A Finite Difference Solver for the D-bar Equation

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Abstract: A finite difference scheme is introduced to solve the D-bar equation. The D-bar equation arises in electrical impedance tomography (EIT) when using the complex geometrical optics solutions to recover the conductivity within a body. This scheme is second-order and is first used on a test equation for error analysis and then used to reconstruct EIT images using the D-bar method.

1 Introduction

In 1996, A. Nachman [1] developed the D-bar method, which proved that the inverse conductivity problem in 2-D, described by A. Calderón [2], has a unique solution. This method reduces to finding the solutions to the D-bar equation

$$\bar{\partial}_k \mu(z,k) - \frac{1}{4\pi\bar{k}}\mathbf{t}(k)e^{\mathrm{i}(kz+\bar{k}\bar{z})}\bar{\mu}(z,k) = 0 \tag{1a}$$

$$\lim_{|z|,|k|\to\infty} \mu(z,k) = 1, \tag{1b}$$

where $z \equiv x+\mathrm{i}y$, $k \equiv k_1 + \mathrm{i}k_2$, $\bar{\partial}_k \equiv \frac{1}{2}(\frac{\partial}{\partial k_1} + \mathrm{i}\frac{\partial}{\partial k_2})$ and $\mathbf{t}(k)$ is the non-physical scattering transform, which contains all data information. The conductivity, γ, can be recovered by the relation $\gamma^{\frac{1}{2}}(x,y) = \lim_{|k|\to 0}\mu(z,k)$. Nachman suggests solving an equivalent integral equation to find μ. Current numerical implementations of the D-bar method solve these integral equations as is done in [3–5]. We seek to solve the D-bar equation (1a) as a partial differential equation using finite differences.

2 Methods

To account for the complex conjugate operator on μ in (1a), we solve the equivalent system of equations found by equating the real and imaginary parts of (1a). Thus, we seek to solve

$$\frac{1}{2}(\frac{\partial}{\partial k_1}u - \frac{\partial}{\partial k_2}v) - (au+bv) = 0 \tag{2a}$$

$$\frac{1}{2}(\frac{\partial}{\partial k_2}u + \frac{\partial}{\partial k_1}v) - (bu+av) = 0 \tag{2b}$$

where $\mu(z,k) \equiv u(z,k) + \mathrm{i}v(z,k)$ and $\frac{1}{4\pi\bar{k}}\mathbf{t}(k)e^{\mathrm{i}(kz+\bar{k}\bar{z})} \equiv a(z,k) + \mathrm{i}b(z,k)$. We approximate the first derivatives in the D-bar operator using centered finite differences. This leads to an $O(h^2)$ truncation error for a uniform mesh spacing, h. We truncate the complex domain to a finite domain $\Omega = [-R,R]^2$ for implementation, and impose the numerical boundary condition approximation $\mu(z,k)|_{\partial\Omega} = 1$. The resulting scheme reduces to solving a $2N^2$x$2N^2$ linear system, as in [3–5], where $h = 2R/(N+1)$. Here the system is sparse with $O(N^2)$ non-zero entries.

3 Results

To test our solver, we use the test function $\mu_1(z,k) = e^{-|k|^2-|z|^2-2\mathrm{i}(k_1k_2+xy)} + 1$ for fixed z and assume no scattering, resulting in the equation $\bar{\partial}_k\mu_1 = f$ with boundary condition $\mu_1|_{\partial\Omega} = g$. Thus, we solve (2) but with the right hand sides of (2a) and (2b) replaced by $\alpha(z,k)$ and $\beta(z,k)$, respectively, where $f = \alpha + \mathrm{i}\beta$ and $a = b \equiv 0$. A plot of the relative errors when solving (1a) can be found in 1a. Note the plateau in error that occurs for $R = 2$. This is caused by the imposed Dirichlet boundary condition on the finite domain, the approach also taken by [3–5]. For small enough h, the other error plots will also plateau, but will decrease with order two until that point. This suggests that the scheme converges with order two, but further analytic work is required to prove this. Figure 1b shows cross-sections of reconstructions of a concentric circle target with radius 0.3 inside a unit circle. The error plateau is evident since the reconstructions are almost identical for larger values of N. Despite this plateau, the conductivity is reconstructed. Improvements in the reconstruction can be made by adjusting R and a regularization parameter in the D-bar method. This solver also works when reconstructing images using experimental data.

References

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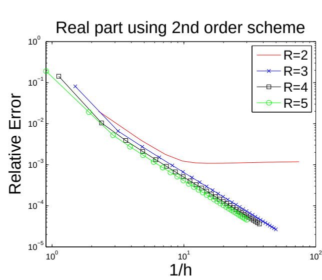
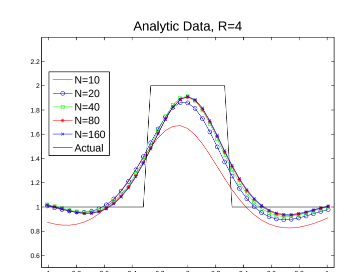


(a) Plateaus in the error occur because of the boundary approximation.



(b) Reconstruction of a concentric target with analytically calculated scattering transform.

Figure 1