# The SVD of the linearized EIT problem on a disk

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# The SVD

The **Singular Value Decomposition**(SVD)

- helps us to understand the **ill-conditioning** of a linear inverse imaging problem
- characterizes **consistent data**
- gives a **basis** for images in order of **how easy their components are to recover**

Some important inverse imaging problems an explicit analytic form of the SVD is known. For example, for the **Radon transform on a disk** the singular functions

- on the **image side** are **Zernike** disk functions
- on the **data side** they are **Fourier basis functions** in two angular coordinates [3, Ch7].

# The quest for SVD of linear 2D EIT

- WL was advised in 1985 'The first thing to do in EIT is calculate the SVD' by Jennifer Scott at Oxford
- First calculated numerically in [2] and presented at *3rd EU Workshop on EIT Copenhagen 1990* as keynote talk 'What can you see with EIT".
- No one has diagonalised linearised EIT analytically yet
- **New** The spectrum is **not discrete**
- **New** For the discrete part of the spectrum right singular functions look a bit like Zernike disk functions.
- **New** The non-discrete part can cause us problems with practical EIT reconstruction

Figure: The conference outing for the Copenhagen meeting was a trip on the schooner Halmø. Photo credit Brian Brown

# SVD of Radon I

Let $R$ be Radon transform integral along lines, $R^*$ backprojection operator then

$$R^* R = (-\Delta)^{-1/2}$$

where $\Delta = \nabla^2$ the Laplacian. On the plane $\mathbb{R}^2$ we can diagonalize using the Fourier transform

$$g(x) = R^* R[f](x) \qquad \hat{g}(\omega) = |\omega|^{-1} \hat{f}(\omega)$$

we say this has a *continuous spectrum* as any non-negative real number is an eigenvalue. On the unit disk $D = \{x : ||x| \leq 1\}$

$$R^* R Z_{n,k} = \frac{4\pi}{n+1} Z_{n,k}$$

so the reciprocal integers for a discrete spectrum and the Zernike disk functions (polynomial in $r$ and Fourier in $\theta$) are the eigenfunctions.

# SVD of Radon II

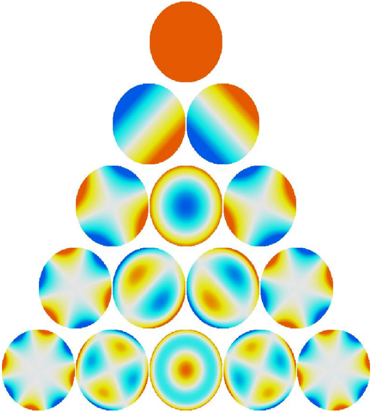

Figure: Zernike disk functions [3]. $Z_{n,r}(r,\theta) = p_{n,k}(r) \left\{ \begin{array}{c} \sin \\ \cos \end{array} \right\} k\theta$ where $p_{n,k}$ is a polynomial of order $n$. They are widely used in optics to describe lenses where they have cutesy names like 'astigmatism and defocus'.

# The Hilbert transform I

The Hilbert transform on the real line $\mathbb{R}$

$$
\begin{aligned}
H[f](y) &= \mathrm{P.V}\int_{-\infty}^{\infty} \frac{1}{y-x} f(x)\mathrm{d}x = \frac{1}{x} * f(x), \\
\widehat{H[f]}(\omega) &= -\mathrm{i}\,\mathrm{sgn}(\omega)\hat{f}(\omega)
\end{aligned}
$$

Restricted Hilbert transform

$$
H[f](y) = \mathrm{P.V}\int_{a}^{b} \frac{1}{y-x} f(x)\,\mathrm{d}x, \quad y \in [c, d]
$$

- If (  ) $a = c,\ b = d$
  $\longrightarrow$ spectrum is discrete.
- If (  ) $c = b$ (intervals abut)
  $\longrightarrow$ we have a continuous spectrum [1].

