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Advances in Applied Probability, Vol. 20, No. 3. (Sep., 1988), pp. 646-676.

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Fri Oct 13 16:15:06 2006

Adv. Appl. Prob. **20**, 646–676 (1988)
Printed in N. Ireland
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STOCHASTIC THEORY OF A FLUID MODEL OF PRODUCERS AND CONSUMERS COUPLED BY A BUFFER

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Abstract

This paper analyzes, derives efficient computational procedures and numerically investigates the following fluid model which is of interest in manufacturing and communications: m producing machines supply a buffer, n consuming machines feed off it. Each machine independently alternates between exponentially distributed random periods in the 'in service' and 'failed' states. Producers/consumers have their own failure/repair rates and working capacities. When the buffer is either full or empty some of the machines in service are not utilized to capacity; otherwise they are fully utilized. Our main result is for the state distribution of the Markovian system in equilibrium which is the solution of a system of differential equations. The spectral expansion for its solution is obtained. Two important decompositions are obtained: the eigenvectors have the Kronecker-product form in lower-dimensional vectors; the characteristic polynomial is factored with each factor an explicitly given polynomial of degree at most 4. All eigenvalues are real. For each of various cases of the model, a system of linear equations is derived from the boundary conditions; their solution complete the spectral expansion. The count in operations of the entire procedure is $O(m^3n^3)$: independence from buffer size exemplifies an important attraction of fluid models. Computations have revealed several interesting features, such as the benefit of small machines and the inelasticity of production rate to inventory. We also give results on the eigenvalues of a more general fluid model, reversible Markov drift processes.

REVERSIBLE MARKOV DRIFT PROCESSES; KRONECKER-PRODUCT FORM

1. Introduction

This paper analyzes and derives efficient computational procedures for the system sketched in Figure 1.1 in which fluid is produced by m machines, transferred to a buffer and consumed by n other machines. In this model, which is of interest in manufacturing and communications, the fluid is produced, transferred and consumed continuously in time and, unlike queueing models, no attempt is made to preserve the identity of individual jobs or packets which, for purposes of analogy, correspond to the atoms of the fluid. Independent random failures of the machines and their repair are allowed: each producing and consuming machine alternates between independent, exponentially distributed random periods in the 'in service' and 'failed' states. The average time spent in the in service state is $1/f_1$ for each

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Received 31 October 1986; revision received 28 May 1987.

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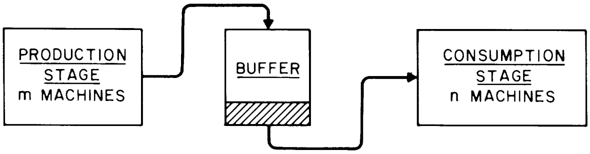

Figure 1.1

producer and $1/f_2$ for each consumer; the average time spent in the failed state is $1/r_1$ for each producer and $1/r_2$ for each consumer. Thus f corresponds to failure rate and r to repair rate. While in service each producer is capable of producing c_1 units of fluid per unit time, and each consumer is capable of consuming c_2 units of fluid per unit time; the fixed constants c_1 and c_2 are termed *working capacities.*

Models with buffers of both finite and infinite capacity are analyzed in the paper. If the buffer is full then the rate of transfer of fluid from the production stage to the buffer is the lesser of two prevailing rates, the cumulative working capacities of the producers in service and the cumulative working capacities of the consumers in service. This is also the transfer rate from the buffer to the consumption stage in the event of an empty buffer. To satisfy the conditions stated, when the buffer is full the working rates of one or more of the producers in service are curtailed as necessary, and when the buffer is empty the same form of control is exerted on the consumers in service. The analysis is insensitive to the discipline and identity of the machines which are not fully utilized. If the buffer is neither full nor empty then all machines in service are utilized to capacity. Clearly the finite buffer introduces quite subtle coupling between the producers and consumers. This leads in a natural way to the study of the dual system which is obtained by reversing the fluid flow; *flow-reversibility* and distributional symmetries between the primary and the dual are reported. Also note that in Figure 1.1 the variety of transfer rates to and from the buffer is large, namely, $(m+1)(n+1)$ which is the number of machine states.

The system is Markovian with the state given by the 3-tuple $(x; i, j)$ where x is a real variable which corresponds to the buffer content, i and j are integers, $0 \leqq i \leqq m$ and $0 \leqq j \leqq n$, which respectively correspond to the number of producers and consumers in service. Our main result is for the distribution of the state of the system in equilibrium. The buffer content distribution and its moments are easily obtained from it. In the finite buffer problem a quantity of interest is the realized mean production (and consumption) rate and this is obtained. An important paradigm in the manufacturing literature is the *balanced* system in which the mean cumulative production and consumption capacities are equal; we give results for both the balanced and unbalanced systems.

Fluid models have been proposed and analyzed in the manufacturing and communications literature at least since 1962 (see below). However, the model and its analysis in this paper differ in important respects from prior work. The usual manufacturing model of a single stage is a single machine subject to failures and repairs. Instances are the models in Wijngaard [36] and Sevastyanov [32]. Our observations on real semiconductor manufacturing is that interruptions, due to a variey of causes, make the transfer rates and the working rates of each stage have considerable variety. The multiple machine model of a stage is a move to correct this shortcoming. It should also be noted that the use of fluid models in manufacturing, especially in modern automated flexible systems, is relatively uncommon and the subject of active research. Future usage will depend on the outcome of current investigations.

The model has applications in communications—integrated services, in particular. Obviously 'machines' and 'failures' have to be interpreted differently, as sources and channels, and service interruptions, respectively. An example is provided by an integrated voice and data system [11], [14], [16], [21] in which the new element is the multiplicity of outgoing channels from a common buffer. Data, unlike voice, is buffered, while voice has priority of use over the outgoing channels. By mapping each consuming machine to an outgoing channel, its failures to the latter's use by voice and the producers to intermittent data sources we obtain an identification with the model given here. We note that the assumption of an infinite buffer has been commonly made in communications; it is justified when buffers are large, overflow probabilities small and the emphasis is on delay. In contrast, the finite buffer is the essential element in manufacturing models.

Returning to the manufacturing models for illustrations, we comment briefly on the presence of *multiple time scales* which provides the *justification* for fluid models. Consider a not atypical situation where the mean time between repair and failure for a machine is measured in several days, perhaps weeks, and mean repair time in days. Another source of randomness is the variability in service times of individual jobs. Eliminating contributions from failures/repairs, the mean service time may be of the order of minutes, or maybe half-hours, and the standard deviation only a fraction of the mean. In such a case attempts at analyzing a composite system are not advisable, first, because of the size of the computational task and, second, because of numerical instability, a natural consequence of the embedded multiple time scales. A second course is to represent the generator of the Markov chain as a function of a small parameter obtained as the ratio of the time scales. The analysis then proceeds along well-studied lines [7], [9], [10] in rough terms, by successively averaging over faster time scales. The computational complexity is still considerable. A third course is provided by fluid models which, in the stationary regime, consist of ordinary differential equations. Kurtz [25] has shown that, under fairly general conditions, the limiting process obtained by a scaling of time and the state space, i.e. job size, of a typical Markov chain is the solution of ordinary differential equations.† Kurtz also gives error estimates.

The *promise* of fluid models is in the computational complexity which is

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† However, the resulting deterministic models, often described as fluid, are to be distinguished from stochastic fluid models such as the one in this paper.

independent of buffer capacities and polynomial in the number of machine states. The *difficulties* associated with fluid models are, first, eigenvalues of both signs and, second, boundary conditions that are implicit and two-point. The former makes it necessary to be careful about numerical errors, as noted in [2] and [22]. The method proposed here requires the computation of $\{a_l, z_l, \phi(l)\}$ in the following typical spectral expansion for the state distribution:

$$\pi(x) = \sum_l a_l \exp(z_l x)\phi(l).$$

Fortunately, as discussed below, decomposition results for our model allow the eigenvalue and eigenvector pairs $\{z_l, \phi(l)\}$ to be accurately computed. The coefficients $\{a_l\}$ are obtained from the boundary conditions.

Our contributions are as follows. A decomposition result is that ϕ, any eigenvector of the system, has the Kronecker-product form $\phi_1 \otimes \phi_2$, where the dimensions of ϕ_1 and ϕ_2 are respectively $(m+1)$ and $(n+1)$. A second decomposition result factors the characteristic polynomial with each factor an explicitly given polynomial of degree at most 4. Thus the problem of computing all the eigenvectors decomposes into one of finding the roots of about $mn/4$ simple polynomials. All eigenvalues are real. For each of the various cases of the model, a system of linear equations is derived from the boundary conditions; their solution gives the coefficients $\{a_l\}$. Solving the linear equations constitutes the main work in the procedure. The count in operations of the entire procedure is $O(m^3n^3)$. We have written two computer programs which implement the procedures for the finite buffer and the infinite buffer. These programs have been tested for mn up to 100 and we report on interesting features that were observed, such as the benefit of small machines and the inelasticity of the mean production rate to inventory.

We also give a result which enumerates the eigenvalues according to sign and specifies all forms of the spectral expansion for a general class of fluid models, which we call *reversible Markov drift processes* and which subsumes the particular model described earlier.

We turn to prior related work. The buffer-independent model of the machines is, in a natural sense, the direct product of two Ehrenfest urn models. The time-dependent behavior in the Ehrenfest urn problem has been solved by Karlin and McGregor [17].

In the manufacturing literature Wijngaard [36] has analyzed the model treated here for the case of one producing machine and one consuming machine. Sevastyanov [32] has given an innovative approximation for a production line model of many stages which incorporates the exact solution of a pair of stages; however, various restrictions are imposed: each stage has one machine, the working capacities of all machines are identical, at most one machine may be in the failed state at any time. See [28] for an evaluation of Sevastyanov's technique. For a fairly recent survey of manufacturing models with two stages see [5]. For a perspective on queueing-theoretic models see [12], [21], [37]. There is a large literature on approximations for multistage queueing models of production lines; see [6] and [1]. For symmetry relations in simple production lines, see [29]. All of the above-mentioned works assume finite buffers.

Latouche and Neuts [26], [30] give a queueing model with algorithmic solutions which has, in tandem, a stage with several parallel producer-servers, a finite buffer and a second stage with several consumer-servers. The point process arriving at the producers is Poisson and the service-time distributions are exponential. Note that this queueing model does not have disparate time scales.

In communications, Kosten's pioneering work [22] analyzes a fluid model with a single fail-proof consumer, infinite buffer and a limiting case of our model for the production stage in which $m \to \infty$, $r_1 \to 0$, mr_1 fixed. Two early and independent analyses of fluid models are due to Cohen [8] and Halfin and Segal [13], [15]. In [14] and in [13], [15] the equations requiring solutions are in transformed variables and the stationary distribution is specified by its Laplace–Stieltjes transform, an approach different from ours. Kosten's work was followed by [2] in which the model for the production stage was replaced by one identical to the present one. Two results were given: the characteristic polynomial was factored with each factor an explicitly given quadratic; the solution of the boundary equations was given in closed-form. Generalization of the second result remains illusive. Weiss [35] has developed a large-deviation theory for the model in [2] and its extensions which, when applicable, is numerically more efficient. McKenna and Mitra [27] have obtained the large-time behavior of the time-dependent model which is a system of partial differential equations. Kosten has recently extended [2] by allowing, in [23], multiple classes of producers and, in [24], the in service distribution to be of phase type; the consumption stage has one fail-proof machine. The eigenvalue calculations have points in common with the results here. The emphasis is on a hybrid of analysis and simulations which gives the large x behavior of the probability of buffer content exceeding x. All the works mentioned above assume an infinite buffer.

