Accelerating Space-Time Regularized Reconstructions

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Abstract: Most modern reconstruction algorithms for EIT use regularization, using a penalty to impose spatial smoothness. Several approaches exist to also impose temporal smoothness. Here, we formulate spatio-temporal reconstruction in a simpler way that helps clarify the impact of parameter choices.

1 Introduction

First, we intruduce notation for spatial and then spatio-temporal (S-T) regularized time difference EIT reconstruction. S-T reconstruction can be formulated in (a) two stages (spatial then temporal) [3], (b) via an augmented S-T matrix [1], or (c) as a Kalman smoother [2]. Here we extend (b) to provide a simplified and efficient calculation.

Regularized time-difference EIT image reconstruction, seeks $\hat{m}$, an optimum image m, to minimize the norm

$$\|d - Sm\|_{\Sigma_n^{-1}} + \|m - m_0\|_{\Sigma_x^{-1}} \quad (1)$$

for data d, and sensitivity matrix S. Measurement noise is Gaussian $\sim \mathcal{N}(0, \Sigma_n)$, and the image $\sim \mathcal{N}(m_0 = 0, \Sigma_x)$.

Using $\Sigma_n^{-1} = W^t W$, and and $\Sigma_x^{-1} = \lambda L^t L$, we introduce auxiliary ("whitened") data, y, and image, x.

$$\hat{m} = L^{-1}\hat{x}, \qquad \hat{x} = (J^t J + \lambda I)^{-1} J y, \qquad y = Wd \quad (2)$$

where $S = W^{-1} J L$. This may be seen from the solution to (1), $\hat{m} = (S^t \Sigma_n^{-1} S + \Sigma_x^{-1})^{-1} S^t \Sigma_n^{-1} d$, and thus $\hat{x} = L[L^t (J^t W^{-t} \Sigma_n^{-1} W^{-1} J + \lambda I) L]^{-1} L^t J^t W^{-t} \Sigma_n^{-1} W^{-1} y$.

A matrix formulation of S-T regularization solves an augmented-matrix forward problem, $\tilde{y} = \tilde{J}\tilde{x}$,

$$\tilde{x} = \begin{bmatrix} x_f \\ x_c \\ x_p \end{bmatrix} \; \tilde{y} = \begin{bmatrix} y_f \\ y_c \\ y_p \end{bmatrix} \; \tilde{J} = \begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix} = I \otimes J, \quad (3)$$

where f, c, p are future, current and past frame values, with respect to the reconstruction frame of interest. Here $\tilde{\Sigma}_n = I \otimes \Sigma_n$ (since noise is independent between frames) and $\tilde{\Sigma}_x = \Gamma \otimes \Sigma_x$, where Γ is symmetric with diagonal 1 and dereasing off-diagonal values. For example

$$\Gamma = \begin{bmatrix} 1 & \gamma & \gamma^2 \\ \gamma & 1 & \gamma \\ \gamma^2 & \gamma & 1 \end{bmatrix}, \; \Gamma^{-1} = \tfrac{1}{1-\gamma^2} \begin{bmatrix} 1 & -\gamma & 0 \\ -\gamma & 1+\gamma^2 & -\gamma \\ 0 & -\gamma & 1 \end{bmatrix} \quad (4)$$

where $0 \leq \gamma < 1$ represents the correlation between frames, and Γ^{-1} is tri-diagonal. The S-T reconstruction is thus

$$\tilde{x} = \left(I \otimes J^t J + \Gamma^{-1} \otimes \lambda I\right)^{-1} (I \otimes J^t)\tilde{y} = \tilde{R}\tilde{y}, \quad (5)$$

where $\tilde{R}$ is the augmented S-T reconstruction matrix. This S-T inverse matrix grows large with the number of frames.

2 Spatio-temporal inverse

The relation $(I + \delta)^{-1} = I - \delta + \delta^2 \ldots$ (valid when the largest eigenvalue of δ is < 1) may be used to simplify (5).

$$\begin{aligned} \tilde{R} &= \left(I \otimes (J^t J + \lambda I) + (\Gamma^{-1} - I) \otimes \lambda I\right)^{-1} (I \otimes J^t) \\ &= \left(I \otimes M^{-1} + D \otimes \lambda I\right)^{-1} (I \otimes J^t) \\ &= \left((I \otimes M^{-1})[I + (I \otimes M)(D \otimes \lambda I)]\right)^{-1} (I \otimes J^t) \\ &= (I + D \otimes \lambda M)^{-1} (I \otimes M^{-1})^{-1} (I \otimes J^t) \\ &= (I + \delta)^{-1} (I \otimes M)(I \otimes J^t) \\ &= \left(I - \delta + \delta^2 - \ldots\right)(I \otimes M J^t) \quad (6) \\ &= I \otimes M J^t - \lambda D \otimes M^2 J^t + (\lambda D)^2 \otimes M^3 J^t - \ldots \end{aligned}$$

where $M = (J^t J + \lambda I)^{-1}$, $D = \Gamma^{-1} - I$ and $\delta = \lambda D \otimes M$. Reconstruction in (6) is first a spatial inverse, followed by temporal smoothing terms, where each time step is successively "filtered" by λM. Contributions from the past and future are thus blurred both in time and space.

Using the singular-value decomposition, $J = U \Sigma V^t$, and $M = V(\Sigma^2 + \lambda I)^{-1} V^t$. Thus each k^{th} term $M^k J^t$ = $V(\Sigma^2 + \lambda I)^{-k} \Sigma U^t$. Finally, the S-T image at frame t, $\tilde{m}_t = L^{-1}\tilde{x}_t$ can be calculated from post-filtering spatial-only images $\hat{x}_{t+i}$ at offsets i from the current frame, as

$$\tilde{x}_t = \hat{x}_t - \sum_{i=-T}^{T} \left([\lambda D]_i M + [(\lambda D)^2]_i M \ldots\right) \hat{x}_{t+i} \quad (7)$$

where $[D]_i$ represents the i^{th} offset on the centre row; for $D \in \mathbb{R}^{2T+1 \times 2T+1}$, $[D]_i$ is the $(T{+}1, T{+}1{+}i)^{\text{th}}$ element.

3 Results and Discussion

Fig. 1 shows sample results. An S-only reconstruction performs equally for moving and still targets, but with worse noise. Using temporal, then spatial regularization [3] offers improvements, like the S-T solution shown last, but the moving target is blurred in space.

In conclusion, we develop an efficient formulation for the S-T regularization of [1]. This approach clarifies how temporal regularization results in blurring in both space and time for each time-offset. This differs from successive S then T regularization [3], which does not introduce the additional S blur.

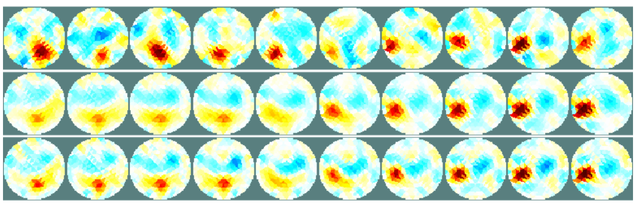

Figure 1: Images of a rotating contrast (with added noise) which stops at the 7^{th} frame. *Top*: Spatial-only solution *Middle*: Temporal then spatial solution, *Bottom*: S-T solution, via (7)

References

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