TRAFFIC CHARACTERIZATION FOR BROADBAND SERVICES.

- Traditional (older) traffic models are based on traffic carried by circuit-switching (telephone) networks or early packet switched networks.

- Traditional traffic model assumptions

  - Poisson Arrivals
  - Exponential holding (service) times
  - Statistical Independence

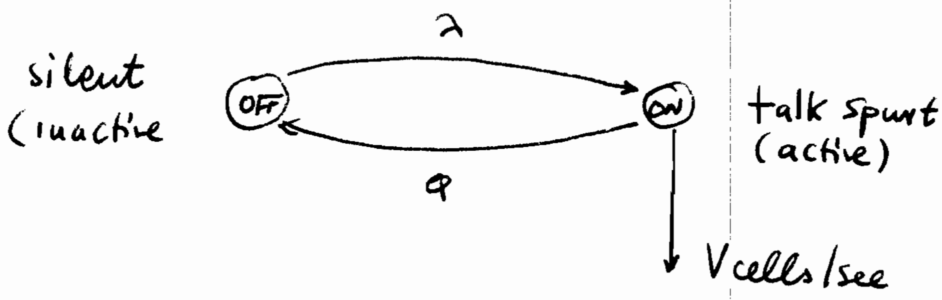
- Integration of different traffic types, such as packet voice, video, images, interactive computer applications etc, with diverse QoS requirements and network handling resulted in traffic streams that

  - Deviate from Poisson or Exponential distributions
  - May not have a Markovian structure
  - May be strongly correlated (e.g. bursty)

- Traffic models need to be devised for simulations and performance analysis studies.

Packet Voice Modelling

- Two state ON-OFF model



- This is a Markovian Model

  Silent and talkspurt periods are EXPONENTIALLY distributed with mean $\frac{1}{\lambda}$ and $\frac{1}{a}$ secs respectively

- Prob [Source is active] = $\frac{\lambda}{a+\lambda}$ (activity factor)

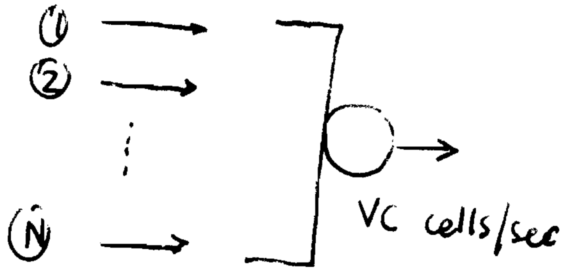
- For a large population N of such minisources there will be $N \cdot \frac{\lambda}{\lambda + a}$ sources that are ON on the average.

  If each minisource requires 1 circuit for transmission we can accomode close to $\frac{N}{N \frac{\lambda}{\lambda+a}} = 1 + \frac{a}{\lambda}$ many users as circuits (trunks) available.

  (TASI advantage)

Voice Packet Modelling.

- Model



Each source emits V cells/sec when ON.

ON periods - Exponential$\left(\frac{1}{a}\right)$

OFF periods - Exponential$\left(\frac{1}{\lambda}\right)$

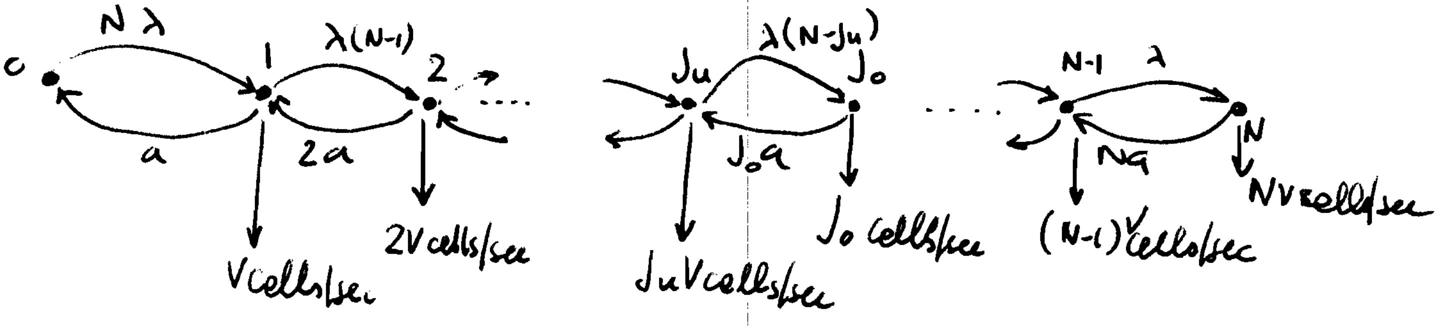
- Stability condition

$$\frac{\lambda}{a+\lambda} N < C.$$

- Loading factor

$$\rho = \left(\frac{\lambda}{a+\lambda}\right) \frac{N}{C} < 1.$$

- Composite model of N minisources.



$J_u = \lfloor C \rfloor$. Underload state. When state $< J_u$ buffer empties

$J_o = \lceil C \rceil$. Overload state. When state $> J_o$ buffer fills.

Packet Voice Modelling

- Steady state probability

  – Binomial

$$\pi_i = \mathbb{P}(i \text{ states are ON})$$

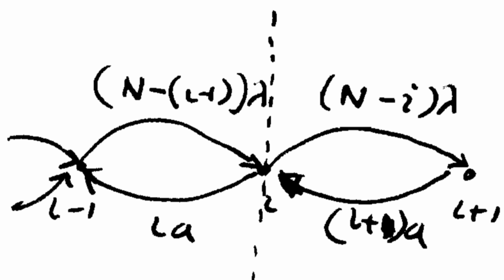
$$= \binom{N}{i}\left(\frac{\lambda}{\lambda+a}\right)^i\left(\frac{a}{\lambda+a}\right)^{N-i}$$

- Same result can be gotten if we write down and solve the <u>birth-death</u> equations.

