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# CONNECTING REVERSIBLE MARKOV PROCESSES

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## Abstract

We provide a framework for interconnecting a collection of reversible Markov processes in such a way that the resulting process has a product-form invariant measure with respect to which the process is reversible. A number of examples are discussed including Kingman's reversible migration process, interconnected random walks and stratified clustering processes.

REVERSIBILITY; QUEUEING NETWORKS; PRODUCT-FORM; INVARIANT MEASURES

## 1. Introduction

In recent times considerable attention has been given to questions concerning the existence of product-form stationary distributions for vector-valued processes and its relationship both to the important notion of partial balance and to that of insensitivity (see, for example, Franken et al. (1981), Jansen and König (1980), Kelly (1979), Walrand and Varaiya (1980) and Whittle (1984)). The questions largely arose out of a study of stochastic systems involving interaction and flow, such as population processes and networks of queues. The classical migration process of Whittle (1967), (1968) and, in particular, the Jackson network (Jackson (1957)) were found to possess product-form equilibrium distributions and, in the open case where individuals may enter or leave the system in question, the state of each of the colonies (queues) was found to be independent. Indeed many aspects of joint equilibrium behaviour of these systems may be studied by considering the behaviour of the individual nodes in isolation, those being birth and death processes with constant birth rates and, in the context of a Jackson network, multiserver queues. Each of these nodes in isolation is a reversible Markov process. In contrast, the so-called BCMP networks (Baskett et al. (1975)) contain nodes that, in isolation, may not be reversible. However, each possesses the intriguing property of *quasireversibility*, first identified by Muntz (1972). Kelly (1976b) coined the name and later refined the notion (Kelly (1979); see also Walrand

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(1981), Walrand and Varaiya (1980)). More recently (Kelly (1981)), the definition has been modified to allow for the possibility that a given node might not conserve customers of a given class. Although this does not occur naturally in queueing networks it is sometimes an essential feature of other vector-valued Markov processes, for example the stratified clustering process (Whittle (1965a), (1980b)).

Product-form phenomena are not restricted to systems consisting of interconnected quasireversible nodes. The reversible migration process of Kingman (1969) (see also Kelly (1979)) illustrates this point. Each of the individual nodes is a birth and death process and thus quasireversible only if the birth rate is constant (Brémaud (1978)). The nodes are connected in such a way that the migration process, like each of the nodes themselves, is reversible. In this paper a general framework is provided for connecting *reversible* Markov processes. The resulting system possesses a product-form invariant measure provided the nodes retain their reversibility under the alterations of transition rates dictated by a set of traffic equations. Although the individual nodes are reversible they *need not* be quasireversible.

In Section 2 we collect together various preliminary results on Markov processes, invariant measures and reversibility. In Section 3 we examine the effect of altering the  $q$ -matrix of a reversible Markov process. Necessary and sufficient conditions are established for a process to retain its reversibility under such an alteration. Section 4 contains the main result of the paper. A general procedure for connecting Markov processes is described and provided they are reversible (with respect to some invariant measure) under modifications dictated by the traffic equations, it is possible to obtain sufficient conditions for the interconnected system to be reversible. If they are satisfied a product-form invariant measure for the system can be determined from invariant measures for the individual nodes. In Section 5 we discuss a number of examples of systems of interconnected reversible nodes. These include a generalisation of Kingman's reversible migration process, interconnected random walks and a discussion of clustering with migration. These examples lie outside the realm of quasireversible (queueing) networks since, although the study of such systems has motivated the methods used in this paper, the conditions necessary to ensure reversibility are usually inappropriate. In particular it is necessary that there be no net circulation of customers in the system; in the queueing network context this can be very restrictive. In the examples we consider this condition is natural and in some cases an essential feature. Finally, in Section 6, we examine the differences between systems consisting of reversible nodes and those consisting of quasireversible nodes, indicating the practical limitations of both.

**2. Preliminaries**

Let  $Q = (q(x, y), x, y \in \zeta)$  be a stable, conservative  $q$ -matrix over a countable set  $\zeta$ , where for convenience  $q(x, x) = 0, x \in \zeta$ . A standard, time-homogeneous Markov chain,  $(X(t), t \geq 0)$ , taking values in  $\zeta$  can be constructed from  $Q$  using the method of Feller. This construction defines the process up to a random time,  $T$ , called the terminal time. If  $T$  is infinite (with probability 1) the process is unique and we say that  $Q$  is *regular*. Reuter (1957) provides necessary and sufficient conditions for regularity. If these are not satisfied  $T$  may be finite, in which case the process explodes by performing infinitely many transitions in a finite time.

We say that  $Q$  admits a positive *invariant measure*  $m = (m(x), x \in \zeta)$ , if there exists a collection of positive quantities,  $(m(x), x \in \zeta)$ , that satisfy the so-called *full-balance equations*

$$(1) \quad m(x)q(x) = \sum_{y \in \zeta} m(y)q(y, x) \quad x \in \zeta,$$

where  $q(x) = \sum_{y \in \zeta} q(x, y)$ . This is in contrast to the definition of an invariant measure for the family,  $\{P(t), t \geq 0\}$ , of transition matrices of the Feller process generated by  $Q$  (see, for example, Kendall (1975)). If  $Q$  is irreducible and recurrent then (1) has a solution, unique up to constant multiples, and this is an invariant measure for  $\{P(t), t \geq 0\}$ . If  $Q$  is transient then the set of solutions to (1) may be empty or may contain linearly independent elements. Any one solution,  $m$ , is an invariant measure for  $\{P(t), t \geq 0\}$  if and only if the  $q$ -matrix  $Q' = (q'(x, y), x, y \in \zeta)$ , the *time-reverse of  $Q$  with respect to  $m$* , defined by

$$q'(x, y) = m(y)q(y, x)/m(x) \quad x, y \in \zeta,$$

is regular (Kelly (1983)).

Observe that  $Q$  and  $Q'$  will be the same if and only if the so-called *detail-balance equations*,

$$(2) \quad m(x)q(x, y) = m(y)q(y, x) \quad x, y \in \zeta,$$

are satisfied. Indeed, if we can find positive quantities  $(m(x), x \in \zeta)$  that satisfy (2) then summing over  $y$  shows that they must be a solution to (1). Thus we have proved the following simple result.

*Lemma 1.* Any positive measure,  $m = (m(x), x \in \zeta)$ , that satisfies (2) is invariant for  $Q$ , and is invariant for  $\{P(t), t \geq 0\}$  if and only if  $Q$  is regular.