2. Mathematical models

2.1. *Buffer-independent, time-dependent model of machines.* Since our interest in the producing machines is restricted to the number in service, let us consider a process with state i, $i = 0, 1, \cdots, m$, to correspond to this number. This process is Markovian, and as the transitions out of state i are only to state $i-1$, with rate if_1, and to state $i+1$, with rate $(m-i)r_1$, the process is birth-and-death. It is identical to the process obtained in the classical Ehrenfest urn model [17]. Let the state probabilities be

$$p_1(t; i) \triangleq \Pr\,[i \text{ producers in service at time } t],\quad 0 \leqq i \leqq m,\ t \geqq 0 \tag{2.1}$$

and $\boldsymbol{p}_1(t) \triangleq \{p_1(t;0)p_1(t;1)\cdots p_1(t;m)\}$. These probabilities satisfy

$$\frac{d}{dt}\boldsymbol{p}_1(t)=\boldsymbol{p}_1(t)\boldsymbol{M}_1 \tag{2.2}$$

where $\boldsymbol{M}_1$ is conveniently specified as a member of the class of generators associated with the Ehrenfest urn model:

$$\boldsymbol{M}_1 \triangleq \boldsymbol{M}(m, r_1, f_1) \tag{2.3}$$

where, generally,

$$\boldsymbol{M}(k,\lambda,\mu) \triangleq \begin{bmatrix} -k\lambda & k\lambda & & & & \\ \mu & -\{\mu+(k-1)\lambda\} & (k-1)\lambda & & & \\ \vdots & & & & & \\ & & & (k-1)\mu & -\{(k-1)\mu+\lambda\} & \lambda \\ & & & & k\mu & -k\mu \end{bmatrix}. \tag{2.4}$$

In a similar way the number of consuming machines in service may be associated with the states $j, j=0,1,\cdots,n,$ of another, independent birth-and-death process. If we let $\{p_2(t;j)\}$ be its state probabilities at time t, then

$$\frac{d}{dt}\boldsymbol{p}_2(t)=\boldsymbol{p}_2(t)\boldsymbol{M}_2, \tag{2.5}$$

where,

$$\boldsymbol{M}_2=\boldsymbol{M}(n, r_2, f_2). \tag{2.6}$$

Next consider the joint machine process which keeps track of the number of producers and consumers in service. We say that the 'machine state is (i,j)' to mean that i producers and j consumers are in service, and let $p(t;i,j)$ denote the probability of this event at time t. Clearly $p(t;i,j)=p_1(t;i)p_2(t;j)$. Now let $\boldsymbol{p}(t)$ denote the arrangement of $\{p(t;i,j)\}$ in lexicographic order, i.e. $\boldsymbol{p}(t)=\{\boldsymbol{p}(t;0,0)\boldsymbol{p}(t;0,1)\cdots\boldsymbol{p}(t;m,n)\}$. The equation satisfied is

$$\frac{d}{dt}\boldsymbol{p}(t)=\boldsymbol{p}(t)\boldsymbol{M} \tag{2.7}$$

where the *generator of the machine process*,

$$\boldsymbol{M} \triangleq \boldsymbol{M}_1 \otimes \boldsymbol{I}(n) + \boldsymbol{I}(m) \otimes \boldsymbol{M}_2, \tag{2.8}$$

$\boldsymbol{I}(k)$ is the $(k+1)$-dimensional identity matrix and $\otimes$ is the Kronecker product [3], [30].

A very brief excursion into Kronecker products: for $\boldsymbol{A}$ M-dimensional and $\boldsymbol{B}$ N-dimensional, $\boldsymbol{A}\otimes\boldsymbol{B}$ is MN-dimensional and $\boldsymbol{A}\otimes\boldsymbol{B}=\{A(k,l)\boldsymbol{B}\}$. Also, $(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{C}\otimes\boldsymbol{D})=(\boldsymbol{AC})\otimes(\boldsymbol{BD})$.

2.2. *Time-dependent model of the buffered system.* The joint machine and buffer process is Markovian in which the state description consists of the buffer content and the machine state. For x real and non-negative, let

$$P(t,x;i,j) \triangleq \Pr[\text{buffer content} \leqq x \text{ and machine state is } (i,j) \text{ at time } t] \tag{2.9}$$

and $\boldsymbol{P}(t,x)$ be the lexicographic arrangement of $\{P(t,x;i,j)\}$. The equation satisfied by it is quite straightforward to derive (see for instance [2]) for $0<x<X$, where X is the *buffer capacity* in the finite buffer problem. In the infinite buffer problem $X=\infty$. The equation is

$$\frac{\partial}{\partial t}\boldsymbol{P}+\frac{\partial}{\partial x}\boldsymbol{PD}=\boldsymbol{PM}, \qquad t\geqq 0, \qquad 0<x<X. \tag{2.10}$$

The matrix $\boldsymbol{D}$ called the *drift matrix* has the representation

$$\boldsymbol{D} \triangleq c_1\boldsymbol{E}(m)\otimes\boldsymbol{I}(n)-c_2\boldsymbol{I}(m)\otimes\boldsymbol{E}(n) \tag{2.11}$$

where, generally, $\boldsymbol{E}(k) \triangleq \operatorname{diag}\{0,1,\cdots,k\}$. The simplicity of $\boldsymbol{D}$ may be concealed in the notation: it is diagonal, its (i,j)th diagonal element,

$$D(i,j)=c_1 i-c_2 j, \tag{2.12}$$

is simply the drift in the buffer when (i,j) is the machine state and the buffer is neither empty nor full.

2.3. *Model of the buffered system in equilibrium.* In this case the stationary state probability distributions satisfy the time-independent part of (2.10). These distributions may have jumps at the boundaries, i.e. at $x=0$ and $x=X$. Hence we let $\boldsymbol{\pi}(x)$ be the continuous solution of the equation

$$\frac{d}{dx}\boldsymbol{\pi}(x)\boldsymbol{D}=\boldsymbol{\pi}(x)\boldsymbol{M}, \qquad 0\leqq x\leqq X, \tag{2.13}$$

and identify it with the distributions at all points not at the boundaries, $0<x<X$,

$$\Pr[\text{buffer content} \leqq x \text{ and machine state is } (i,j)]=\pi(x;i,j), \tag{2.14}$$

where $\{\pi(x;i,j)\}$ are the lexicographically ordered elements of $\boldsymbol{\pi}(x)$. The distributions are completely specified by noting that

$$\Pr[\text{empty buffer and machine state is } (i,j)]=\pi(0;i,j), \tag{2.15i}$$

$$\begin{aligned}&\Pr[\text{full buffer and machine state is } (i,j)]\\ &\quad=\Pr[\text{machine state is } (i,j)]-\pi(X;i,j).\end{aligned} \tag{2.15ii}$$

The first term on the right is obtained from the stationary distribution of the machine process and it is given in the next section. Equation (2.13) is the main concern of this paper.

Note that the problem of solving for π cannot be specified as an initial value problem since the initial conditions are not *a priori* known. In Section 6 we treat the solution of (2.13) as a two-point boundary-value problem with conditions to be satisfied at $x = 0$ and $x = X$.

It should be clear from the progression to (2.13) that its form carries over to the description of stationary distributions in quite general stochastic fluid models. The essential components are, an underlying continuous-time Markov process on a discrete state space, the machine process in our case, and a controlled quantity represented by a real variable, the buffer content in our case, which drifts, when not at a boundary, at a constant rate determined by the state of the underlying Markov process. In this general setting, $\boldsymbol{M}$ is the generator of the underlying Markov process and the diagonal of $\boldsymbol{D}$ gives the drift rates. The next section gives some results for the eigenvalue problem in this general framework when the underlying Markov process is time-reversible.

Later in the paper we shall need to know the number of zero, positive and negative elements on the diagonal of the drift matrix $\boldsymbol{D}$. Let

$$\begin{aligned} d_0 &\triangleq \#\{(i, j) \mid c_1 i - c_2 j = 0,\ 0 \leqq i \leqq m,\ 0 \leqq j \leqq n\}, \\ d_+ &\triangleq \#\{(i, j) \mid c_1 i - c_2 j > 0,\ 0 \leqq i \leqq m,\ 0 \leqq j \leqq n\}, \\ d_- &\triangleq \#\{(i, j) \mid c_1 i - c_2 j < 0,\ 0 \leqq i \leqq m,\ 0 \leqq j \leqq n\}. \end{aligned} \tag{2.16}$$

Also, $d \triangleq d_+ + d_-$. Thus, d_+ is the number of distinct machine states which give positive drift. Obviously, $d_0 + d_+ + d_- = (m + 1)(n + 1)$. Since $d_0 > 0$, (2.13) is automatically a system of differential equations with redundancies. We may eliminate the redundancies by eliminating d_0 variables, the number of algebraic relations in (2.13). This is done in Section 3. Elsewhere we prefer to retain the redundancies and also the neat structure of the equation.

2.4. *Stationary distributions and time-reversibility of the machine process.* The stationary probability of i producing machines being in service is given by $w_1(i)$, $0 \leqq i \leqq m$. Similarly, the stationary probability of j consuming machines being in service is given by $w_2(j)$, $0 \leqq j \leqq n$. Let the vectors obtained by arranging these quantities in the usual manner be $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ respectively. The equations satisfied are

$$\boldsymbol{w}_1 \boldsymbol{M}_1 = 0; \qquad \boldsymbol{w}_2 \boldsymbol{M}_2 = \boldsymbol{0}. \tag{2.17}$$

$$w_1(i) = \frac{1}{(1 + r_1/f_1)^m} \binom{m}{i} \left(\frac{r_1}{f_1}\right)^i; \qquad w_2(j) = \frac{1}{(1 + r_2/f_2)^n} \binom{n}{j} \left(\frac{r_2}{f_2}\right)^j. \tag{2.18}$$

In general we let $\boldsymbol{w1}$ denote $\sum w(k)$. Hence we have $\boldsymbol{w}_1\boldsymbol{1} = \boldsymbol{w}_2\boldsymbol{1} = 1$. The stationary distribution of the machine process is $\boldsymbol{w}$ where

$$\boldsymbol{w} = \boldsymbol{w}_1 \otimes \boldsymbol{w}_2, \tag{2.19}$$

which satisfies $\boldsymbol{wM} = \boldsymbol{0}$ and $\boldsymbol{w1} = 1$.

The *mean production capacity*, the mean cumulative rate at which the producing machines are capable of delivering, is given by

$$c_1 \boldsymbol{w}_1 \boldsymbol{E}(m)\mathbf{1} = c_1 \sum_{i=0}^{m} i w_1(i) = m c_1 r_1/(f_1 + r_1). \tag{2.20}$$

Similarly the *mean consumption capacity* is given by

$$c_2 \boldsymbol{w}_2 \boldsymbol{E}(n)\mathbf{1} = c_2 \sum_{j=0}^{n} j w_2(j) = n c_2 r_2/(f_2 + r_2). \tag{2.21}$$

The realized mean rate of production is referred to as the *mean production rate*; in the case of a finite buffer it is less than both of the above quantities.

