Resolution guarantees in EIT including random and systematic errors

Bastian Harrach[1] and Marcel Ullrich[1]

[1]Department of Mathematics, University of Stuttgart, Stuttgart, Germany, harrach@math.uni-stuttgart.de and marcel.ullrich@mathematik.uni-stuttgart.de

Abstract: To improve the practical applicability of electrical impedance tomography is a great ongoing challenge. Theoretical identifiability results exist for noiseless continuous boundary measurements. However, little is known about what can be achieved with a finite number of realistically modelled electrodes in a setting including modelling and measurement errors. In this paper, we sketch how to derive rigorous resolution guarantees for such settings.

1 Introduction

Notation: χ_M denotes the characteristic function of a set M and $\mathrm{eig}(A)$ the set of eigenvalues of a square matrix A.

We consider a conductive object $\Omega \subseteq \mathbb{R}^n$ ($n \in \{2,3\}$) with a conductivity distribution

$$\sigma : \Omega \to \mathbb{R}, \quad \sigma(x) = \sigma_B(x)\chi_{\Omega\setminus D}(x) + \sigma_D(x)\chi_D(x), \quad (1)$$

where $\sigma_B(x)$ is the background conductivity and $\sigma_D(x)$ the inclusion conductivity of an inclusion $D \subseteq \Omega$. The inclusion is characterized by a contrast to the background with

$$\inf_{x\in D} \sigma_D(x) \geq \sigma_{D\min} > \sup_{y\in\Omega\setminus D} \sigma_B(y), \quad \sigma_{D\min} \in \mathbb{R}. \quad (2)$$

Furthermore, let $(\omega_1, \omega_2, \cdots, \omega_N)$ be a resolution partition of Ω (see Figure 1). In Section 3, we sketch how to verify if a realistically modelled measurement setting (see Section 2) yields enough information to design an inclusion detection method that fulfils the following guarantee.

Resolution guarantee (RG):

(a) A resolution element ω_i will be marked if $\omega_i \subseteq D$.

(b) No resolution element will be marked if $D = \emptyset$.

2 The measurement setting

The setting is given by current-voltage measurements on a finite number of (almost perfectly conductive) electrodes $E_1, E_2, \ldots, E_L$. We assume that a contact layer between each electrode E_i and Ω leads to a contact impedance $z^{[i]}$.This setting is mathematically modelled by the complete electrode model (CEM), cf. [1]. For a conductivity distribution σ and contact impedances given by the components of $z \in \mathbb{R}^L$, the measurement matrix is defined by

$$R(\sigma,z) = \left(R^{[i,j]}(\sigma,z)\right)_{i,j=1}^{L-1} \in \mathbb{R}^{L-1\times L-1}, \quad (3)$$

where the components $R^{[i,j]}(\sigma,z)$ are given by the measurements as in Fig. 1. The matrix $R(\sigma,z)$ is symmetric, cf. [1].

To allow for **modelling and measurement errors**:

(a) The background conductivity $\sigma_B(x)$ is given approximately by $\sigma_0(x)$ with $\|\sigma_B - \sigma_0\|_\infty \leq \varepsilon \in \mathbb{R}$.

(b) The vector z (contact impedances) is given approximately by z_0 with $\|z - z_0\|_\infty \leq \gamma \in \mathbb{R}$.

(c) There are noisy measurements $R_\delta(\sigma,z)$ given with an absolute noise level $\delta \geq \|R(\sigma,z) - R_\delta(\sigma,z)\|_2$, $\delta \in \mathbb{R}$. Possibly replacing $R_\delta(\sigma,z)$ by its symmetric part, we can assume that $R_\delta(\sigma,z)$ is symmetric.

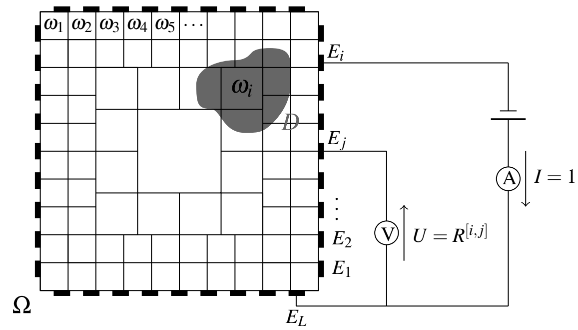

Figure 1: Setting with a sample resolution for $\Omega = [-1,1]^2$.

3 Verification of the resolution guarantee

Let $\sigma_0(x)$, z_0, ε, γ, δ and $\sigma_{D\min}$ be given. We define

$$\sigma_{B\min}(x) := \sigma_0(x) - \varepsilon, \quad \sigma_{B\max}(x) := \sigma_0(x) + \varepsilon, \quad (4)$$

$$z_{\min} := z_0 - \gamma(1,\ldots,1), \quad z_{\max} := z_0 + \gamma(1,\ldots,1), \quad (5)$$

$$\tau_i(x) := \sigma_{B\min}(x)\chi_{\Omega\setminus\omega_i}(x) + \sigma_{D\min}\chi_{\omega_i}(x) \quad (6)$$

for $i \in \{1,2,\ldots,N\}$. Then the **RG is possible** if

$$\max_{i=1}^{N} \min \mathrm{eig}\left(R(\tau_i, z_{\max}) - R(\sigma_{B\max}, z_{\min})\right) < -2\delta. \quad (7)$$

The proof is based on the monotonicity relation

$$\sigma_1 \leq \sigma_2,\ z_1 \geq z_2 \quad \Rightarrow \quad R(\sigma_1,z_1) - R(\sigma_2,z_2) \geq 0. \quad (8)$$

The main idea is to consider (7) as a worst-case scenario test for Algorithm 1 (cf. [2] for $\gamma = 0$).

Algorithm 1: Mark element ω_i if

$$\min \mathrm{eig}\left(R(\tau_i, z_{\max}) - R_\delta(\sigma,z)\right) \geq -\delta. \quad (9)$$

3.1 Numerical results

Let Ω be given with a resolution partition as in Fig. 1. Furthermore, let $\sigma_0 \equiv 1$ and $z_0 = (1,\ldots,1) \in \mathbb{R}^L$ be approximations of the background conductivity $\sigma_B(x)$ and the vector z (contact impedances), respectively. Additionally, let $\sigma_{D\min} = 2$ be a lower bound of the inclusion conductivity.

Then (7) is fulfilled for a background error of $\varepsilon = 1\%$, an absolute measurement noise level of $\delta = 0.9\%$ and exactly given contact impedances ($\gamma = 0$). Hence, the RG is possible. In particular, Algorithm 1 fulfils the RG.

The results can be extended to the case of approximately known contact impedances.

4 Conclusion

This paper presents the possibility of a rigorous resolution guarantee for a realistically modelled electrode measurement setting including modelling and measurement errors. The resolution guarantee can be verified by a simple test.

References

[1] Somersalo E, Cheney M, Isaacson D. *SIAM J Appl Math* **52**(4):1023–1040, 1992

[2] Harrach B, Ullrich M. In *Journal of Physics: Conference Series*, vol. 434, 012076. IOP Publishing, 2013