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The SVD of the linearized EIT problem on a disk

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Abstract: In this paper we calculate the right singular functions to the linearized EIT problem on homogeneous disk. We note the similarity to Zernike disk functions and the dependence on the mesh.

1 Introduction

The Singular Value Decomposition (SVD) helps us to understand the ill-conditioning of a linear inverse imaging problem, it characterizes consistent data and gives a basis for images in order of how easy their components are to recover. For some important inverse imaging problems an explicit analytic form of the SVD is known. For example, for the Radon transform on a disk the singular functions on the image side are Zernike disk functions while on the data side they are Fourier basis functions in two angular coordinates [1, Ch7]. Often symmetry conditions and commutation with a partial differential operator are the key to finding these explicitly. In many cases the forward operator is compact and therefore the spectrum is discrete, that is the singular values σ_i are indexed by an integer. Many important operators that appear in inverse imaging problems, such as restricted Hilbert transforms, do not have a discrete spectrum [2]. In that case numerical calculation of the SVD will depend heavily on the discretization used.

In this paper we calculate the singular functions of the linearized EIT forward problem on a uniform disk, discretized using a triangular mesh. We find that the right singular functions (on the image side) appear as two series. One very close to Zernike disk functions and the other highly oscillatory and concentrated near the boundary. The singular functions are sorted by their decreasing singular value and the two series are interspersed in a way that is highly dependent on the mesh.

2 Methods

We use a triangular mesh of the disk, and for maximal symmetry use trigonometric current drive and measurement patterns. In this discrete context the Fréchet derivative of the forward problem is approximated by the Jacobian matrix J (sensitivity), and is calculated using EIDORS v3.10 [4] with a unit background conductivity. The singular values of J are σ_i and right singular vectors v_i where

$$J^T J v_i = \sigma_i^2 v_i$$

and the σ_i are arranged in non decreasing order. The right singular vectors are approximately the average of the right singular functions in each triangle of the mesh and form an orthonormal basis for the discrete image space. The significance of the right singular vectors is that they are components of a conductivity image ordered by how difficult they are to recover from EIT measurement.

3 Results

As shown in fig. 1 we see that the right singular functions appear as two series: the most obvious being very close to the Zernike disk functions, the same as the right singular functions of the Radon transform on the disk. The other series is highly oscillatory and concentrated near the boundary. If the mesh is changed qualitatively the same functions appear but way the second series is interposed in the first varies.

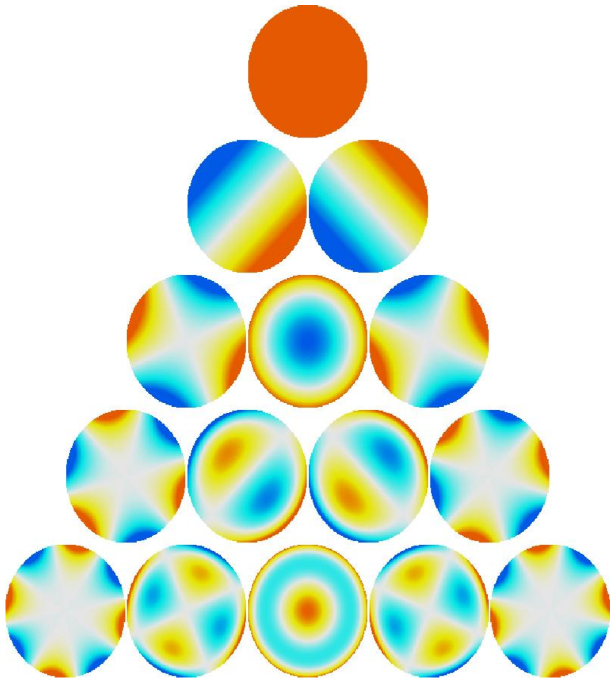

Figure 1: Images of right singular functions of EIT Jacobian of the unit disk, using a 17424-element 2D model with the four low-order sinusoidal drive and measurement patterns. *Left*: Zernike-like singular functions, corresponding to the singular values [1; 2,3; 4,12,5; 6,15,16,7; 8,19,21,20,9; 10,22,26,27,23,11; 17,28,30,32,31,29,18]. *Right*: sample of the mesh dependent singular functions concentrated near the boundary [37,40,46,49,58,105,126].

4 Conclusions

The identification numerically of the right singular functions as close to Zernike disk functions settles a problem raised when the SVD of EIT was first calculated in the second author's PhD [3]. The rotationally symmetric case can be shown to be equivalent to a Hilbert transform on adjacent intervals and has a continuous spectrum[2]. We conjecture that the variability of the singular values of the second series is due to partly continuous spectrum and it sounds a note of caution for the mesh dependence of EIT reconstruction.

References

[1] P.C. Hansen, J. S. Jørgensen, W.R.B. Lionheart, *Computed Tomography: Algorithms, Insight and Just Enough Theory*, SIAM 2021

[2] M. Bertola, E. Blackstone, A. Katsevich, and A. Tovbis. *Anal. Math. Phys*, 10:1–59, 2020.

[3] W.R. Breckon, Image Reconstruction in Electrical Impedance Tomography by , PhD Thesis Oxford Polytechnic, 1990

[4] A Adler and W.R.B Lionheart, Physiol. Meas., 27:S25-S42,2006.