If we can find an invariant measure,  $m$ , that satisfies (2) then we say that the Markov process is (*symmetrically*) *reversible with respect to  $m$* . This nomenclature is suggested by the observation that if  $m$  is summable we can extend the

definition of the process to the whole of the real line, in such a way that the event  $\{X(t) = x\}$  has measure  $m(x)$ , to produce a Markov process,  $(X(t), t \in \mathbb{R})$ , that has the same law as  $(X(-t), t \in \mathbb{R})$ . If  $Q$  is regular and  $\sum_{x \in \zeta} m(x)$  converges then  $Q$  is positive recurrent and  $m$  may be normalised to produce the (unique) stationary distribution of both processes. If  $\sum_{x \in \zeta} m(x)$  diverges then  $Q$  is either null recurrent, in which case  $m$  is essentially unique, or transient, in which case there may be other invariant measures that satisfy (2) and one must say with respect to which invariant measure the process is reversible.

Notice that since  $m$  is positive the detail-balance equations imply that

$$q(x, y) > 0 \Leftrightarrow q(y, x) > 0 \quad x, y \in \zeta.$$

This means that if the process is reversible the irreducible sub-classes will not communicate with one another, and the process remains within the class in which it started. Thus for a reversible Markov process it is often convenient to assume that the  $q$ -matrix is irreducible.

The detail-balance equations, (2), provide a convenient means for deciding whether or not reversibility ensues. Since an invariant measure is determined by  $Q$  it is natural to ask whether or not reversibility can be established considering  $Q$  alone. The Kolmogorov criteria (Kolmogorov (1936), Kendall (1959)) provide a means: there exists an invariant measure satisfying (2) if and only if for each state  $y$  and each finite sequence of states  $z_1, z_2, \dots, z_r$ , where  $r \geq 2$ , distinct from one another and from  $y$  we have

$$(3) \quad q(y, z_1)q(z_1, z_2) \cdots q(z_r, y) = q(y, z_r)q(z_r, z_{r-1}) \cdots q(z_1, y).$$

In practice it is not necessary to test these criteria for all closed paths in  $\zeta$  since their truth can be established by proving that they are satisfied for certain simple paths (see, for example, Kelly (1979), Exercise 1.5.2).

The definition of symmetric reversibility given here is slightly more general than that given by Kendall (1975). However, the two definitions coincide provided  $Q$  is regular. If  $Q$  is not regular then it may be possible that  $Q$  admits an invariant measure with respect to which the process is reversible, but the argument of Kendall (1975) shows that it must be strictly subinvariant for  $P(t)$ . An indication of the extent of this subinvariance can be obtained by considering the particle system studied by Derman (1955) and Brown (1970). It also serves to provide an alternative interpretation of reversibility and invariance, one which remains valid even when the process is not stationary (see Kelly (1983)).

### 3. Altering the $q$ -matrix

Let  $(X(t), 0 \leq t < T)$  be a standard, time-homogeneous Markov process with a stable, conservative, irreducible  $q$ -matrix,  $Q$ , over a countable state space,  $\zeta$ .

Suppose that  $Q$  admits an invariant measure,  $m$ , with respect to which the process is reversible.

Let  $A(c), c \in C$  be a countable collection of mutually disjoint subsets of

$$\xi = \{(x, y) \in \zeta \times \zeta : q(x, y) > 0\}$$

and define

$$\hat{Q} = (\hat{q}(x, y), x, y \in \zeta)$$

by

$$\hat{q}(x, y) = \begin{cases} \eta(c)q(x, y) & (x, y) \in A(c), \quad c \in C \\ q(x, y) & (x, y) \notin U \end{cases}$$

where  $(\eta(c), c \in C)$  is a collection of positive constants and  $U = \bigcup_{c \in C} A(c)$ . The matrix  $\hat{Q}$  is thus a  $q$ -matrix formed by altering the rates of a collection of types of transitions; the rate of a type  $c$  transition is altered by a factor  $\eta(c)$ . It is instructive to think of  $A(c)$  as consisting of those transitions which correspond to arrivals of individuals or units of type  $c$  and the reverse transitions as being type  $c$  departures, although this interpretation is not essential to the arguments which follow. An alteration of the rate of a type  $c$  transition would then correspond to a change in the rate at which type  $c$  units arrive. In the next section we shall see that the operation of connecting one or more reversible Markov processes will usually mean that the rates of certain transitions will be altered from what they were in isolation. In particular these rates will be determined by the communication mechanism through a set of traffic equations. Further, if the specification of this mechanism is altered, or if the demands on the system from outside change, the system may be rendered intractable, for the simplifying feature of reversibility may be destroyed. For this reason we shall establish conditions under which a process remains reversible after a specified alteration of its transition rates, that is, conditions under which  $\hat{Q}$  is reversible with respect to some invariant measure.

*Lemma 2.* If there exists a function  $f: \zeta \rightarrow \mathbb{R}^+$  that satisfies the recurrence

$$(4) \quad f(y) = \begin{cases} \eta(c)f(x) & (y, x) \notin U \\ \frac{\eta(c)}{\eta(d)}f(x) & (y, x) \in A(d) \\ \frac{1}{\eta(c)}f(x) & (y, x) \in A(c) \\ f(x) & (y, x) \notin U \end{cases} \quad (x, y) \in A(c), c \in C$$

then  $\hat{Q}$  admits an invariant measure,  $\hat{m} = (\hat{m}(x), x \in \zeta)$ , given by  $\hat{m}(x) = f(x)m(x)$  and the Markov process constructed from  $\hat{Q}$  is reversible with respect

to  $\hat{m}$ . Conversely, if  $\hat{Q}$  admits an invariant measure,  $\hat{m}$ , with respect to which a Markov process constructed from it is reversible, then  $f(x) = \hat{m}(x)/m(x)$  satisfies the recurrence (4).

*Proof.* Define  $\hat{m}(x) = f(x)m(x)$  and consider the equations

$$\hat{m}(x)\hat{q}(x, y) = \hat{m}(y)\hat{q}(y, x).$$

By using (2) it is straightforward to show these will be satisfied if and only if  $f$  satisfies (4). The fact that  $\hat{m}$  is invariant for  $\hat{Q}$  follows from Lemma 1.

Often it is not a simple matter to establish the existence of a function satisfying (4). The next result provides *sufficient* conditions for reversibility to be preserved under the alteration.

*Lemma 3.* If a function,  $n : C \times \zeta \rightarrow \mathbb{Z}$ , exists such that for all  $c$  in  $C$  and  $(x, y)$  in  $\zeta$

$$(5) \quad n(c, y) = \begin{cases} n(c, x) + 1 & (x, y) \in A(c) \quad (y, x) \notin A(c) \\ n(c, x) - 1 & (x, y) \notin A(c) \quad (y, x) \in A(c) \\ n(c, x) & \text{otherwise} \end{cases}$$

then  $f$ , given by

$$f(x) = \prod_{c \in C} [\eta(c)]^{n(c,x)} \quad x \in \zeta,$$

satisfies (4).