Stationary birth-and-death processes are time-reversible [18], [19], consequently, $\boldsymbol{W}_k \boldsymbol{M}_k = \boldsymbol{M}'_k \boldsymbol{W}_k$, $k = 1, 2$, where $\boldsymbol{W}_k \triangleq \operatorname{diag}(\boldsymbol{w}_k)$ and the prime denotes transposition. The stationary machine process, while not birth-and-death, is also time-reversible. This is a consequence of the following easily verified relation:

$$(\boldsymbol{W}_1 \otimes \boldsymbol{W}_2)\boldsymbol{M} = \boldsymbol{M}'(\boldsymbol{W}_1 \otimes \boldsymbol{W}_2). \tag{2.22}$$

A corollary to (2.22) is that $\boldsymbol{M}$ is *essentially symmetric* [3], i.e. applying a similarity transformation makes it symmetric, as the following shows:

$$(\boldsymbol{W}_1^{\frac{1}{2}} \otimes \boldsymbol{W}_2^{\frac{1}{2}})\boldsymbol{M}(\boldsymbol{W}_1^{-\frac{1}{2}} \otimes \boldsymbol{W}_2^{-\frac{1}{2}}) = (\boldsymbol{W}_1^{-\frac{1}{2}} \otimes \boldsymbol{W}_2^{-\frac{1}{2}})\boldsymbol{M}'(\boldsymbol{W}_1^{\frac{1}{2}} \otimes \boldsymbol{W}_2^{\frac{1}{2}}). \tag{2.23}$$

The left-hand matrix is $\tilde{\boldsymbol{M}}$, the symmetrized form of $\boldsymbol{M}$.

One effect is that the differential equation in (2.13), and its associated eigenvalue problem, may be examined in their symmetrized forms. With

$$\tilde{\boldsymbol{\pi}}(x) \triangleq \boldsymbol{\pi}(x)(\boldsymbol{W}_1^{-\frac{1}{2}} \otimes \boldsymbol{W}_2^{-\frac{1}{2}}),$$

$$\frac{d}{dx}\tilde{\boldsymbol{\pi}}(x)\boldsymbol{D} = \tilde{\boldsymbol{\pi}}(x)\tilde{\boldsymbol{M}}. \tag{2.24}$$

Section 3 makes use of the symmetrized forms.

For future reference, note that $\boldsymbol{M}$ is an irreducible generator.

3. Some general results on reversible Markov drift processes

We give some algebraic results for the ordinary differential equations

$$\frac{d}{dx}\boldsymbol{\pi}(x)\boldsymbol{D} = \boldsymbol{\pi}(x)\boldsymbol{M}, \tag{3.1}$$

and the corresponding eigenvalue problem

$$z\boldsymbol{\phi}\boldsymbol{D} = \boldsymbol{\phi}\boldsymbol{M}. \tag{3.2}$$

These results apply to the problem at hand but are more general since the special structure of $\boldsymbol{D}$ and $\boldsymbol{M}$ is not utilized.

Theorem 1. In (3.1) let $\boldsymbol{M}$ denote the generator of an irreducible, time-reversible, finite-state, continuous-time Markov process and let $\boldsymbol{D}$ denote the diagonal drift matrix with d_0, d_+ and d_- zero, positive and negative diagonal elements. Let $\boldsymbol{w}$ denote the stationary probabilities of the Markov process, i.e. $\boldsymbol{w} > \boldsymbol{0}$, $\boldsymbol{w1} = 1$ and

$$\boldsymbol{wM} = \boldsymbol{0}. \tag{3.3}$$

(i) The proper degree of the characteristic polynomial $|z\boldsymbol{D} - \boldsymbol{M}|$ is $(d_+ + d_-)$ and the zeros of this polynomial, the eigenvalues, are all real. (Following Szego [34] we distinguish between the (apparent) degree and the proper degree of a polynomial: for $\sum_{i=0}^{N} c_i z^i$, N is called the degree and if $c_N \neq 0$, the proper degree.)

(ii) Interpret $\boldsymbol{wD1}$ as the *mean drift*. There are

	# (negative eigenvalues)	multiplicity of eigenvalue at 0	# (positive eigenvalues)
$\boldsymbol{wD1} < 0$	d_+	1	$d_- - 1$
$\boldsymbol{wD1} = 0$	$d_+ - 1$	2	$d_- - 1$
$\boldsymbol{wD1} > 0$	$d_+ - 1$	1	d_-

(iii) If $\boldsymbol{wD1} \neq 0$ then there are $(d_+ + d_-)$ solutions to the eigenvalue problem (3.2) in which the eigenvectors $\{\boldsymbol{\phi}\}$ form a linearly independent set, and the independent solutions to (3.1) are of the form $e^{zx}\boldsymbol{\phi}$. If $\boldsymbol{wD1} = 0$ then there exists a non-trivial solution $\boldsymbol{\xi}$ to the equation

$$\boldsymbol{wD} = \boldsymbol{\xi M} \tag{3.4}$$

and the independent solutions to (3.1) are $\boldsymbol{w}$, $(\boldsymbol{\xi} + x\boldsymbol{w})$ and $e^{zx}\boldsymbol{\phi}$, $z \neq 0$, where $\boldsymbol{w}$, $\boldsymbol{\xi}$ and $\{\boldsymbol{\phi}\}$ form a linearly independent set.

The proof of the Theorem is in Appendix A. A restricted version of this theorem is stated in [16]; [33] gives partial results without the reversibility assumption but the results are not sufficient to meet the needs of a complete solution procedure.

In the problem specific to this paper the mean drift obtained from (2.11) and (2.19) is

$$\boldsymbol{wD1} = \frac{mc_1r_1}{f_1 + r_1} - \frac{nc_2r_2}{f_2 + r_2}. \tag{3.5}$$

The *stable* system, which is considered later in conjunction with the infinite buffer, has $\boldsymbol{wD1} < 0$, the *balanced* system, which is of special importance in the case of the finite buffer, has $\boldsymbol{wD1} = 0$, and the *unstable* system has $\boldsymbol{wD1} > 0$. If $\boldsymbol{wD1} \neq 0$ then the system is *unbalanced*.

4. Eigenvalues and eigenvectors

In this section the specific structure of the matrices $\boldsymbol{D}$ and $\boldsymbol{M}$ is utilized in the eigenvalue problem

$$z\boldsymbol{\phi D} = \boldsymbol{\phi M}. \tag{4.1}$$

4.1. *Kronecker-product form of eigenvectors.* In (4.1) consider the following form for the eigenvector:

$$\phi = \phi_1 \otimes \phi_2 \tag{4.2}$$

where ϕ_1 and ϕ_2 are respectively $(m+1)$- and $(n+1)$-dimensional. On substituting in (4.1) the expression for $\boldsymbol{D}$, $\boldsymbol{M}$ and ϕ in (2.11), (2.8) and (4.2), respectively, we obtain

$$\phi_1 \otimes \{\phi_2 \boldsymbol{M}_2 + zc_2\phi_2 \boldsymbol{E}(n)\} = \{zc_1\phi_1\boldsymbol{E}(m) - \phi_1\boldsymbol{M}_1\} \otimes \phi_2.$$

Hence, for (4.1) and (4.2) to be true it is sufficient that there exist some real number v such that both sides of the above equation equal $zv\phi_1 \otimes \phi_2$, i.e.

$$z\phi_1[c_1\boldsymbol{E}(m) - v] = \phi_1\boldsymbol{M}_1, \tag{4.3i}$$

$$z\phi_2[v - c_2\boldsymbol{E}(n)] = \phi_2\boldsymbol{M}_2. \tag{4.3ii}$$

Consider (4.3i). It is similar to the basic eigenvalue problem considered in [2], the only difference being that in [2] the counterpart of v is given *a priori*. Let $\Phi_1(x)$ be the generating function of ϕ_1, i.e.

$$\Phi_1(x) \triangleq \sum_{i=0}^{m} x^i \phi_1(i). \tag{4.4}$$

From (4.3i) it is found that

$$\frac{\Phi_1'(x)}{\Phi_1(x)} = \frac{vz - mr_1 - mr_1x}{r_1x^2 + (c_1z + f_1 - r_1)x - f_1}. \tag{4.5}$$

The quadratic in x in the denominator of the right-hand side has two real and distinct roots, say τ_1 and τ_2 where $\tau_1 > 0 > \tau_2$:

$$\tau_{1,2} \triangleq \frac{-(c_1z + f_1 - r_1) \pm \sqrt{Q_1(z)}}{2r_1}, \tag{4.6i}$$

$$Q_1(z) \triangleq (c_1z + f_1 - r_1)^2 + 4f_1r_1. \tag{4.6ii}$$

Equation (4.5) may now be written as

$$\frac{\Phi_1'(x)}{\Phi_1(x)} = \frac{a}{x - \tau_1} + \frac{m - a}{x - \tau_2} \tag{4.7i}$$

where the residue

$$a = \frac{1}{\sqrt{Q_1(z)}}\left\{vz - \frac{m}{2}(c_1z + f_1 + r_1) + \frac{m}{2}\sqrt{Q_1(z)}\right\}. \tag{4.7ii}$$

A solution to (4.7i) is

$$\Phi_1(x) = (x - \tau_1)^a(x - \tau_2)^{m-a}. \tag{4.8}$$

An important observation is that, for compatibility with the definition of $\Phi_1(x)$ in (4.4), the right-hand quantity must be a polynomial in x of degree m, and hence the quantity a must be an integer in $[0, m]$. Denoting this integer by k_1 we have $0 \leqq k_1 \leqq m$ and

$$\Phi_1(x) = (x - \tau_1)^{k_1}(x - \tau_2)^{m-k_1}. \tag{4.9}$$

Next consider (4.3ii) and obtain by similar reasoning that $\Phi_2(x)$, the generating function of ϕ_2, is

$$\Phi_2(x) = (x - \sigma_1)^{k_2}(x - \sigma_2)^{n-k_2}, \tag{4.10}$$

where

$$\sigma_{1,2} \triangleq \frac{-(-c_2 z + f_2 - r_2) \pm \sqrt{Q_2(z)}}{2r_2}, \tag{4.11i}$$

$$Q_2(z) \triangleq (-c_2 z + f_2 - r_2)^2 + 4f_2 r_2, \tag{4.11ii}$$

and k_2, $0 \leqq k_2 \leqq n$, is an integer satisfying

$$k_2 \triangleq \frac{1}{\sqrt{Q_2(z)}}\left\{-vz - \frac{n}{2}(-c_2 z + f_2 + r_2) + \frac{n}{2}\sqrt{Q_2(z)}\right\}. \tag{4.11iii}$$

To now impose the requirement that a common value of v appears in both (4.3i) and (4.3ii) we proceed by noting that from (4.7ii) and (4.11iii),

$$\begin{aligned} vz &= \frac{m}{2}(c_1 z + f_1 + r_1) + \left(k_1 - \frac{m}{2}\right)\sqrt{Q_1(z)} \\ &= -\frac{n}{2}(-c_2 z + f_2 + r_2) - \left(k_2 - \frac{n}{2}\right)\sqrt{Q_2(z)}. \end{aligned} \tag{4.12}$$

At this stage we have that every solution z of (4.12) is also a solution of (4.1) for which the corresponding ϕ has the Kronecker-product form in (4.2). Let

$$\begin{aligned} f(z; k_1, k_2) \triangleq{}& \left(k_1 - \frac{m}{2}\right)\sqrt{Q_1(z)} + \left(k_2 - \frac{n}{2}\right)\sqrt{Q_2(z)} \\ &+ \frac{m}{2}(c_1 z + f_1 + r_1) + \frac{n}{2}(-c_2 z + f_2 + r_2). \end{aligned} \tag{4.13}$$

Proposition 4.1. For each integral 2-tuple (k_1, k_2), $0 \leqq k_1 \leqq m$, $0 \leqq k_2 \leqq n$, the real roots of $f(z; k_1, k_2)$ are eigenvalues. The corresponding eigenvector is $\phi_1 \otimes \phi_2$ where ϕ_1 and ϕ_2 may be identified by either one of two equivalent ways: (i) from their generating functions in (4.8) and (4.10); (ii) by solving the three-term recurrence implicit in each of the pair of tridiagonal systems of equations, (4.3i) and (4.3ii), in which v is obtained from (4.12).