# Linearized EIT on a disk with trig patterns I

On a unit disk

- Potential for uniform conductivity $\phi_k^{\mathrm{trig}} = r^k \mathrm{trig} k\theta$ where $\mathrm{trig}$ is sin or cos.
- Fréchet derivative of forward problem at unit conductivity with perturbation in conductivity $\eta(r,\theta)$

$$K_{km}^{\cos}[\eta] = \int_D \eta(r,\theta) \nabla \phi_k^{\cos} \cdot \nabla \phi_k^{\cos} r \mathrm{d}r \mathrm{d}\theta$$

- to get the normal operator $N = K^* K$ we sum over $k, m$ this can be explicitly summed to give a kernel function for $N$ as a function of $r, \theta, r', \theta'$
- for the case $\eta(r)$ independent of theta this reduces to

$$\frac{4\,(rr')^2\left((rr')^2+1\right)}{\left(1-(rr')^2\right)^3}$$

# Linearized EIT on a disk with trig patterns II

- With some change of variables this can be written in terms of a restricted Hilbert transform from $[0, 1]$ to $[1, \infty)$ so we expect spectrum of $N$ to have a **continuous part**.
- The continuous part of the spectrum means that the numerical SVD will **depend on the discretization** of the disk.

# Numerical results

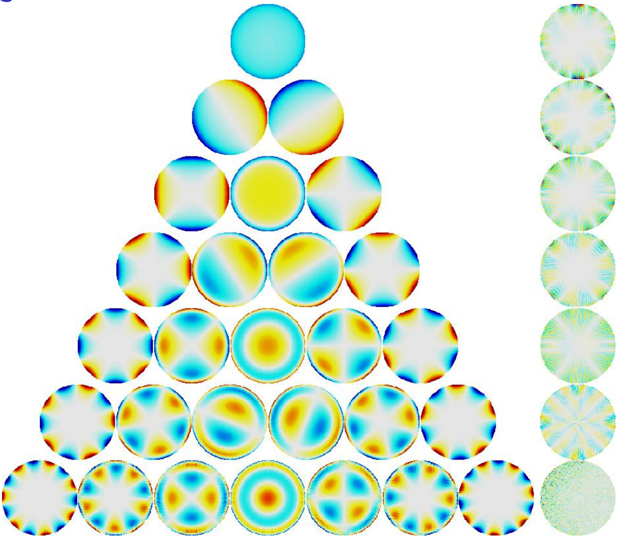

Figure: Singular functions of EIT Jacobian of the unit disk low-order sinusoidal patterns *Left*: Zernike-like *Right*: mesh-dependent

# Numerical results + Zernike

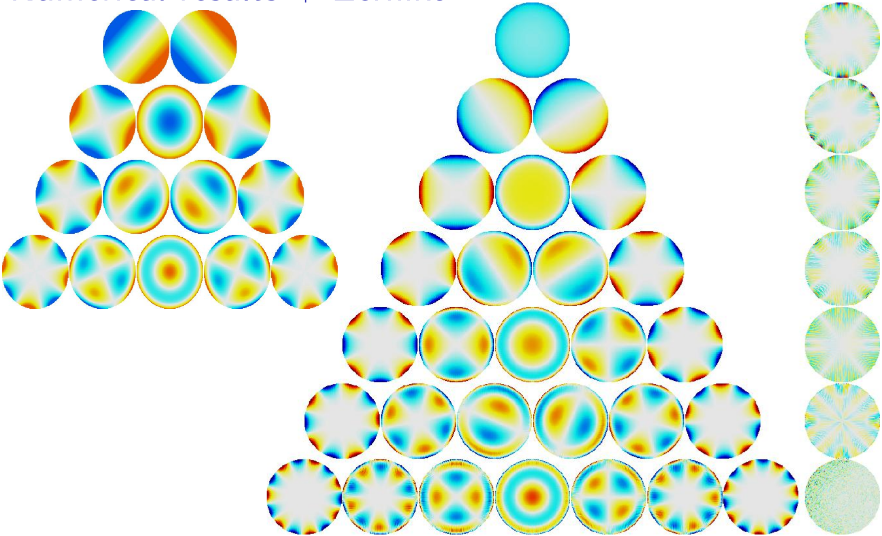

Figure: Singular functions of EIT Jacobian of the unit disk low-order sinusoidal patterns *Left*: Zernike-like *Right*: mesh-dependent

# Conclusions I

- There are right singular functions of linearized EIT on the disk that are like Zernike functions
- There are also some mesh dependent singular functions concentrated near the boundary that we think reflect a non discrete part of the spectrum
- With a regularized linear solution your images are likely to be mesh dependent
- More details in forthcoming preprint...

# References I


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