*Proof.* Suppose that there exists an  $n$  with the required property. Then it is easy to show that

$$f(x) = \prod_{c \in C} [\eta(c)]^{n(c,x)}$$

satisfies (4), by considering  $f(y)/f(x)$  for all choices of  $(x, y)$  in  $\zeta$ .

The condition of Lemma 3, that there exists a quantity which is invariant under ‘internal’ transitions, is precisely analogous to that provided by Kelly (1981) for preserving quasireversibility. In many practical situations there is an obvious candidate for  $n$ , where  $n(c, x)$  can be interpreted as the number of units in the node when its state is  $x$ .

We shall conclude this section with an example for which the function  $f$  may be calculated explicitly. Let us suppose that for each  $c \in C$  there exists a subset,  $\zeta(c)$ , of  $\zeta$  such that

$$A(c) = \{(x, y) : x \in \zeta(c) \text{ and } y \in \zeta - \zeta(c)\}.$$

Thus a type  $c$  transition indicates that the process has left  $\zeta(c)$ . It can be

shown, although the details are somewhat tedious, that  $f$ , given by

$$f(y) = \prod_{c \in C} \beta(c; y) \quad y \in \zeta,$$

where

$$\beta(c; y) = \begin{cases} 1 & y \in \zeta(c) \\ \eta(c) & y \in \zeta - \zeta(c), \end{cases}$$

satisfies (4). If  $C$  contains only one element then this amounts to Lemma 1.9 of Kelly (1979). The implicit assumption that the  $A(c)$ ,  $c \in C$  are mutually disjoint places some restriction on the transition rates. In particular it means that each transition can result in the exit of at most one of  $\zeta(c)$ ,  $c \in C$ . This condition appears in the context of insensitivity (see Whittle (1985) for a recent economical and elegant treatment). If we assume further that for a single transition the entry of more than one of  $\zeta(c)$ ,  $c \in C$  is forbidden then  $m$  is *insensitive* to the distribution of nominal sojourn times in  $\zeta(c)$ ,  $c \in C$ , owing to the reversibility of  $Q$ . We have shown that this insensitivity is preserved under *any* alteration of the rates of transition from  $\zeta(c)$  into its complement. This observation is extremely important since such an alteration precisely amounts to changing the rate (work rate) at which nominal sojourns in  $\zeta(c)$  are completed.

#### 4. Interconnected processes

We now examine the behaviour of a Markov process constructed by linking together a finite collection, labelled  $1, 2, \dots, J$ , of reversible nodes. Each node,  $j$ , is a standard, time-homogeneous Markov process,  $(X_j(t), t \geq 0)$ , constructed from a stable, conservative  $q$ -matrix,  $Q_j$ , (not necessarily irreducible) over a countable state space,  $\zeta_j$ . Associated with  $\zeta_j$  is the set of all pairs of states corresponding to possible transitions,  $\bar{\zeta}_j$ , together with a countable, mutually disjoint collection,  $(A_j(c), c \in C_j)$ , of subsets of  $\bar{\zeta}_j$  whose union is  $U_j$ . We shall assume that if  $(x, y) \in A_j(c)$  for some  $c \in C_j$  then  $(y, x) \notin U_j$  so that a reverse type  $c$  transition at node  $j$  is not a transition of another type.

Now define a  $q$ -matrix,  $Q$ , over the state space  $\zeta = \zeta_1 \times \zeta_2 \times \dots \times \zeta_J$  with elements  $q(x, y)$ ,  $x, y \in \zeta$ , defined as follows. If

$$x = (x_1, x_2, \dots, x'_j, \dots, x_k, \dots, x_J)$$

and

$$x' = (x_1, x_2, \dots, x_j, \dots, x'_k, \dots, x_J)$$

where

$$(x_j, x'_j) \in A_j(c), c \in C_j, \quad \text{and} \quad (x_k, x'_k) \in A_k(d), d \in C_k,$$

then set

$$(6) \quad q(\mathbf{x}, \mathbf{x}') = q_j(x'_j, x_j)q_k(x_k, x'_k)\lambda_{j,k}(c, d)$$

where  $(\lambda_{j,k}(c, d), j, k = 1, 2, \dots, J, c \in \mathbf{C}_j, d \in \mathbf{D}_k)$  is a collection of non-negative constants such that, for convenience,  $\lambda_{j,j}(c, d) = 0$  for all  $c, d \in \mathbf{C}_j, j = 1, 2, \dots, J$ . If

$$\mathbf{x} = (x_1, x_2, \dots, x_j, \dots, x_J)$$

and

$$\mathbf{x}' = (x_1, x_2, \dots, x'_j, \dots, x_J)$$

then set

$$(7) \quad q(\mathbf{x}, \mathbf{x}') = q_j(x_j, x'_j)v_j(c)$$

if  $(x_j, x'_j) \in \mathbf{A}_j(c), c \in \mathbf{C}_j$ , or

$$(8) \quad q(\mathbf{x}, \mathbf{x}') = q_j(x_j, x'_j)\mu_j(c)$$

if  $(x'_j, x_j) \in \mathbf{A}_j(c), c \in \mathbf{C}_j$ , where  $(v_j(c), \mu_j(c), j = 1, 2, \dots, J, c \in \mathbf{C}_j)$  is a collection of non-negative constants; if neither  $(x_j, x'_j)$  nor  $(x'_j, x_j)$  is an element of  $\mathbf{U}_j$  then set

$$(9) \quad q(\mathbf{x}, \mathbf{x}') = q_j(x_j, x'_j).$$

If a transition from one state,  $\mathbf{x}$ , to another,  $\mathbf{y}$ , is of none of the above forms then set  $q(\mathbf{x}, \mathbf{y}) = 0$ . Thus a *reverse* type  $c$  transition at node  $j$  may trigger a transition of some type,  $d$ , at another node,  $k$ , and the parameter  $\lambda_{j,k}(c, d)$  may be regarded as measuring the tendency for this to happen. However, this tendency will usually be affected by the states of both nodes  $j$  and  $k$ . It might be that such a transition results in no transition elsewhere, and the tendency for this to happen is measured by  $\mu_j(c)$ . We have also allowed for the possibility that transitions might be triggered, as it were, exogenously. The parameter  $v_j(c)$  measures the tendency for a type  $c$  transition at node  $j$  to be triggered in this way.