In the next section the above procedure for obtaining the eigenvalues will be simplified and it will be shown that the procedure yields all the eigenvalues.

4.2. *Factors of the characteristic polynomial.* A simple step transforms the problem of finding the roots of $\{f(z; k_1, k_2)\}$ into one of finding the zeros of a family of polynomials $\{P(z; k_1, k_2)\}$. The equation $P(z; k_1, k_2)=0$ is obtained from $f(z; k_1, k_2)=0$ by rearranging terms and squaring as necessary, which is not more than twice, to eliminate square roots. For $0 \leqq k_1 \leqq m$, $0 \leqq k_2 \leqq n$, with

$$L(z) \triangleq \frac{m}{2}(c_1 z + f_1 + r_1) + \frac{n}{2}(-c_2 z + f_2 + r_2), \tag{4.14}$$

and $Q_1(z)$, $Q_2(z)$ defined in (4.6ii), (4.11ii), respectively, we obtain

$$\begin{aligned} P(z; k_1, k_2) &= \left\{\left(\frac{m}{2} - k_1\right)^2 Q_1(z) + \left(\frac{n}{2} - k_2\right)^2 Q_2(z) - L^2(z)\right\}^2 \\ &\quad - 4\left(\frac{m}{2} - k_1\right)^2 \left(\frac{n}{2} - k_2\right)^2 Q_1(z) Q_2(z), && \text{(4.15i)} \\ &\qquad\qquad \text{if } k_1 \neq \frac{m}{2} \text{ and } k_2 \neq \frac{n}{2}, \\ &= \left(\frac{m}{2} - k_1\right)^2 Q_1(z) - L^2(z), \quad \text{if } k_1 \neq \frac{m}{2} \text{ and } k_2 = \frac{n}{2}, && \text{(4.15ii)} \\ &= \left(\frac{n}{2} - k_2\right)^2 Q_2(z) - L^2(z), \quad \text{if } k_1 = \frac{m}{2} \text{ and } k_2 \neq \frac{n}{2}, && \text{(4.15iii)} \\ &= L(z), \quad \text{if } k_1 = \frac{m}{2} \text{ and } k_2 = \frac{n}{2}. && \text{(4.15iv)} \end{aligned}$$

For each 2-tuple (k_1, k_2), the roots of $f(z; k_1, k_2)$, $f(z; m-k_1, k_2)$, $f(z; m-k_1, n-k_2)$, $f(z; k_1, n-k_2)$ are precisely the zeros of $P(z; k_1, k_2)$. For this reason and, also, directly from (4.15) we have the symmetry relations

$$\begin{aligned} P(z; k_1, k_2) &= P(z; m-k_1, k_2) = P(z; m-k_1, n-k_2) \\ &= P(z; k_1, n-k_2). \end{aligned} \tag{4.16}$$

So roughly only a quarter of the polynomials are distinct. Hence, for computational and enumeration purposes we need only consider $\{P(z; k_1, k_2)\}$ with $0 \leqq k_1 \leqq m/2$ and $0 \leqq k_2 \leqq n/2$.

The reader will find it helpful to establish a correspondence between the integral (i, j) used in (2.14) to index the machine states, and the integral (k_1, k_2) used to index the above polynomials. A simple calculation shows that the sum of the degrees of the polynomials $\{P(z; k_1, k_2)\}$, $0 \leqq k_1 \leqq m/2$, $0 \leqq k_2 \leqq n/2$, is $(m+1)(n+1)$, the dimension of the system (4.1). The above is not a statement on the proper degrees of the polynomials which is considered next.

Note that the highest degree of the polynomials in (4.15) is 4, which is obtained from (4.15i), and that the coefficient of z^4 is

$$(k_1 c_1 - k_2 c_2)\{(m-k_1)c_1 - k_2 c_2\}\{(m-k_1)c_1 - (n-k_2)c_2\}\{k_1 c_1 - (n-k_2)c_2\}.$$

For each (k_1, k_2), Appendix 2 establishes a correspondence between the number of non-zero elements in

$$\{k_1c_1 - k_2c_2, (m-k_1)c_1 - k_2c_2, (m-k_1)c_1 - (n-k_2)c_2, k_1c_1 - (n-k_2)c_2\}$$

and the proper degree of $P(z; k_1, k_2)$. In particular, it is proven that the sum of the proper degrees of the family of polynomials $\{P(z; k_1, k_2)\}$, $0 \leqq k_1 \leqq m/2$, $0 \leqq k_2 \leqq n/2$, equals $(d_+ + d_-)$ which, see Section 3, is the proper degree of the characteristic polynomial.

Since each zero of the above family of polynomials is an eigenvalue, we have the following result.

Theorem 2. The set of eigenvalues coincide with the set of zeros of the family of polynomials $\{P(z; k_1, k_2)\}$, $0 \leqq k_1 \leqq m/2$, $0 \leqq k_2 \leqq n/2$.

This result achieves a factorization of the characteristic polynomial in which each factor is a polynomial of degree at most 4. From the computational viewpoint, the eigenvalue problem has been decomposed into roughly $mn/4$ sub-problems each of complexity independent of m and n.

Another consequence of the theorem is that all solutions $\boldsymbol{\phi}$ to the eigenvalue problem (4.1), the eigenvectors, have the product form given in (4.2).

We already know from Section 3 that all the eigenvalues are real. This can also be shown directly by examining each $P(z; k_1, k_2)$. The argument is not revealing and is omitted. It does however show that the quantities v, see (4.3) and (4.12), are real and non-negative, and positive for $(k_1, k_2) \neq (0, 0)$, a fact useful in interpreting (4.3) which is done in Section 5.4.

4.3. *The balanced system.* Recall from Section 3 that the balanced system occurs when the mean drift $= 0$, and in this case there exists an independent solution to (2.13), different from all other independent solutions, of the form $\boldsymbol{\xi} + x\boldsymbol{w}$ where

$$\boldsymbol{wM} = \boldsymbol{0}, \quad \text{and} \quad \boldsymbol{\xi M} = \boldsymbol{wD}. \tag{4.17}$$

The procedure for computing eigenvectors given in Section 4.1 does not extend to the computation of $\boldsymbol{\xi}$ which is now undertaken.

Proposition 4.2. For the balanced system let

$$v = \frac{c_1 m r_1}{f_1 + r_1} = \frac{c_2 n r_2}{f_2 + r_2}, \tag{4.18}$$

and let $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$ solve

$$\boldsymbol{\eta}_1 \boldsymbol{M}_1 = \boldsymbol{w}_1[c_1\boldsymbol{E}(m) - v], \qquad \boldsymbol{\eta}_2 \boldsymbol{M}_2 = \boldsymbol{w}_2[v - c_2\boldsymbol{E}(n)]. \tag{4.19}$$

Then

$$\boldsymbol{\xi} = \boldsymbol{\eta}_1 \otimes \boldsymbol{w}_2 + \boldsymbol{w}_1 \otimes \boldsymbol{\eta}_2 \tag{4.20}$$

solves (4.17).

Proof. By direct verification on using the expressions for $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{D}$ and $\mathbf{M}$ in (2.17), (2.11) and (2.8).

It can be shown that the following solve (4.19):

$$\eta_1(i) = w_1(i)\frac{c_1(m-i)}{(f_1+r_1)}, \qquad 0 \leqq i \leqq m, \tag{4.21i}$$

$$\eta_2(j) = -w_2(j)\frac{c_2(n-j)}{(f_2+r_2)}, \qquad 0 \leqq j \leqq n. \tag{4.21ii}$$

From (4.20) and (4.21) we obtain the following explicit characterization

$$\xi(i,j) = w_1(i)w_2(j)\left[\frac{c_1(m-i)}{f_1+r_2} - \frac{c_2(n-j)}{f_2+r_2}\right] \tag{4.22}$$

and

$$\boldsymbol{\xi 1} = \frac{c_1 m f_1}{(f_1+r_1)^2} - \frac{c_2 n f_2}{(f_2+r_2)^2}. \tag{4.23}$$

5. Further properties of the eigenvalues and eigenvectors

5.1. *The dominant non-zero eigenvalue in stable systems.* In (4.13) there is given a family of functions $f(z; k_1, k_2)$ the roots of which are the eigenvalues of the system. Here we investigate $f(z; 0, 0)$ which will be shown to be special. From (4.13),

$$\begin{aligned} f(z;0,0) = &-\frac{m}{2}\sqrt{Q_1(z)} - \frac{n}{2}\sqrt{Q_2(z)} + \frac{m}{2}(c_1 z + f_1 + r_1) \\ &+ \frac{n}{2}(-c_2 z + f_2 + r_2). \end{aligned} \tag{5.1}$$

Note that 0 is a root. Also,

$$f'(0;0,0) = \frac{mc_1r_1}{f_1+r_1} - \frac{nc_2r_2}{f_2+r_2}, \tag{5.2}$$

and

$$\tfrac{1}{2}f''(z;0,0) = -\frac{mc_1^2 f_1 r_1}{Q_1^{\frac{3}{2}}(z)} - \frac{nc_2^2 f_2 r_2}{Q_2^{\frac{3}{2}}(z)}. \tag{5.3}$$

Hence $f(z; 0, 0)$ is a concave function. Moreover, $f(z; 0, 0) \to -\infty$ as $|z| \to \infty$.

If the system is balanced then $f'(0; 0, 0) = 0$, and it follows that $f(z; 0, 0)$ has a repeated root at 0 and no other real roots. Otherwise, the function has only one non-zero real root which is negative if the system is stable and positive if it is unstable. Let this root be $-r$.

Now observe that

$$f(z;k_1,k_2) - f(z;0,0) = k_1\sqrt{Q_1(z)} + k_2\sqrt{Q_2(z)} \geqq 0 \tag{5.4}$$

with the inequality strict if either k_1 or k_2 is different from 0. Hence, we have the following result.

Proposition 5.1. If the system is stable then there are no eigenvalues in $(-r, 0)$.

This result provides the basis for referring to $-r$ as the dominant eigenvalue in the stable system. The procedure given in Section 4.2 allows $-r$ to be obtained as a zero of the polynomial $P(z; 0, 0)$ which has the form $z(Az^2 + Bz + C)$. Hence the dominant eigenvalue is obtained as a root of this quadratic component.

In conclusion, we have seen that the simple function $f(z; 0, 0)$ is special in connection with the dominant non-zero eigenvalue and in explicating the enumeration of eigenvalues given earlier in Theorem 1, Section 3.

5.2. *The reversed system.* The reversed system is obtained by reversing the flow. Equivalently, the reversed system is obtained by reversing the production/consumption roles of the machines while holding fixed the capacity of the buffer.

