EIT papers citing the *reciprocity theorem*, the best and original reference is [2].

References

[1] DB Geselowitz, "An Application of Electrocardiographic Lead Theory to Impedance Plethysmography" IEEE T Biomed Eng, 18:38–41, 1971

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[3] RJ Potton, "Reciprocity in optics" Prog Physics 67:717–754, 2004.