*Theorem 1.* If:

(i) there exist strictly positive quantities  $\eta_j(c), c \in \mathbf{C}_j, j = 1, 2, \dots, J$ , that satisfy

$$\eta_j(c)\mu_j(c) = v_j(c)$$

and

$$\eta_j(c)\lambda_{j,k}(c, d) = \eta_k(d)\lambda_{k,j}(d, c),$$

for all  $j, k = 1, 2, \dots, J, c \in \mathbf{C}_j$  and  $d \in \mathbf{C}_k$ , and

(ii) for each  $j$ , the modification

$$\hat{q}_j(\mathbf{x}, \mathbf{x}') = \begin{cases} \eta_j(c)q_j(\mathbf{x}, \mathbf{x}') & (\mathbf{x}, \mathbf{x}') \in \mathbf{A}_j(c), c \in \mathbf{C}_j \\ q_j(\mathbf{x}, \mathbf{x}') & (\mathbf{x}, \mathbf{x}') \notin \mathbf{U}_j \end{cases}$$

produces a  $q$ -matrix,  $\hat{Q}_j$ , that admits an invariant measure,  $\hat{m}_j = (\hat{m}_j(x), x \in \zeta_j)$ , with respect to which the Markov process constructed from  $\hat{Q}_j$  is reversible, then  $Q$  admits an invariant measure,  $m = (m(x), x \in \zeta)$ , given by

$$m(x) = m(x_1, x_2, \dots, x_J) = \prod_{j=1}^J \hat{m}_j(x_j)$$

and the Markov process,  $(X(t), t \geq 0)$ , constructed from  $Q$  is reversible with respect to  $m$ .

*Remarks.* We have made no assumptions about each of the  $Q_j, j = 1, 2, \dots, J$ , regarding reversibility, only that the modified  $q$ -matrices yield reversibility with respect to some invariant measure. Nor have we made any assumptions about regularity, recurrence, etc., of the modified  $q$ -matrices. Questions regarding  $Q$ , although undoubtedly related, are essentially independent of those pertaining to the individual nodes.

Condition (i) places some restrictions on the way the individual nodes are linked, but these are necessary for the interconnected system to be reversible. For example, we have that

$$v_j(c) > 0 \Leftrightarrow \mu_j(c) > 0 \quad c \in C_j$$

and

$$\lambda_{j,k}(c, d) > 0 \Leftrightarrow \lambda_{k,j}(d, c) > 0 \quad c \in C_j, \quad d \in D_k,$$

and there is an obvious symmetry in the way the nodes interact. Although this restriction does not arise naturally in, say, queueing networks, it does in many other vector-valued processes. We shall consider some examples in the next section.

Condition (ii) requires that the modified  $q$ -matrices be reversible with respect to some invariant measure. However, if each of the nodes in isolation exhibit reversibility and for each  $j = 1, 2, \dots, J$  there exists a function  $n_j: C_j \times \zeta_j \rightarrow \mathbb{Z}$  satisfying the conditions of Lemma 3, then (ii) follows immediately and  $m$  has the particular form given by

$$(10) \quad m(x) = \prod_{j=1}^J m_j(x_j) \prod_{c \in C_j} [\eta_j(c)]^{n_j(c, x_j)}.$$

Of course this is not the only way of verifying condition (ii). It may be that for some  $j$  we cannot find a function  $n_j$  satisfying (5), yet node  $j$  might still be reversible under the alteration. This will happen if *and only if* a function  $f_j: \zeta_j \rightarrow \mathbb{R}^+$  exists satisfying the conditions of Lemma 2 and then the  $j$ th term in the product (10) will be  $m_j(x_j)f_j(x_j)$ .

As a final remark it should be noted that in some applications the number of nodes,  $J$ , might not be finite (see, for example, Kelly (1976a)). Indeed the theorem holds good when  $J$  is infinite provided  $\zeta$  is countable.

*Proof of Theorem 1.* We show that the ratio

$$(11) \quad m(\mathbf{x})q(\mathbf{x}, \mathbf{x}')/m(\mathbf{x}')q(\mathbf{x}', \mathbf{x})$$

is 1 for all  $\mathbf{x}, \mathbf{x}' \in \zeta$  such that  $q(\mathbf{x}, \mathbf{x}') > 0$ . If  $q(\mathbf{x}, \mathbf{x}')$  is of the form (6) then the ratio is

$$\begin{aligned} & \frac{\hat{m}_j(x'_j)q_j(x'_j, x_j)\hat{m}_k(x_k)q_k(x_k, x'_k)\lambda_{j,k}(c, d)}{\hat{m}_j(x_j)q_j(x_j, x'_j)\hat{m}_k(x'_k)q_k(x'_k, x_k)\lambda_{k,j}(d, c)} \\ &= \frac{\hat{m}_j(x'_j)q_j(x'_j, x_j)\hat{m}_k(x_k)\eta_k(d)q_k(x_k, x'_k)}{\hat{m}_j(x_j)\eta_j(c)q_j(x_j, x'_j)\hat{m}_k(x'_k)q_k(x'_k, x_k)} \text{ by (i).} \end{aligned}$$

By using (ii) and expression (2) applied to nodes  $j$  and  $k$  we see that the ratio is unity. If  $q(\mathbf{x}, \mathbf{x}')$  takes any of forms (7), (8) or (9) then the argument is very similar; if, for example, the form taken is (7), then the ratio is

$$\frac{\hat{m}_j(x_j)q_j(x_j, x'_j)v_j(c)}{\hat{m}_j(x'_j)q_j(x'_j, x_j)\mu_j(c)} = \frac{\hat{m}_j(x_j)\eta_j(c)q_j(x_j, x'_j)}{\hat{m}_j(x'_j)q_j(x'_j, x_j)} = 1.$$

### 5. Examples

In this section we consider various examples to illustrate Theorem 1.

(i) *The reversible migration process.* The model presented is a generalisation of one studied by Kingman (1969) (see also Kelly (1979)). We shall start by describing the individual nodes. Let us suppose that each node,  $j$ , is a multi-class birth and death process defined as follows. Let  $\mathbf{C}_j$  be a countable collection of types of individual and let

$$\zeta_j = \{(n(c), c \in \mathbf{C}_j) : n(c) \in \mathbb{N}\}.$$

Write  $\mathbf{e}_c \in \zeta_j$  for the unit vector with 1 in the  $c$ th position and for  $\mathbf{n} \in \zeta_j$  suppose that the only non-zero elements of  $\mathbf{Q}_j$  are given by

$$q_j(\mathbf{n}, \mathbf{n} + \mathbf{e}_c) = \psi_{j,c}(n(c)) \quad c \in \mathbf{C}_j$$

and

$$q_j(\mathbf{n}, \mathbf{n} - \mathbf{e}_c) = \Phi_{j,c}(n(c)) \quad c \in \mathbf{C}_j,$$

where  $\Phi_{j,c}(0) = 0$ . We shall assume that  $\Phi_{j,c}(n) > 0$  if  $n > 0$  and  $\psi_{j,c}(n) > 0$  if  $n \geq 0$  so that  $\mathbf{Q}_j$  is irreducible over  $\zeta_j$ . The quantity  $\psi_{j,c}(n(c))$  may be regarded as the birth rate of type  $c$  individuals when there are  $n(c)$  present, while  $\Phi_{j,c}(n(c))$  may be regarded as the death rate. In accordance with this we associate transitions that lie in

$$\mathbf{A}_j(c) = \{(\mathbf{n}, \mathbf{n} + \mathbf{e}_c) : \mathbf{n} \in \zeta_j\}$$

with births and the reverse transitions with deaths.