The reader may verify the following facts: for each eigenvalue z of the given system there is an eigenvalue $-z$ in the reversed system; v is invariant; where $\phi_1 \otimes \phi_2$ is the eigenvector corresponding to z in the given system, $\phi_2 \otimes \phi_1$ is the eigenvector corresponding to $-z$ in the reversed system.

If the given system is stable then the reversed system is unstable. Hence the reversed system is of little interest in the infinite buffer problem. In the finite buffer problem, however, the reversed system is of considerable interest [29], [32] and we give our results on it in Section 6.4.

5.3. *The eigenvectors as Krawtchouck polynomials.* We examine in greater detail the elements of ϕ_1 and ϕ_2 and show that they possess a polynomial structure which identifies them as Krawtchouk polynomials [17], [27], [34]. Our analysis does not exploit the properties of these polynomials and the correspondence is given for completeness.

For any $p > 0$, $q > 0$, $p + q = 1$, Szego [34] defines these polynomials thus:

$$K_i(q; \zeta) \triangleq \binom{m}{i}^{-\frac{1}{2}} (pq)^{-\frac{1}{2}} \sum_{\mu=0}^{i} (-1)^{i-\mu} \binom{m-\zeta}{i-\mu} \binom{\zeta}{\mu} p^{i-\mu} q^{\mu}, \tag{5.5}$$

$$i = 0, 1, \cdots, m.$$

We are only interested in cases where ζ is an integer, $0 \leqq \zeta \leqq m$. Szego [34] has demonstrated the orthogonality of these polynomials and also has given an expression for their generating function.

A simple change of variables shows that the generating functions for $\{K_i\}$ and for $\{\phi_1(i)\}$ in (4.9) are equal and thus we obtain the following correspondence:

$$\frac{\phi_1(i)}{\phi_1(0)} = \sqrt{\frac{w_1(i)}{w_1(0)}} K_i\left(\frac{\tau_1}{\tau_1 - \tau_2}; m - k_1\right), \qquad 0 \leqq i \leqq m. \tag{5.6}$$

(Recall from (4.6) and (4.9) that τ_1, τ_2 and k_1 are elements in the algebraic construction of ϕ_1.) Since an explicit expression for K_i is given in (5.5), we obtain as a by-product,

$$\frac{\phi_1(i)}{\phi_1(0)} = \left(\frac{r_1}{f_1}\right)^i \sum_{\mu=0}^{i} \binom{k_1}{i-\mu}\binom{m-k_1}{\mu} \tau_2^{i-\mu}\tau_1^{\mu}, \qquad 0 \leqq i \leqq m. \tag{5.7}$$

Equations (5.6)-(5.7) have their obvious counterparts when ϕ_2 is considered.

Evaluating the sums in (5.7) is not the most efficient numerical procedure for calculating ϕ_1; the recursive procedure stated in Proposition 4.1, in which the number of multiplications is linear in m, is more efficient. Also, we have not found the orthogonality properties of Krawtchouk polynomials useful, primarily because in the representation given in (5.6) the values of τ_1 and τ_2 are not the same for the various eigenvalues.

5.4. *An interpretation of the decomposition implied by the product form.* The object here is to interpret (4.3) which is the decomposition implied by the product form of the solution of (4.1) and which we now know to hold. Consider Figure 5.1(a) which is obtained from the system of this paper (see Figure 1.1) by aggregating the consuming machines into one *failure-proof* machine of working capacity v. A similar aggregation of the producing machines is reflected in Figure 5.1(b). The duality of the figures is reflected in the common value of v.

Let $\boldsymbol{\pi}_1(x) = \{\pi_1(x; i)\}$ and $\boldsymbol{\pi}_2(x) = \{\pi_2(x; j)\}$ respectively denote the equilibrium state distributions for the two systems. Then, for $0 \leqq x \leqq X$,

$$\frac{d}{dx}\boldsymbol{\pi}_1(x)[c_1\boldsymbol{E}(m) - v] = \boldsymbol{\pi}_1(x)\boldsymbol{M}_1, \tag{5.8}$$

$$\frac{d}{dx}\boldsymbol{\pi}_2(x)[v - c_2\boldsymbol{E}(n)] = \boldsymbol{\pi}_2(x)\boldsymbol{M}_2. \tag{5.9}$$

We seek independent solutions to the above equations which are capable of yielding $\boldsymbol{\pi}_1(x)\mathbf{1} \equiv \boldsymbol{\pi}_2(x)\mathbf{1}$, i.e. identical 'buffer content distributions'. Since this implies identical dependence on x, the independent solutions are of the forms $\boldsymbol{\pi}_1(x) = e^{zx}\boldsymbol{\phi}_1$ and $\boldsymbol{\pi}_2(x) = e^{zx}\boldsymbol{\phi}_2$; the distributional condition may be satisfied by normalizing $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$. For these modal solutions to exist we must have (4.3).

In view of this interpretation we may call v the 'aggregated working capacity' for the mode. Also, not surprisingly, v is invariant under flow reversal, see Section 5.2.

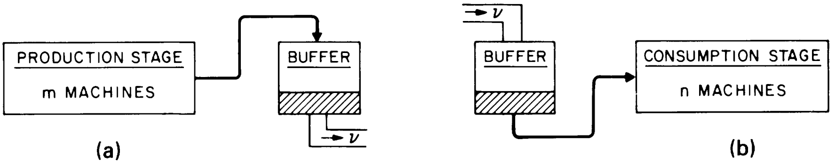


Figure 5.1

6. Boundary conditions and solutions

First we give an indexing scheme for the eigenvalues. Recall that $d = d_+ + d_-$ where d_+ and d_- have been defined in (2.16). We adopt the following convention which is consistent with the enumeration in Section 3 and the investigation in Section 5.1 on the dominant non-zero eigenvalue.

$$(6.1)\quad \begin{aligned} &\textit{stable}: && z_{d_+} \leqq \cdots \leqq z_2 < z_1 < z_d = 0 < z_{d-1} \leqq \cdots \leqq z_{d_++1} \\ &\textit{balanced}: && z_{d_+} \leqq \cdots \leqq z_2 < z_1 = z_d = 0 < z_{d-1} \leqq \cdots \leqq z_{d_++1} \\ &\textit{unstable}: && z_{d_+} \leqq \cdots \leqq z_2 < z_d = 0 < z_1 < z_{d-1} \leqq \cdots \leqq z_{d_++1} \end{aligned}$$

The eigenvector corresponding to the eigenvalue z_l is denoted by $\boldsymbol{\phi}(l)$. Let $\boldsymbol{\phi}(l) = \boldsymbol{\phi}_1(l) \otimes \boldsymbol{\phi}_2(l)$ and denote the scalar components of $\boldsymbol{\phi}_1(l)$ and $\boldsymbol{\phi}_2(l)$ by $\phi_1(l; i)$, $0 \leqq i \leqq m$, and $\phi_2(l; j)$, $0 \leqq j \leqq n$, respectively.

The general form of $\boldsymbol{\pi}(x)$ is obtained from Theorem 1, Section 3; the stationary state distributions are obtained from it by (2.14) and (2.15).

$$(6.2\text{i})\qquad \boldsymbol{\pi}(x) = a_d \boldsymbol{w}_1 \otimes \boldsymbol{w}_2 + \sum_{l=1}^{d-1} a_l \exp(z_l x) \boldsymbol{\phi}_1(l) \otimes \boldsymbol{\phi}_2(l): \quad \textit{unbalanced}$$

$$(6.2\text{ii})\qquad \begin{aligned} &= a_d \boldsymbol{w}_1 \otimes \boldsymbol{w}_2 + a_1(\boldsymbol{\xi} + x \boldsymbol{w}_1 \otimes \boldsymbol{w}_2) \\ &\quad + \sum_{l=2}^{d-1} a_l \exp(z_l x) \boldsymbol{\phi}_1(l) \otimes \boldsymbol{\phi}_2(l): \quad \textit{balanced} \end{aligned}$$

The coefficients a_l, $1 \leqq l \leqq d$, are undetermined, $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ have been explicitly given in (2.18), $\boldsymbol{\xi}$ has been explicitly given in (4.22).

In the following two sections the coefficients $\{a_l\}$ are determined from the distinct boundary conditions that exist in the finite buffer problem and the infinite buffer problem.

6.1. *The finite buffer problem.* Let (i, j) be the instantaneous machine state. If $c_1 i - c_2 j < 0$ then the drift in buffer content is negative and it is therefore impossible that {the machine state is (i, j) *and* the buffer is full}, except at isolated points in time. From (2.15ii), (2.18) and (2.19) the stationary probability of the event in braces is $\{w_1(i) w_2(j) - \pi(X; i, j)\}$, and hence we have that

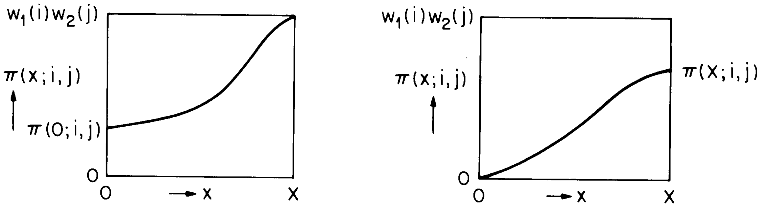
$$(6.3)\qquad c_1 i - c_2 j < 0 \Rightarrow \pi(X; i, j) = w_1(i) w_2(j).$$

We note that there are d_- such conditions.

Similarly, if $c_1 i - c_2 j > 0$ then the drift is positive and it is therefore impossible that {the machine state is (i, j) *and* the buffer is empty}, except at isolated points in time. From (2.15i) the stationary probability of the event in braces is $\pi(0; i, j)$, and hence we also have that

$$(6.4)\qquad c_1 i - c_2 j > 0 \Rightarrow \pi(0; i, j) = 0.$$

There are d_+ such conditions.



(a) $c_1i - c_2j < 0$ (b) $c_1i - c_2j > 0$

Figure 6.1

The sketches in Figure 6.1 contrast the typical behaviors of the functions $\pi(x; i, j)$ at $x = 0$ and $x = X$.

We now have $d = d_+ + d_-$ conditions in (6.3) and (6.4) which is exactly the number of undetermined coefficients in the general solution (6.2). Hence, we have the following result.

Proposition 6.1

(i) For the *unbalanced* system $\pi(x)$, $0 \leqq x \leqq X$, is given by (6.2i) in which the constants a_l, $1 \leqq l \leqq d$, solve the following system of d equations:

$$(6.5)\qquad \begin{aligned} w_1(i)w_2(j) &= a_d w_1(i)w_2(j) + \sum_{l=1}^{d-1} a_l \exp(z_l X)\phi_1(l; i)\phi_2(l; j), \quad c_1 i - c_2 j < 0, \\ 0 &= a_d w_1(i)w_2(j) + \sum_{l=1}^{d-1} a_l \phi_1(l; i)\phi_2(l; j), \quad c_1 i - c_2 j > 0. \end{aligned}$$

(ii) For the *balanced* system $\pi(x)$, $0 \leqq x \leqq X$, is given by (6.2ii) in which the constants a_l, $1 \leqq l \leqq d$, solve the following system of d equations:

$$(6.6)\qquad \begin{aligned} w_1(i)w_2(j) &= \{a_d + a_1(\alpha(i, j) + X)\} w_1(i)w_2(j) \\ &\quad + \sum_{l=2}^{d-1} a_l \exp(z_l X)\phi_1(l; i)\phi_2(l; j), \quad c_1 i - c_2 j < 0, \\ 0 &= \{a_d + a_1\alpha(i, j)\} w_1(i)w_2(j) + \sum_{l=2}^{d-1} a_l \phi_1(l; i)\phi_2(l; j), \quad c_1 i - c_2 j > 0, \end{aligned}$$

where $\alpha(i, j) \triangleq c_1(m - i)/(f_1 + r_1) - c_2(n - j)/(f_2 + r_2)$.

