Inequality Constrained EIT Modelling and Inversion

Nick Polydorides[1], Taweechai Ouypornkochagorn[1], Hugh McCann[1]
[1]University of Edinburgh, Edinburgh, UK. N.Polydorides@ed.ac.uk

Abstract: We consider modelling and imaging in EIT under some a priori inequality constraints on conductivity. Instead of constrained optimisation, we reformulate the model equations with respect to some monotone functions that encompass these constraints. We address the cases of positivity and boundness, posing the inverse problem using regularised nonlinear least squares. The results demonstrate significant spatial resolution improvements.

1 Introduction

The motivation for this work is to introduce a robust and simple to implement computational scheme appropriate for the cases where inequality constraints on the electrical conductivity are available a priori. Although several constrained optimisation algorithms are readily available [1] the methods presented here are appealing for their implementation simplicity. In the medical EIT setting, this type of prior information is likely attained through the literature on the targeted physiological phenomenon [2].

2 Methods

The scalar conductivity function $\sigma : B \to \Re$, is related to a finite set of real valued observations $\zeta \in \Re^m$ through the model

$$\zeta = M(\sigma) + n, \tag{1}$$

where $M : \sigma \mapsto \zeta$ is the nonlinear forward EIT mapping and n is some additive noise corrupting the data. Suppose further that σ is a priori known to belong within a subspace S. To enforce this assumption we introduce the injective mapping $\nu : \Re(B) \to S(B)$ from the space of real functions over the domain onto a subspace $S(B) \subseteq \Re(B)$, such that

$$\sigma(\mathbf{x}) \doteq \nu[\gamma(\mathbf{x})], \tag{2}$$

and conversely $\gamma(\mathbf{x}) = \nu^{-1}[\sigma(\mathbf{x})]$, where $\nu^{-1} : \sigma \mapsto \gamma$ always exists and it is continuous. Based on this one may formulate another forward model $F : \gamma \mapsto \zeta$, such as

$$\zeta = F(\gamma) + n, \quad \text{where} \quad F(\gamma) = (M \circ \nu)(\gamma). \tag{3}$$

2.1 Positivity

To impose positivity prompts to consider the subspace $S \doteq \{\sigma(\mathbf{x}) \in B | 0 < \sigma \leq \infty\}$ where a suitable choice for ν is the exponential function scaled by a relaxation factor $\kappa \neq 0$

$$\nu[\gamma(\mathbf{x})] \doteq e^{\gamma(\mathbf{x})/\kappa}, \quad \mathbf{x} \in B, \tag{4}$$

Under this transformation notice that the perturbations in the original and surrogate unknown functions, from reference points σ_*, γ_* are related by

$$\delta\sigma \doteq e^{\gamma/\kappa}(e^{\delta\gamma/\kappa} - 1). \tag{5}$$

To linearise the model (3) at (σ_*, γ_*), we appeal to the chain differentiation rule,

$$\partial_\gamma F(\gamma_*)\, \delta\gamma = \partial_\gamma (M \circ \nu)(\gamma_*)\, \delta\gamma = \partial_\sigma M(\sigma_*)\, \sigma_* \kappa^{-1}\, \delta\gamma, \tag{6}$$

where $\partial_\sigma M(\sigma_*)$ is the Jacobian of M. In this way, the linear approximation of the inverse problem for γ becomes

$$\delta\zeta = \partial_\sigma M(\sigma_*)\, \sigma_* \kappa^{-1}\, \delta\gamma + n. \tag{7}$$

2.2 Boundness

As an extension of the above scheme we consider mapping the conductivity into the subspace $S \doteq \{\sigma(\mathbf{x}) \in B | 0 < p \leq \sigma \leq t < \infty\}$ for some a priori known bounds $p < t$, using the scaled logistic regression function

$$\nu[\gamma(\mathbf{x})] \doteq p + \frac{t-p}{1+e^{-\gamma(\mathbf{x})/\kappa}}. \tag{8}$$

In this instance the perturbations $\delta\sigma$ and $\delta\gamma$ from a fixed reference are related via

$$\delta\sigma = \nu(\gamma + \delta\gamma) - \nu(\gamma) = w(\gamma, \delta\gamma)\nu_1(-\gamma), \tag{9}$$

where $w(\gamma, \delta\gamma) \doteq \frac{(t-p)(1-e^{-\delta\gamma/\kappa})}{(1+e^{-(\gamma+\delta\gamma)/\kappa})}$, and $\nu_1(\gamma) \doteq \frac{1}{1+e^{-\gamma/\kappa}}$. Appealing to the chain differentiation now yields the linearised problem for $\delta\gamma$ as

$$\delta\zeta = \partial_\sigma M(\sigma_*)\left(\nu(\gamma_*) - p\right)\nu_1(-\gamma_*)\kappa^{-1}\, \delta\gamma + n. \tag{10}$$

3 Results

To test the performance of our scheme we formulate the inverse problems as least squares problems based on (1) and (3) respectively. We then apply the Gauss-Newton algorithm for a few iterations while we regularise the linear problems using smoothness imposing regularisation.

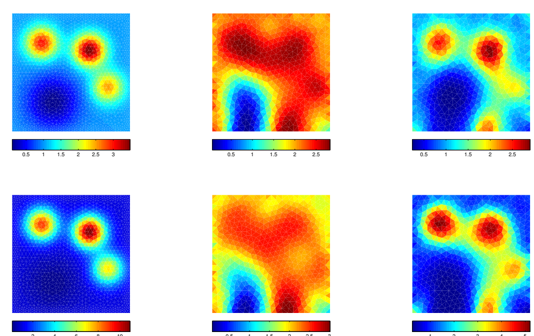

Figure 1: Top row, the target σ and the reconstructions using two Gauss-Newton iterations with smooth priors [2] on the original (middle) and positivity preserving model (right). Below, the corresponding images for a different target by implementing two iterations on the bound preserving model. Regularisation matrices and parameters are kept fixed to aid comparison of the results.

4 Conclusions

This work demonstrates how to obtain a constrained solution of the inverse problem in EIT using unconstrained optimisation. The proposed framework is computationally simple and can be used in conjunction with various inversion algorithms.

References

[1] D. P. Bertsekas, Constrained optimization and Lagrange multiplier methods, Athena Scientific, 1996.

[2] D. S. Holder (ed), Electrical Impedance Tomography: methods, history and applications, Institute of Physics, 2004.