It is well known that the birth and death process is reversible. This can be established, using the Kolmogorov criteria, by considering only those paths with  $r = 3$  that form a closed loop in the state space, namely,  $\mathbf{n} \rightarrow \mathbf{n} + \mathbf{e}_c \rightarrow \mathbf{n} + \mathbf{e}_c + \mathbf{e}_d \rightarrow \mathbf{n} + \mathbf{e}_d \rightarrow \mathbf{n}$  for arbitrary  $c, d \in C_j, d \neq c$ . The truth of the Kolmogorov criteria for such paths follows from the identity

$$(12) \quad \begin{aligned} \psi_{j,c}(n(c))\psi_{j,d}(n(d))\Phi_{j,c}(n(c) + 1)\Phi_{j,d}(n(d) + 1) \\ = \psi_{j,d}(n(d))\psi_{j,c}(n(c))\Phi_{j,d}(n(d) + 1)\Phi_{j,c}(n(c) + 1) \quad d \neq c. \end{aligned}$$

Thus we may determine an invariant measure,  $m_j = (m_j(\mathbf{n}), \mathbf{n} \in \xi_j)$ , by solving the detail-balance equations,

$$m_j(\mathbf{n})\psi_{j,c}(n(c)) = m_j(\mathbf{n} + \mathbf{e}_c)\Phi_{j,c}(n(c) + 1) \quad c \in C_j,$$

and the process generated from  $Q_j$  will be reversible with respect to  $m_j$ . Their (essentially unique) solution is given by

$$m_j(\mathbf{n}) = \prod_{c \in C_j} \chi_{j,c}(n(c)) \quad \mathbf{n} \in \xi_j,$$

where

$$\chi_{j,c}(n) = \prod_{r=1}^n \{ \psi_{j,c}(r-1) / \Phi_{j,c}(r) \}.$$

It is clear that we can allow the transition rates to depend, in a more general sense, on the whole vector  $\mathbf{n}$  and still retain reversibility. All we require is that the Kolmogorov criteria be satisfied for the simple paths described above. For example, we can set

$$q_j(\mathbf{n}, \mathbf{n} + \mathbf{e}_c) = g_j(N)\psi_{j,c}(n(c)) \quad c \in C_j$$

and

$$q_j(\mathbf{n}, \mathbf{n} - \mathbf{e}_c) = h_j(N)\Phi_{j,c}(n(c)) \quad c \in C_j$$

for  $\mathbf{n} \in \xi_j$ , where the functions  $g_j$  and  $h_j$  are strictly positive and  $N = \sum_{c \in C_j} n(c)$ . It is elementary to check that the essentially unique invariant measure, with respect to which the process is reversible, is given by

$$m_j(\mathbf{n}) = \prod_{r=1}^N \frac{g_j(r-1)}{h_j(r)} \prod_{c \in C_j} \chi_{j,c}(n(c)).$$

The system resulting from connecting birth and death processes together according to the procedure described in Section 4 is known as a migration process. A transition of the type (6) indicates that an individual of type  $c \in C_j$  at node  $j$  has immigrated to node  $k$  and has changed his type to  $d \in C_k$ . A transition of type (7) indicates the immigration of a type  $c \in C_j$  individual to node  $j$  from outside the system while transitions of type (8) indicate type  $c \in C_j$  emigration from node  $j$  to the outside. There are no transitions of type (9). Thus

the system may be thought of as a set of  $J$  colonies. Individuals may enter the system and move between the various colonies according to a stochastic rule, spending a period of time at each colony they visit, before possibly leaving the system.

The state space for the migration process may be written

$$\zeta = \{(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_J) : \mathbf{n}_j \in \zeta_j, j = 1, 2, \dots, J\};$$

if  $J$  is infinite we shall mean the subset comprising those elements which have only finitely many non-zero terms. Let us suppose that we can find quantities  $\eta_j(c)$ ,  $c \in C_j$ ,  $j = 1, 2, \dots, J$ , satisfying condition (i) of the theorem. Then it is clear that condition (ii) will be satisfied since  $\hat{Q}_j$  defines a birth and death process with the same death rates, and birth rates given by

$$\hat{q}_j(\mathbf{n}, \mathbf{n} + \mathbf{e}_c) = \eta_j(c)q_j(\mathbf{n}, \mathbf{n} + \mathbf{e}_c) \quad \mathbf{n} \in \zeta_j, \quad c \in C_j.$$

We may determine an appropriate invariant measure for  $\hat{Q}_j$  using Lemma 2 and Lemma 3, since there is an obvious candidate for a function which satisfies the recurrence (5), namely  $n(c, \mathbf{n}) = n(c)$ , and therefore  $\hat{m}_j$  is given by

$$\hat{m}_j(\mathbf{n}) = m_j(\mathbf{n}) \prod_{c \in C_j} [\eta_j(c)]^{n(c)} \quad \mathbf{n} \in \zeta_j.$$

Thus the  $q$ -matrix,  $Q$ , of the migration process admits the invariant measure

$$m(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_J) = \prod_{j=1}^J m_j(\mathbf{n}_j) \prod_{c \in C_j} [\eta_j(c)]^{n_j(c)} \quad \mathbf{n}_j \in \zeta_j, \quad j = 1, 2, \dots, J,$$

and the Markov process,  $(X(t), t \geq 0)$ , constructed from  $Q$  is reversible with respect to this measure.

The reversible migration process admits a number of important special cases, some of which have been studied extensively in their own right. These include models of social grouping behaviour (Cohen (1971), (1972)), flow models and electrical networks (Spitzer (1970), Kingman (1969), Kelly (1979), Nash-Williams (1959), Kemeny et al. (1976) and Kelly (1976c)), and models in genetics, in particular the neutral allele model (Moran (1958), Trajstman (1974), Kelly (1976a), Kelly (1977), Watterson (1976), Watterson and Guess (1977)). In all of these, the assumption that the nodes interact in a symmetric fashion is very natural and in no sense contrived. However, this is not always the case and in some applications, for example queuing networks with blocking, the symmetry condition is imposed to ensure analytical truncatability and is often far too restrictive.