6.2. *The infinite buffer problem.* The assumption here is that the system is stable. Since $\pi(x; i, j)$ for each x is a probability, and therefore bounded by 1, it is immediate that in the general solution (6.2) the modes associated with positive eigenvalues cannot be excited, i.e.

$$(6.7)\qquad a_l = 0 \quad \text{for} \quad d_+ < l < d.$$

Moreover, $\boldsymbol{\pi}(x) \to \boldsymbol{w}_1 \otimes \boldsymbol{w}_2$ as $x \to \infty$; hence,

$$(6.8)\qquad a_d = 1.$$

This leaves only the coefficients a_l, $1 \leqq l \leqq d_+$, to be determined. The argument used in the finite buffer problem which led to (6.4) is intact in the present context. There are exactly as many such conditions as there are coefficients remaining to be determined. Hence, we have the following result.

Proposition 6.2. For a buffer of infinite capacity and a stable system,

$$\boldsymbol{\pi}(x) = \boldsymbol{w}_1 \otimes \boldsymbol{w}_2 + \sum_{l=1}^{d_+} a_l \exp(z_l x) \boldsymbol{\phi}_1(l) \otimes \boldsymbol{\phi}_2(l), \qquad 0 \leqq x,$$

where the coefficients a_l, $1 \leqq l \leqq d_+$, solve the following system of d_+ equations,

$$0 = w_1(i) w_2(j) + \sum_{l=1}^{d_+} a_l \phi_1(l; i) \phi_2(l; j), \qquad c_1 i - c_2 j > 0. \tag{6.9}$$

6.3. *Mean values for the finite buffer problem.* Explicit expressions are given for the basic performance measures. These expressions require knowledge of $\pi(0; i, j)$ and $\pi(X; i, j)$ for each machine state (i, j), and these are obtained from the expression for $\{\pi(x; i, j)\}$ in Proposition 6.1.

Proposition 6.3.

(i) Mean production rate

$$= \frac{mc_1 r_1}{f_1 + r_1} - \sum_{(i,j): c_1 i - c_2 j > 0} \{w_1(i) w_2(j) - \pi(X; i, j)\}(c_1 i - c_2 j) \tag{6.10i}$$

$$= \frac{nc_2 r_2}{f_2 + r_2} + \sum_{(i,j): c_1 i - c_2 j < 0} \pi(0; i, j)(c_1 i - c_2 j). \tag{6.10ii}$$

$$\Pr(\text{empty buffer}) = \sum_{(i,j): c_1 i - c_2 j \leqq 0} \pi(0; i, j). \tag{6.11i}$$

$$\Pr(\text{full buffer}) = \sum_{(i,j): c_1 i - c_2 j \geqq 0} \{w_1(i) w_2(j) - \pi(X; i, j)\}. \tag{6.11ii}$$

The two equivalent expressions for the realized mean production rate are associated with the transfer rates between, first, the producers and the buffer and, second, the buffer and the consumers. From the first,

$$\begin{aligned} \text{mean production rate} &= \text{mean production capacity} \\ &\quad - \text{mean rate of lost production due to full buffer.} \end{aligned} \tag{6.12}$$

Now the mean production capacity is $mc_1 r_1 / (f_1 + r_1)$, and the

$$\begin{aligned} &\text{mean rate of lost production due full buffer} \\ &\quad = \sum_{(i,j): c_1 i - c_2 j > 0} \Pr[\text{full buffer and machine state is } (i, j)](c_1 i - c_2 j). \end{aligned}$$

From (2.15ii) we obtain (6.10i). The expression for the second transfer rate takes into account the mean rate of lost consumption due to an empty buffer and leads to (6.10ii).

The equivalence of the two expressions in (6.10) follows directly from the identity

$$\sum_{(i,j)} \pi(x;i,j)(c_1 i - c_2 j) = \text{constant}, \ \forall x, \qquad 0 \leqq x \leqq X,$$

which is easy to obtain from the differential equation (2.13) and the fact that $\boldsymbol{M1} = \boldsymbol{0}$.

6.4. *The reversed system.* The reversed system was described in Section 5.2 where it was also stated that a simple correspondence exists between its eigenvalues and eigenvectors and those of the original system. Here, in the context of the finite buffer problem, we establish distributional correspondences and end by showing that the systems considered in this paper are flow-reversible.

The symbols for the various quantities of interest in the reversed system are specified by the superscript (R). Consider the unbalanced system. Reflecting the observations made in Section 5.2 we have, in correspondence to (6.2i),

$$\boldsymbol{\pi}^{(R)}(x) = a_d^{(R)} \boldsymbol{w}_2 \otimes \boldsymbol{w}_1 + \sum_{l=1}^{d-1} a_l^{(R)} \exp(-z_l x) \boldsymbol{\phi}_2 \otimes \boldsymbol{\phi}_1. \tag{6.13}$$

The boundary conditions are, for $0 \leqq i \leqq m$ and $0 \leqq j \leqq n$,

$$\begin{aligned} c_1 i - c_2 j < 0 &\Rightarrow \pi^{(R)}(0; j, i) = 0; \\ c_1 i - c_2 j > 0 &\Rightarrow \pi^{(R)}(X; j, i) = w_1(i) w_2(j). \end{aligned} \tag{6.14}$$

On comparing the equations obtained from (6.13) and (6.14) with (6.5) we obtain that

$$\begin{aligned} a_d + a_d^{(R)} &= 1, \\ a_l + a_l^{(R)} \exp(-z_l X) &= 0, \qquad 1 \leqq l \leqq d-1. \end{aligned} \tag{6.15}$$

From this we obtain

Proposition 6.4

(i)

$$\pi(x; i, j) + \pi^{(R)}(X - x; j, i) = w_1(i) w_2(j), \tag{6.16}$$

for $0 \leqq i \leqq m$, $0 \leqq j \leqq n$, $0 \leqq x \leqq X$.

(ii)

$$\pi(x) + \pi^{(R)}(X - x) = 1, \qquad 0 \leqq x \leqq X, \tag{6.17}$$

i.e. the buffer content of the reversed system in equilibrium is identical in distribution to the empty space in the buffer of the original system in equilibrium.

(iii)

$$\text{mean production rate} = (\text{mean production rate})^{(R)}. \tag{6.18}$$

Relation (6.18) was obtained by Muth for the queueing system he considers in [29].

6.5. *Properties of the solution for the infinite buffer problem.* Here $\pi(0; i, j)$ is obtained from Proposition 6.2 by setting $x = 0$. Obviously, here the mean production rate = mean production capacity. The expression in (6.11i) for the Pr [buffer empty] remains intact. The expression for the Mth moment ($M \geqq 1$) of the buffer contents is

$$(6.19) \qquad E[x^M] = \frac{M!}{(-1)^{M+1}} \sum_{l=1}^{d_+} \frac{a_l\{\phi_1(l)\mathbf{1}\}\{\phi_2(l)\mathbf{1}\}}{z_l^M}.$$

The asymptotic behavior of the distribution of the buffer content is as follows:

$$(6.20) \qquad \Pr[\text{buffer content} \geqq x] \sim A \exp(-rx) \quad \text{as} \quad x \to \infty,$$

where $-r = z_1$ is the negative real root of the quadratic discussed in Section 5.1, and $A = a_1\{\phi_1(1)\mathbf{1}\}\{\phi_2(1)\mathbf{1}\}$.

7. Numerical investigations

We have written two computer programs which implement the procedures that have been described; one program (fluid.i.c) is for the infinite buffer problem, and the other (fluid.f.c) is for the finite buffer problem. We report below on some of the results. In the course of the numerical work we have observed various phenomena which are of practical importance and the report draws attention to them. The practical import of the analysis is by no means exhaustively treated—the objective here is to provide a sampling.

The computation time on a VAX 11/750 for problems with $mn \leqq 100$ is of the order of a few seconds. The computational procedure is limited by the conditioning of the linear equations associated with the boundary conditions.

Typical failure and repair rates chosen for the machines in the systems investigated are respectively 0·14 and 1·00. Thus if the unit of time is a day, then, on average, the machines represented work continuously without failing for about a week and require a day to repair.

7.1. *Eigenvalues.* Figure 7.1 is a plot of the computed eigenvalues for a particular system. The proper degree of the characteristic polynomial for this stable system is 49; also, $d_+ = 20$, $d_- = 29$ and $d_0 = 5$.

7.2. *Infinite buffer*

7.2.1. *Distributions of buffer content and their asymptotic approximations.* Figure 7.2 plots the buffer content distributions for three systems. We have consistently observed (but not proven) that the asymptotic approximation is an upper bound on the probability of the buffer content exceeding a particular value.

7.2.2. *The advantage of small machines.* For equal failure/repair characteristics it is always preferable to have many, small machines rather than few, large machines

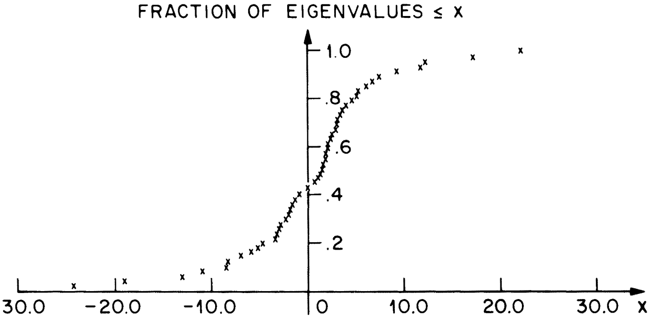


Figure 7.1. $m=8$, $n=5$; $c_1=0{\cdot}5$, $c_2=1{\cdot}0$; $f_1=f_2=0{\cdot}14$; $r_1=r_2=1{\cdot}0$. Mean production capacity = 3·5087 and mean consumption capacity = 4·3859

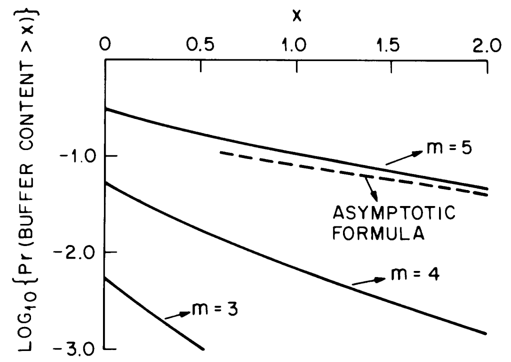


Figure 7.2. $n=6$; $c_1=c_2=1{\cdot}0$, $f_1=f_2=0{\cdot}14$; $r_1=r_2=1{\cdot}0$. Mean production capacity = 2·6315, 3·5087, 4·3859 for $m=3, 4, 5$, respectively. Mean consumption capacity = 5·2631

with the same mean capacities. This is so for both producing and consuming machines. However, the advantage is more pronounced here in the case of consuming machines. The advantage that is being referred to is in lower inventory, i.e. mean buffer content. Figure 7.3 illustrates this phenonemon.

In Figure 7.3 the number and capacity of the machines are varied while the mean production and consumption capacities are held fixed. While the mean is fixed, the statistical variability of the production and consumption capacities, as measured by their variances for instance, decrease with increased number of machines. This phenomenon causes the inventory to decrease monotonically.

7.3. *Finite buffer*

7.3.1. *The effect of buffer size on production.* Figure 7.4 illustrates the trade-off for a balanced system. The mean lost production rate is defined as the difference between the mean production capacity and realized mean production rate. Observe the diminishing return from increased buffer size and the increasing cost in inventory.

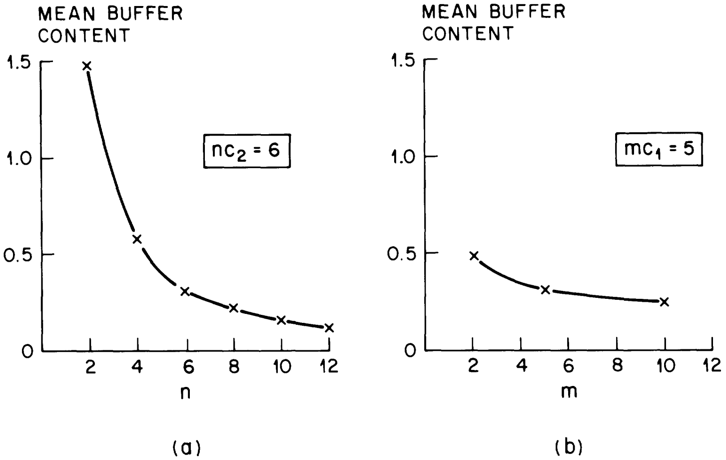


Figure 7.3. In (a), $m=5$ and $c_1=1{\cdot}0$; in (b), $n=6$ and $c_2=1{\cdot}0$. For each system $f_1=f_2=0{\cdot}14$; $r_1=r_2=1{\cdot}0$; the mean production capacity is 4·3859 and the mean consumption capacity is 5·2631

7.3.2. *The bufferless system.* The reader will observe that in Figure 7.4 data are given for the case of buffer capacity $=0$; in fact, for this case the mean production rate $=0{\cdot}6445$. In general, this quantity may be arrived at in two different ways. One, of course, is by setting $X=0$ in the boundary equations given in (6.5) and (6.6). The other, direct way is by observing that conditioned on the machine state $=(i,j)$, the production rate for the bufferless system is $\min(c_1 i, c_2 j)$. Hence,

$$\text{mean production rate} = \sum_{(i,j)} w_1(i)w_2(j)\min(c_1 i, c_2 j).$$

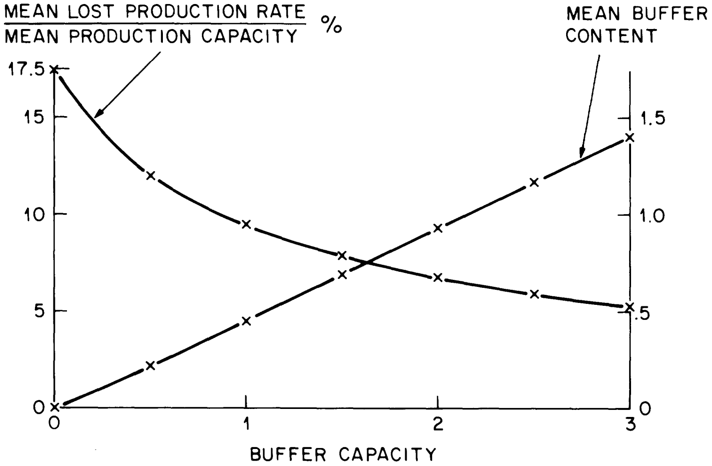


Figure 7.4. $m=5$, $n=2$; $c_1=0{\cdot}2$, $c_2=0{\cdot}5$; $f_1=f_2=0{\cdot}14$; $r_1=r_2=0{\cdot}5$. The system is balanced: mean production capacity = mean consumption capacity = 0·7812

7.3.3. *The advantage of small buffers in certain systems*

TABLE 7.1

$m = 5$, $n = 2$; $c_1 = 1{\cdot}0$, $c_2 = 3{\cdot}0$; $f_1 = f_2 = 0{\cdot}14$; $r_1 = r_2 = 1{\cdot}0$. Mean production capacity = 4·3859, mean consumption capacity = 5·2631. The data in the last column is from fluid.i.c.

Buffer capacity	0	1·0	5·0	10·0	∞
Mean production rate	4·0177	4·1569	4·3252	4·3705	4·3859
Mean buffer content	0	0·2130	0·8412	1·2311	1·4832
Pr [buffer full]	0·2185	0·1216	0·0292	0·0073	—
Pr [buffer empty]	0·7815	0·6781	0·5692	0·5416	0·5322

A rather striking feature of Table 7.1 is the inelasticity of the mean production rate to buffer capacity. Even more important is the inelasticity of the mean production rate to mean buffer content. For example, in going from a buffer of capacity 1·0 to one of capacity 10·0 there is a gain of about 5% in the mean production rate bought at the cost of an almost sixfold increase in inventory. The above system is in contrast to the system in Section 7.3.1 in which the elasticity is somewhat greater. The main distinguishing feature between the systems is that c_1 and c_2 are substantially bigger here.

8. Conclusion

The computational task has been shown to be in two parts: (i) the calculations related to eigenvalues, and (ii) the calculation of the coefficients $\{a_l\}$ from the boundary conditions. The analytic results have made the former quite easy and, as matters stand, the latter surprisingly dominates the computational burden. Efficient and stable algorithms, such as is reported in [2] for a simpler model, which solve the boundary equations by exploiting its special structure will be helpful.

It is straightforward to extend the analysis to allow multiple classes of producers and consumers [23]. Each machine class is distinguished by the number of constituent machines and their working capacities and failure and repair rates. In particular, the model with multiple machine classes remains subsumed in the class of reversible Markov drift processes, the Kronecker-product form of eigenvectors remains intact and a natural extension of the factorization result (Theorem 2) holds. The factors of the characteristic polynomial are polynomials of degree $2(p+q)$ where p and q are respectively the number of classes of producers and consumers. (The scheme of representations based on the Kronecker product extends naturally and proves extremely useful in this context.) The boundary equations are natural extensions of the equations in Section 6. Since the number of boundary equations grows exponentially with the number of classes, the full value of such an extension will be realized only with the development of efficient algorithms for solving its boundary equations.

An important direction for future work is the extension to more than two stages in fluid models of production lines. There are two categories of such work: first, an exact treatment of the resulting system of first-order partial differential equations in which there are as many independent variables as buffers and rather complex boundary conditions; second, an approximate treatment based on decomposition and assumed independence, such as the work of Sevastyanov [32]. At present the former appears to be difficult, while the latter seems quite feasible and promising.

Acknowledgment

We are grateful for the benefit of discussions with Dr L. Kaufman and Dr J. McKenna of AT & T Bell Laboratories.

Appendix A: Reversible Markov drift processes

We gave here the proof of Theorem 1 which applies for the differential equation in (3.1) and its associated eigenvalue problem in (3.2).

A.1. *Redundancies in the differential equations.* Equation (3.1) represents a system of $(d_+ + d_-)$ differential equations and a supplementary set of d_0 algebraic equations. Now, we may eliminate d_0 variables to obtain a reduced system of $(d_+ + d_-)$, first-order, homogeneous differential equations in the variables to which correspond non-zero diagonal entries in $\boldsymbol{D}$. We propose to show here that for our purposes the two forms of (3.1), the redundant and the reduced, are equivalent.

After an appropriate permutation let

$$\text{(A1.1)} \qquad \boldsymbol{D} = \begin{bmatrix} \boldsymbol{0} & \\ & \boldsymbol{D}_1 \end{bmatrix}, \qquad \boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{00} & \boldsymbol{M}_{01} \\ \boldsymbol{M}_{10} & \boldsymbol{M}_{11} \end{bmatrix}$$

where $\boldsymbol{D}_1$ is diagonal with non-zero diagonal elements and is therefore of dimension $(d_+ + d_-)$ and $\boldsymbol{M}_{11}$ has dimensions identical to $\boldsymbol{D}_1$. Now a consequence of the irreducibility of $\boldsymbol{M}$ is that $\boldsymbol{M}_{00}$, and all other proper principal submatrices of $\boldsymbol{M}$, is non-singular [4] and the reduced form of (3.1) is

$$\text{(A1.2)} \qquad \begin{aligned} \frac{d}{dx}\boldsymbol{\pi}_1(x)\boldsymbol{D}_1 &= \boldsymbol{\pi}_1(x)[\boldsymbol{M}_{11} - \boldsymbol{M}_{10}\boldsymbol{M}_{00}^{-1}\boldsymbol{M}_{01}], \\ \boldsymbol{\pi}_0(x) &= -\boldsymbol{\pi}_1(x)\boldsymbol{M}_{10}\boldsymbol{M}_{00}^{-1}, \end{aligned}$$

where $\boldsymbol{\pi}(x) = [\boldsymbol{\pi}_0(x), \boldsymbol{\pi}_1(x)]$. The eigenvalue problem associated with (A1.2) is

$$\text{(A1.3)} \qquad z\boldsymbol{\phi}^{(1)}\boldsymbol{D}_1 = \boldsymbol{\phi}^{(1)}[\boldsymbol{M}_{11} - \boldsymbol{M}_{10}\boldsymbol{M}_{00}^{-1}\boldsymbol{M}_{01}],$$

and the characteristic polynomial is $|z\boldsymbol{D}_1 - (\boldsymbol{M}_{11} - \boldsymbol{M}_{10}\boldsymbol{M}_{00}^{-1}\boldsymbol{M}_{01})|$, which has proper degree $(d_+ + d_-)$.

Now consider the redundant form in (A1.1) and observe that its characteristic polynomial may be factored:

$$|z\boldsymbol{D}-\boldsymbol{M}|=|-\boldsymbol{M}_{00}|\,|z\boldsymbol{D}_1-(\boldsymbol{M}_{11}-\boldsymbol{M}_{10}\boldsymbol{M}_{00}^{-1}\boldsymbol{M}_{01})|. \tag{A1.4}$$

Hence its proper degree is also equal to $(d_+ + d_-)$, and the eigenvalues of the two forms coincide. Also note that (3.2) is true if and only if $\boldsymbol{\phi}=[\boldsymbol{\phi}^{(0)}\boldsymbol{\phi}^{(1)}]$ where $\boldsymbol{\phi}^{(1)}$ satisfies (A1.3) and $\boldsymbol{\phi}^{(0)}=-\boldsymbol{\phi}^{(1)}\boldsymbol{M}_{10}\boldsymbol{M}_{00}^{-1}$.