(iii) *Interconnected random walks.* In this example we suppose that each

node,  $j$ , is a random walk on the integers whose  $q$ -matrix,  $\mathcal{Q}_j$ , is defined by

$$q_j(n, m) = \begin{cases} \psi_j g_j(n) & m = n + 1 \\ \Phi_j g_j(n) & m = n - 1 \\ 0 & \text{otherwise} \end{cases}$$

for  $n \in \mathbb{Z}$ , where (without loss of generality)  $\psi_j \geq \Phi_j \geq 0$ ,  $\psi_j + \Phi_j = 1$  and  $g_j(n) > 0$  for all  $n \in \mathbb{Z}$ .

Results for this random walk are well known (for a recent exposition see Kelly (1983)). It is clearly reversible if and only if  $\Phi_j > 0$ , in which case the detail-balance equations yield the essentially unique solution

$$m_j(n) = \rho_j^n / g_j(n) \quad n \in \mathbb{Z},$$

where  $\rho_j = \psi_j / \Phi_j \geq 1$ . However,  $\mathcal{Q}_j$  admits other invariant measures, for example,

$$([g_j(n)]^{-1}, n \in \mathbb{Z}).$$

This is different from the first provided  $\psi_j > \Phi_j$ , but the process generated from  $\mathcal{Q}_j$  is not reversible with respect to this measure. It is also clear that  $\mathcal{Q}_j$  is recurrent if and only if  $\rho_j = 1$ . If  $\mathcal{Q}_j$  is transient ( $\rho_j > 1$ ) then it is possible to show that a necessary and sufficient condition for regularity is that

$$S_j^+ \triangleq \sum_{n=0}^{\infty} [g_j(n)]^{-1} = \infty.$$

If  $\mathcal{Q}_j$  is recurrent, regularity is assured but, since  $m_j$  is not summable, positive recurrence is impossible.

Let us now examine the effect of causing these random walks to interact. Define the  $q$ -matrix,  $\mathcal{Q}$ , of the interconnected process as follows. For  $\mathbf{n} \in \mathbb{Z}^J$  and  $j, k = 1, 2, \dots, J$  define

$$q(\mathbf{n}, \mathbf{n} - \mathbf{e}_j + \mathbf{e}_k) = \Phi_j g_j(n_j) \psi_k g_k(n_k) \lambda_{j,k},$$

where  $\lambda_{j,j} = 0$ ,

$$q(\mathbf{n}, \mathbf{n} + \mathbf{e}_j) = \nu_j \psi_j g_j(n_j)$$

and

$$q(\mathbf{n}, \mathbf{n} - \mathbf{e}_j) = \mu_j \Phi_j g_j(n_j).$$

Thus a down-jump at a given node,  $j$ , may trigger an up-jump at node  $k$  with probability

$$\rho_{j,k}(\mathbf{n}) = \psi_k g_k(n_k) \lambda_{j,k} / \left\{ \mu_j + \sum_{i=1}^J \psi_i g_i(n_i) \lambda_{j,i} \right\}$$

or cause no trigger with probability

$$1 - \sum_{k=1}^J \rho_{j,k}(\mathbf{n}).$$

If there exist positive quantities  $\eta_j, j = 1, 2, \dots, J$ , such that

$$\eta_j \mu_j = \nu_j$$

and

$$\eta_j \lambda_{j,k} = \eta_k \lambda_{k,j}$$

then, by Theorem 1,

$$m(\mathbf{n}) = \prod_{j=1}^J (\eta_j \rho_j)^{n_j} / g_j(n_j), \quad \mathbf{n} \in \mathbb{Z}^J,$$

is a  $\mathcal{Q}$ -invariant measure with respect to which the interconnected process is reversible.

Observe that under the modification dictated by condition (ii) of Theorem 1 the properties of transience or recurrence of the isolated nodes might be affected but *not* the regularity or otherwise. We have assumed, without loss of generality, that  $\rho_j \geq 1$  so that each node suffers no net downwards drift. However, the modification may bring about a change in this situation. We shall now illustrate, with reference to a simple example, how various properties of  $\mathcal{Q}$  relate to those of the individual nodes in isolation. Consider the system consisting of only two nodes ( $J = 2$ ) and with  $\nu_j = \mu_j = 0, j = 1, 2$ . For this system the state space,  $\mathbb{Z}^2$ , decomposes into irreducible sub-classes ( $\zeta_N, N \in \mathbb{Z}$ ) where

$$\zeta_N = \{\mathbf{n} \in \mathbb{Z}^2 : n_1 + n_2 = N\},$$

and for fixed  $N$  the interconnected process is itself a random walk of the type described, but on the line  $\zeta_N$ . It is clearly equivalent to the random walk on  $\mathbb{Z}$  with parameters  $\psi = \psi_1 \Phi_2 \lambda_{2,1}, \Phi = \psi_2 \Phi_1 \lambda_{1,2}$  and  $g(n) = g_1(n)g_2(-n), n \in \mathbb{Z}$ , and so the task of testing for regularity, etc., is simple. We shall assume, without loss of generality, that the modified parameters  $\hat{\rho}_j = \eta_j \rho_j$  are such that  $\hat{\rho}_1 \geq \hat{\rho}_2 (\geq 0)$ , and so

$$\rho \triangleq \psi / \Phi = \hat{\rho}_1 / \hat{\rho}_2 \geq 1.$$

We see that if, under the modification, node 2 is recurrent the system is recurrent if and only if node 1 is recurrent. If both nodes are transient then the system is recurrent if and only if  $\hat{\rho}_1 = \hat{\rho}_2$ . Thus all combinations of transience and recurrence are possible. Further, by considering the series  $S_j^+, j = 1, 2$ , together with

$$S^+ = \sum_{n=0}^{\infty} [g(n)]^{-1},$$

we can show that any combination of regularity or otherwise is possible. For

example, suppose that  $\hat{\rho}_1 > \hat{\rho}_2 > 1$ . Then if  $g_1(n) = g_2(n) = |n|$  each node is regular but the system is not, or if  $g_1(n) = g_2(n) = e^n$  then the system is regular but the individual nodes are not.

(iii) *Stratified clustering processes.* We shall conclude by considering a process where individuals or units move between various localities and, at each of these, group themselves into clusters. The basic model arises naturally from the study of polymerisation processes occupying a particular volume that can be divided into regions within which polymerisation occurs and between which migration may occur. Such processes have received an elegant treatment by Whittle (1980b) and touched upon in Whittle (1965a), (1983) and Kelly (1981). They allow for the notion of space and locality to be introduced into models for clustering and, in some respects, simplify their analysis. Clustering processes appear in a variety of diverse contexts including the study of social grouping behaviour (Coleman and James (1961), White (1962) and Goodman (1964)), of aggregation of slime mould (Keller and Segel (1970) and Cohen and Robertson (1971)) and of chemical reactions (Whittle (1965a), (1965b), (1972), (1980a), (1980b), (1980c)). The model presented here for the clustering nodes encapsulates all of these and is a generalisation of Whittle's (1965a) model suggested by Exercise 8.5.6 of Kelly (1979).

For each node,  $j$ , let  $\mathbf{R}_j$  be a countable collection of cluster types and let  $\zeta_j$  be the subset of  $\mathbb{N}^{\mathbf{R}_j}$  whose elements have only finitely many non-zero terms. Define the  $q$ -matrix,  $\mathbf{Q}_j$ , for clustering node,  $j$ , by

$$\begin{aligned} q_j(\mathbf{n}, \mathbf{n} - \mathbf{e}_r + \mathbf{e}_u) &= t_{j,r,u} \Phi_{j,r}(n(r)) \psi_{j,u}(n(u)), \\ q_j(\mathbf{n}, \mathbf{n} - \mathbf{e}_u + 2\mathbf{e}_r) &= d_{j,r,r,u} \Phi_{j,u}(n(u)) \psi_{j,r}(n(r)) \psi_{j,r}(n(r) - 1), \\ q_j(\mathbf{n}, \mathbf{n} - 2\mathbf{e}_r + \mathbf{e}_u) &= a_{j,r,r,u} \Phi_{j,r}(n(r)) \Phi_{j,r}(n(r) - 1) \psi_{j,u}(n(u)) \end{aligned}$$

and for  $r \neq s$ ,

$$\begin{aligned} q_j(\mathbf{n}, \mathbf{n} - \mathbf{e}_u + \mathbf{e}_r + \mathbf{e}_s) &= d_{j,r,s,u} \Phi_{j,u}(n(u)) \psi_{j,r}(n(r)) \psi_{j,s}(n(s)), \\ q_j(\mathbf{n}, \mathbf{n} - \mathbf{e}_r - \mathbf{e}_s + \mathbf{e}_u) &= a_{j,r,s,u} \Phi_{j,r}(n(r)) \Phi_{j,s}(n(s)) \psi_{j,u}(n(u)), \end{aligned}$$

where

$$a_{j,r,s,u} = a_{j,s,r,u} \quad d_{j,r,s,u} = d_{j,s,r,u} \quad \text{and} \quad \Phi_{j,r}(0) = 0.$$

The quantity  $d_{j,r,s,u}$  may be regarded as the tendency for which a cluster of type  $u$  divides into one of type  $r$  and one of type  $s$ , while  $a_{j,r,s,u}$  represents the tendency for which a given type  $r$  cluster and a given type  $s$  cluster join to form a type  $u$  cluster. The parameter  $t_{j,r,u}$  represents the tendency for which an  $r$ -cluster transmutes into a  $u$ -cluster. The possibility of such an occurrence is useful, for example, in modelling isomerisation reactions in chemistry. The quantities  $\Phi_{j,r}$  and  $\psi_{j,r}$  measure the extent to which these tendencies are affected by the numbers of clusters of each relevant type. They are most often

linear functions but there are examples of chemical reactions where empirical observations lead to the conclusion that some other form is more appropriate (see, for example, McQuarrie (1967)), one which does not reflect the stoichiometry of the reaction. Observe that we have allowed the rates of association, dissociation and transmutation to depend upon the number of clusters of the cluster type(s) produced by these occurrences. This is often useful when modelling a chemical reaction that involves autocatalysis (see Atkins (1982)). It can be shown (Whittle (1965a), Morgan (1976), Kelly (1979)) that the clustering node described is reversible if and only if there exist positive quantities  $(\xi_{j,r}, r \in \mathbf{R}_j)$  such that, for all  $r, s$  and  $u$ ,

$$(13) \quad \begin{aligned} \xi_{j,r} \xi_{j,s} a_{j,r,s,u} &= \xi_{j,u} d_{j,r,s,u} \\ \xi_{j,r} t_{j,r,u} &= \xi_{j,u} t_{j,u,r} \end{aligned}$$

in which case  $\mathbf{Q}_j$  admits the essentially unique invariant measure  $\mathbf{m}_j$  given by

$$\mathbf{m}_j(\mathbf{n}) = \prod_{r \in \mathbf{R}_j} \xi_{j,r}^{n(r)} \chi_{j,r}(n(r)) \quad \mathbf{n} \in \zeta_j$$

where

$$\chi_{j,r}(n) = \prod_{l=1}^n \{ \psi_{j,r}(l-1) / \Phi_{j,r}(l) \}.$$

It is often difficult to establish whether or not the relations (13) admit a positive solution although in chemistry applications the existence of a solution is dictated by established physical laws (Whittle (1965a)). Further, although the system described is closed in the sense that we have not allowed for immigration and emigration of clusters, it might be that with positive probability cluster numbers become unbounded, and it may not be a straightforward task to determine the irreducible subclasses of  $\zeta_j$ .

It is possible to provide for immigration and emigration of clusters from outside the system, thus making it open, by amending the  $q$ -matrix as follows: define, for all  $r \in \mathbf{R}_j$ ,

$$\begin{aligned} q_j(\mathbf{n}, \mathbf{n} + \mathbf{e}_r) &= a_{j,r} \psi_{j,r}(n(r)) \\ q_j(\mathbf{n}, \mathbf{n} - \mathbf{e}_r) &= d_{j,r} \Phi_{j,r}(n(r)). \end{aligned}$$

Under this amendment we shall require, in addition to (13), that for all  $r$

$$(14) \quad a_{j,r} = \xi_{j,r} d_{j,r}$$

in order to retain reversibility and the stated invariant measure. This extra condition may of course be too restrictive, but generally, and, in particular, with reference to chemical reactions, it is possible to allow migration of at least one cluster type. To illustrate this consider the case where clusters are made up of basic units and the type of a cluster determines the number of such units

comprising that cluster. Thus  $R_j = \{1, 2, \dots\}$  and if a solution to (13) exists it is determined by the recursion

$$\xi_{j,r} d_{j,1,r-1,r} = \xi_{j,r-1} \xi_{j,1} a_{j,1,r-1,r} \quad r = 2, 3, \dots,$$

where  $\xi_{j,1}$  is chosen arbitrarily, provided  $a_{j,1,r,r+1} > 0$  for each  $r \geq 1$ . It is clear, therefore, that we can allow the migration of only one cluster type. If we choose to allow units (1-clusters) to migrate then  $\xi_{j,1}$  is set to  $a_{j,1}/d_{j,1}$ .