Finally, the reduced matrix $(\boldsymbol{M}_{11}-\boldsymbol{M}_{10}\boldsymbol{M}_{00}^{-1}\boldsymbol{M}_{01})$ is also symmetrized by a diagonal similarity transformation. Specifically, where $\boldsymbol{W}=\operatorname{diag}(\boldsymbol{w})$ and

$$\boldsymbol{W}\boldsymbol{M}=\boldsymbol{M}'\boldsymbol{W}, \tag{A1.5}$$

it follows that

$$\boldsymbol{W}_1(\boldsymbol{M}_{11}-\boldsymbol{M}_{10}\boldsymbol{M}_{00}^{-1}\boldsymbol{M}_{01})=(\boldsymbol{M}_{11}-\boldsymbol{M}_{10}\boldsymbol{M}_{00}^{-1}\boldsymbol{M}_{01})'\boldsymbol{W}_1 \tag{A1.6}$$

where

$$\boldsymbol{W}=\begin{bmatrix}\boldsymbol{W}_0 & \\ & \boldsymbol{W}_1\end{bmatrix}.$$

Consequently $\boldsymbol{W}_1^{\frac{1}{2}}(\boldsymbol{M}_{11}-\boldsymbol{M}_{10}\boldsymbol{M}_{00}^{-1}\boldsymbol{M}_{01})\boldsymbol{W}_1^{-\frac{1}{2}}$ is symmetric. Note that the nullspace of this matrix is of dimension 1 and that it is spanned by $\boldsymbol{w}_1^{\frac{1}{2}}$, where $\boldsymbol{w}=(\boldsymbol{w}_0,\boldsymbol{w}_1)$.

We digress to observe that $(\boldsymbol{M}_{11}-\boldsymbol{M}_{10}\boldsymbol{M}_{00}^{-1}\boldsymbol{M}_{01})$ is called the Schur complement of $\boldsymbol{M}_{00}$ in $\boldsymbol{M}$ [4], and that $\boldsymbol{M}$ and the Schur complements are irreducible, essentially symmetric and such that their symmetrized form (see Section 2.4) are negative semi-definite. The properties that are needed for the analysis follow directly from this observation. Some of these properties are also easily obtained by noting that, on changing the signs of $\boldsymbol{M}$ and the Schur complements, the resulting matrices are elements of the larger class of singular, irreducible M-matrices [4].

A.2. *Enumeration of modes.* We now consider the eigenvalue problem on $(\boldsymbol{D},\boldsymbol{M})$,

$$z\boldsymbol{\phi}\boldsymbol{D}=\boldsymbol{\phi}\boldsymbol{M} \tag{A1.7}$$

in which it is assumed that the reduction and symmetrization procedures just described have been carried out. Hence in (A1.7), $\boldsymbol{D}$ has no zero diagonal elements, d_+ positive diagonal elements and d_- negative elements; also, $\boldsymbol{M}$ is symmetric, negative semi-definite with a 1-dimensional nullspace spanned by $\boldsymbol{w}^{\frac{1}{2}}$.

Note that $z=0$, $\boldsymbol{\phi}=\boldsymbol{w}^{\frac{1}{2}}$ is a solution of (A1.7). Now, there exists a non-singular matrix $\boldsymbol{K}$ such that [31],

$$\boldsymbol{K}\boldsymbol{M}\boldsymbol{K}'=\begin{bmatrix}0 & \boldsymbol{0}\\ \boldsymbol{0}' & -\boldsymbol{I}\end{bmatrix}. \tag{A1.8}$$

Let

$$\boldsymbol{A}\triangleq\boldsymbol{K}\boldsymbol{D}\boldsymbol{K}'=\begin{bmatrix}e & \boldsymbol{f}\\ \boldsymbol{f}' & \boldsymbol{G}\end{bmatrix}. \tag{A1.9}$$

It is easy to see that e is the inner-product of $\boldsymbol{w}^{\frac{1}{2}}\boldsymbol{D}$ and $\boldsymbol{w}^{\frac{1}{2}}$, and hence that

$$e = \boldsymbol{wD1}, \tag{A1.10}$$

which will be recognized to be the *mean drift.* Further development requires us to treat separately the cases of $e \neq 0$ ('the unbalanced system') and $e = 0$ ('the balanced system').

A.2.1. *The unbalanced system.* From (A1.7)–(A1.9) we have a reduced eigenvalue problem

$$(-z)\psi^{(1)}\left[\boldsymbol{G} - \frac{1}{e}\boldsymbol{f}'\boldsymbol{f}\right] = \psi^{(1)}, \tag{A1.11}$$

where $\psi = (\psi^{(0)}, \psi^{(1)}) = \phi\boldsymbol{K}^{-1}$. At this point we have for the eigenvalue problem (A1.7) on $(\boldsymbol{D}, \boldsymbol{M})$ that one of the eigenvalues is 0 and the remainder, which remain to be ascertained, are the negative reciprocals of the eigenvalues of $[\boldsymbol{G} - (1/e)\boldsymbol{f}'\boldsymbol{f}]$. Since this matrix is symmetric we have established that all eigenvalues of (A1.7) are real.

Now consider the non-singular matrix

$$\boldsymbol{L} \triangleq \begin{bmatrix} 1 & \boldsymbol{0} \\ -\frac{1}{e}\boldsymbol{f}' & \boldsymbol{I} \end{bmatrix} \tag{A1.12}$$

and let

$$\boldsymbol{B} \triangleq \boldsymbol{LAL}' = \begin{bmatrix} e & \boldsymbol{0} \\ \boldsymbol{0}' & \boldsymbol{G} - \frac{1}{e}\boldsymbol{f}'\boldsymbol{f} \end{bmatrix}. \tag{A1.13}$$

By Sylvester's Law of Inertia [3], [31] the signature (by which is meant the number of positive, negative and zero eigenvalues) of $\boldsymbol{D}$ is the same as that of $\boldsymbol{A}$ and $\boldsymbol{B}$. Since the signature of $\boldsymbol{D}$ is known, it follows from (A1.13) that $[\boldsymbol{G} - (1/e)\boldsymbol{f}'\boldsymbol{f}]$ has

d_+ positive eigenvalues and $(d_- - 1)$ negative eigenvalues if $e < 0$,
$(d_+ - 1)$ positive eigenvalues and d_- negative eigenvalues if $e > 0$.

From the comment below (A1.11) it follows that the problem in (A1.7) on $(\boldsymbol{D}, \boldsymbol{M})$ admits one eigenvalue at 0 and

$$\begin{aligned} &d_+ \text{ negative eigenvalues and } (d_- - 1) \text{ positive eigenvalues if } e < 0, \\ &(d_+ - 1) \text{ negative eigenvalues and } d_- \text{ positive eigenvalues if } e > 0. \end{aligned} \tag{A1.14}$$

A.2.2. *The balanced system.* In this case $e = 0$ and the steps given above for the unbalanced system are not defined. On the other hand it is easy to see that in lieu of the reduced eigenvalue problem in (A1.11) we have that

$$(-z)\psi^{(1)}\boldsymbol{P}_f\boldsymbol{G}\boldsymbol{P}_f = \psi^{(1)} \tag{A1.15}$$

where $\psi^{(1)}$ is, as before, part of $\boldsymbol{\phi K}^{-1}$ and $\boldsymbol{P}_f$ is the projection operator

$$\boldsymbol{P}_f = \left[\boldsymbol{I} - \frac{1}{(\boldsymbol{f}, \boldsymbol{f})} \boldsymbol{f}'\boldsymbol{f}\right]. \tag{A1.16}$$

An important observation is that the matrix $\boldsymbol{P}_f\boldsymbol{G}\boldsymbol{P}_f$ in (A1.15) is symmetric.

The enumeration of negative and positive eigenvalues is essentially similar except that now there is an eigenvalue of multiplicity 2 at 0. This is equivalent to observing that there exists a non-trivial solution $\boldsymbol{\xi}$ to the equation

$$\boldsymbol{w}^{\frac{1}{2}}\boldsymbol{D} = \boldsymbol{\xi M}, \tag{A1.17}$$

where $(\boldsymbol{D}, \boldsymbol{M})$ define (A1.7).

To see that $\boldsymbol{\xi}$ exists recall that the nullspace of $\boldsymbol{M}$ is one-dimensional and spanned by $\boldsymbol{w}^{\frac{1}{2}}$. We also know that since $\boldsymbol{M}$ is symmetric, its range space is orthogonal to its nullspace. Now the vector to the left of (A1.17) is orthogonal to $\boldsymbol{M}$'s nullspace since $0 = e = (\boldsymbol{w}^{\frac{1}{2}}\boldsymbol{D}, \boldsymbol{w}^{\frac{1}{2}})$. Therefore it is an element of the range space of $\boldsymbol{M}$ and $\boldsymbol{\xi}$ exists.

A.3. *Characterization of the modes of the differential equation.* The purpose here is to observe that the forms of the reduced eigenvalue problems in (A1.11) and (A1.15) imply simple forms for the independent solutions of (3.1). In the case of the unbalanced line, the important observation is that $[\boldsymbol{G} - (1/e)\boldsymbol{f}'\boldsymbol{f}]$ is symmetric. Even if it has eigenvalues of multiplicity greater than 1, it is known [3], [31] that it may be diagonalized by an orthogonal transformation. Specifically, for each eigenvalue of multiplicity k there exists k linearly independent eigenvectors which may be taken to be mutually orthogonal.

On tracing the effect of the above observation on (3.1) we find, for the unbalanced system, that there are $(d_- + d_+)$ solutions $(z, \boldsymbol{\phi})$ to the eigenvalue problem (3.2) in which the eigenvectors $\{\boldsymbol{\phi}\}$ form an independent set, and that the independent solutions to (3.1) are of the form $e^{zx}\boldsymbol{\phi}$.

For the balanced system the eigenvalue of multiplicity 2 at 0 give independent solutions $\boldsymbol{w}$ and $(\boldsymbol{\xi} + x\boldsymbol{w})$ where

$$\boldsymbol{wD} = \boldsymbol{\xi M}, \tag{A1.18}$$

which is identical to (A1.17) in the original coordinate system. From the rank and symmetry of $\boldsymbol{P}_f\boldsymbol{G}\boldsymbol{P}_f$ in (A1.15) we also deduce that there are $(d_- + d_+ - 2)$ other independent solutions $e^{zx}\boldsymbol{\phi}$, $z \neq 0$.

Appendix B: The proper degrees of the polynomials $P(z; k_1, k_2)$.

The degree of the polynomial $P(z; k_1, k_2)$ in (4.15i) for $k_1 \neq m/2$, $k_2 \neq n/2$, is 4. The proper degree of this polynomial is obtained by subtracting from 4 the number of conditions from the following set which are true:

(i) $c_1k_1 - c_2k_2 = 0$

(ii) $c_1(m - k_1) - c_2k_2 = 0$,

(iii) $c_1(m-k_1)-c_2(n-k_2)=0$,
(iv) $c_1k_1-c_2(n-k_2)=0$.

To see this consider first the case of $mc_1 \neq nc_2$. In this case, at most one of the conditions may be true and then it can be readily shown that the proper degree of $P(z; k_1, k_2)$ is 3. If $mc_1 = nc_2$ then either none or a pair of the above conditions is true, and in the latter case the proper degree is 2.

Similar reasoning extends to the case where $k_1 = m/2$ or $k_2 = n/2$. If $k_1 = m/2$, $k_2 \neq n/2$ then, see (4.15ii), $P(z; k_1, k_2)$ has degree 2. The proper degree is obtained by subtracting from 2 the number of occurrences of

(i) $c_1k_1-c_2k_2=0$,
(ii) $c_1k_2-c_2(n-k_2)=0$.

We note parenthetically that at most one of the conditions may be true. A related result holds for $k_1 \neq m/2$, $k_2 = n/2$. Finally, for the case $k_1 = m/2$, $k_2 = n/2$, the proper degree is either 1 or 0, where the latter holds if and only if $c_1k_1 - c_2k_2 = 0$.

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