It is interesting to observe that it is possible to allow the transition rates of a clustering process to depend upon the number of clusters present and, in the open system just described, upon the number of basic units present. If  $g_j$  and  $h_j$  are functions of  $N = \sum_r n(r)$  then the immigration and dissociation rates can be multiplied by a factor  $g_j(N)$  and the emigration and association rates by a factor  $h_j(N)$ . The resulting invariant measure will include a factor

$$\prod_{r=1}^N \frac{g_j(r-1)}{h_j(r)}$$

and reversibility is preserved. Observe that since the transmutations do not change  $N$  their rates may depend arbitrarily on  $N$  without affecting the invariant measure. In the example, where basic units may enter or leave the system, the immigration and emigration rates can be modified by factors  $\alpha_j(M)$  and  $\beta_j(M)$ , where  $M = \sum_r rn(r)$  is the number of units and a further factor

$$\prod_{r=1}^M \frac{\alpha_j(r-1)}{\beta_j(r)}$$

is introduced into the invariant measure, and again reversibility is preserved. If the only possible transmutations of an  $r$ -cluster are to  $r + 1$  and  $r - 1$  ( $r > 0$ ) then the corresponding rates may be altered by factors  $\alpha_j(M)$  and  $\beta_j(M)$  without affecting the invariant measure.

Before proceeding to discuss the stratified process let us briefly examine the effect of varying the immigration rates. Suppose that we are presented with an open reversible clustering node of the type described. Let  $C_j$  be the set of types,  $r \in R_j$ , such that  $a_{j,r} > 0$  and define, for each  $r$  in  $C_j$ ,

$$A_j(r) = \{(\mathbf{n}, \mathbf{n} + \mathbf{e}_r) : \mathbf{n} \in \zeta_j\}$$

and suppose that the rates of transitions that lie in  $A_j(r)$  are multiplied by a positive factor  $\eta_j(r)$ . It is clear that in order to preserve reversibility it is necessary and sufficient that  $(\eta_j(r), r \in C_j)$  satisfy

$$\begin{aligned} \eta_j(r)\eta_j(s)a_{j,r,s,u} &= \eta_j(u)d_{j,r,s,u} \\ \eta_j(r)t_{j,r,u} &= \eta_j(u)t_{j,u,r} \end{aligned} \tag{15}$$

for all  $r, s$  and  $u$  in  $C_j$ . Under these conditions  $\xi_{j,r}$  is replaced by  $\hat{\xi}_{j,r} = \eta_j(r)\xi_{j,r}$

for  $r \in C_j$  and, for  $r \notin C_j$ ,  $\hat{\xi}_{j,r}$  bears the same functional relationship to  $(\hat{\xi}_{j,u}, u \in C_j)$  as did  $\xi_{j,r}$  to  $(\xi_{j,u}, u \in C_j)$ .

We shall now consider the stratified process consisting of clustering nodes  $1, 2, \dots, J$ . For each  $j$  and  $k$ ,  $r \in C_j$  and  $u \in C_k$  let  $\lambda_{j,k}(r, u)$  be the probability intensity that a type  $r$  cluster at node  $j$  migrates to node  $k$  and becomes a cluster of type  $u$ , and for each  $j$  and  $r \in C_j$  let  $v_j(r)$  and  $\mu_j(r)$  be, respectively, the exogeneous immigration and emigration intensities of type  $r$  clusters at node  $j$ . Theorem 1 shows that provided there exists positive quantities  $\eta_j(r)$ ,  $r \in R_j$ ,  $j = 1, 2, \dots, J$  that satisfy (15) for all  $j$  and  $r, s, u \in R_j$  and

$$\begin{aligned} \eta_j(r)\mu_j(r) &= v_j(r) \\ \eta_j(r)\lambda_{j,k}(r, u) &= \eta_k(u)\lambda_{k,j}(u, r) \end{aligned}$$

for all  $j, k, r \in C_j$  and  $u \in C_k$ , then an invariant measure for the stratified process is given by

$$m(n_1, n_2, \dots, n_J) = \prod_{j=1}^J \prod_{r \in R_j} [\eta_j(r)]^{n_j(r)} \chi_{j,r}(n_j(r)), \quad n_j \in \zeta_j, \\ j = 1, 2, \dots, J.$$

### 6. Quasireversibility

By summing the equations in condition (i) of Theorem 1 we see it is necessary that the collection  $(\eta_j(c), c \in C_j, j = 1, 2, \dots, J)$  satisfies

$$(16) \quad \eta_j(c)\lambda_j(c) = v_j(c) + \sum_{k=1}^J \sum_{d \in C_k} \eta_k(d)\lambda_{k,j}(d, c),$$

where

$$\lambda_j(c) = \mu_j(c) + \sum_{k=1}^J \sum_{d \in C_k} \lambda_{j,k}(c, d),$$

which are the usual traffic equations for an interconnected system (see, for example, Baskett et al. (1975)). If we were to replace condition (ii) of the theorem by the condition that each node be quasireversible, in the sense of Walrand and Varaiya (1980)), subject to the specified modification, then  $m$  would be invariant for  $Q$  under the weaker assumption (16). This condition would require that for each  $j$  there exists a  $\hat{Q}_j$ -invariant measure,  $\hat{m}_j$ , and a collection  $(\alpha_j(c), \beta_j(c), c \in C_j, j = 1, 2, \dots, J)$  satisfying

$$(17) \quad \sum_{y: (x,y) \in A_j(c)} \hat{q}_j(x, y) = \alpha_j(c)$$

and

$$\sum_{y:(x,y) \in A_j(c)} \hat{m}_j(y) \hat{q}_j(y, x) = \beta_j(c) \hat{m}_j(x)$$

for all  $x$  such that  $(x, y) \in A_j(c)$  for some  $y$ . However this would imply that the rate out of state  $\mathbf{x} = (x_1, x_2, \dots, x_j)$  due to a type  $c$  transition being triggered at node  $j$ ,

$$\sum_{y \in \zeta : (x_j, y_j) \in A_j(c)} q(\mathbf{x}, \mathbf{y}),$$

might depend on  $x_k$  for some  $k \neq j$ , but *not* on  $x_j$ . Theorem 1 requires no such restriction. Indeed we have seen that it is often desirable to allow dependence on the state of a node that is being triggered. Thus although we have a traffic restriction more stringent than (16) there is less of a restriction on the transition rates of the individual nodes.

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