$i=0$ $\qquad -N\lambda\pi_0 = a\pi_1$

$1 \le i \le N-1$ $\qquad \pi_i\left[ia + (N-i)\lambda\right] = \left[N-(i-1)\right]\lambda\pi_{i-1} + (i+1)a\pi_{i+1}$

$i=N$ $\qquad Na\pi_N = \lambda\pi_{N-1}$



- Above equations can be written in matrix form as

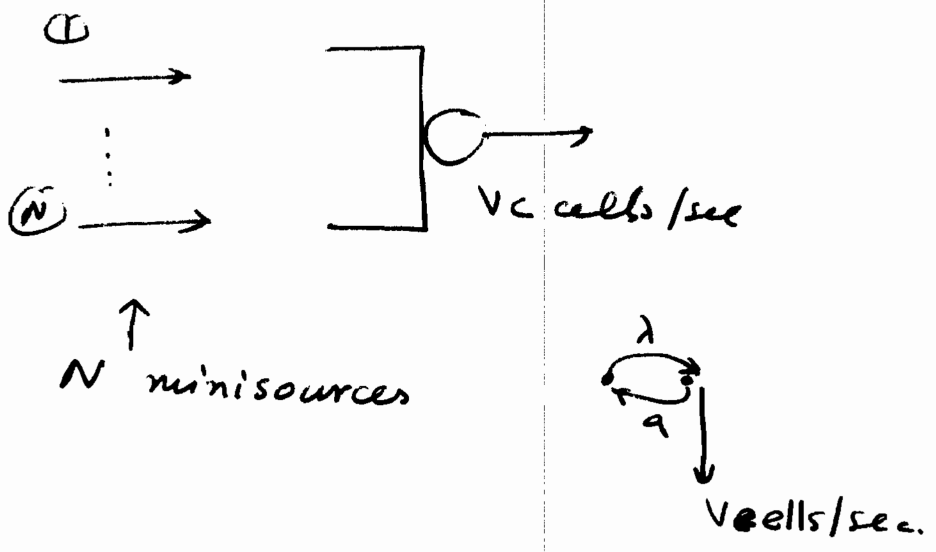
$\pi M = 0$ where

$$M = \begin{bmatrix} -N\lambda & N\lambda & \cdots 0 & \cdots \\ a & -[a+(N-1)\lambda] & (N-1)\lambda & \cdots \\ 0 & 2a & -(N-2)\lambda-2a & \cdots \\ 0 & 0 & 3a & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Packet Voice Modeling.

• **Goal:**

Given the minisource model study the behavior of the buffer



i.e determine delay, delay jitter & packet loss.

Equivalently find the distribution of the queuesize

i.e.

$$\mathbb{P}\left(\text{Queuesize} \le x\right) = F(x).$$

Fluid Modelling of Packet Voice

- When an engineer is phased with a systems analysis problem his FIRST action is to estimate the gross behavior of the system.

- Once this is done his task is then to refine his estimates and his approx. analysis.

- Queues in BISDN's can be treated as "continous" fluid flow systems rather than as discrete customer flow under suitable assumptions:

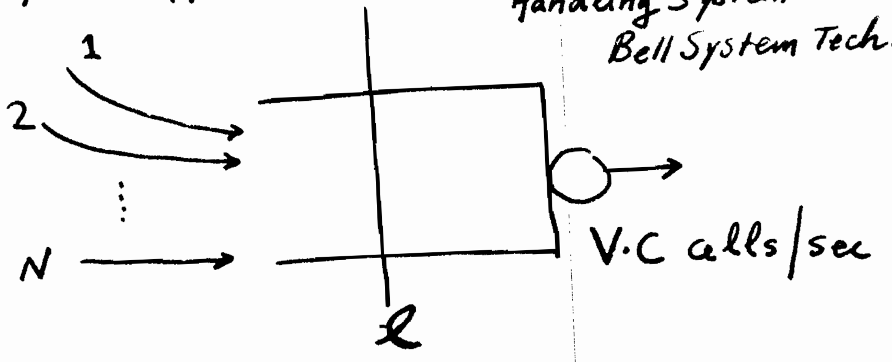
  - Heavy traffic conditions
    (quesizes large compared to unity, delays large compared to average SERVICE times $= \frac{1}{c}$)
  - N is assumed large
  - C is assumed large

- Magnitude of Discontinuities due to arrivals/departures is small compared to quesize

- Therefore queuesize behaves like a continous quantity!

Fluid Flow Approximation - Model

(Presentation of Fluid Approx is based on: Anick D. et al: Stochastic Theory of a Data Handling System with Multiple Sources" Bell System Tech. J., 61, 8, Oct 1982).



- Voice source, generates cells at a rate of <u>V cells/sec</u> during a talk spurt of average length of <u>$\frac{1}{a}$ secs.</u>
- Service rate is VC cells/sec
- Average "increment" of queue is

$$\boxed{V \cdot \frac{1}{a} \text{ cells during talk spurt (\underline{Unit of Information})}}$$

- Normalized service rate is:

$$\frac{VC}{(V/a)} = \boxed{\begin{array}{l} aC \text{ units of information/sec} \\ \quad\quad\text{(Normalized service rate)} \end{array}}$$

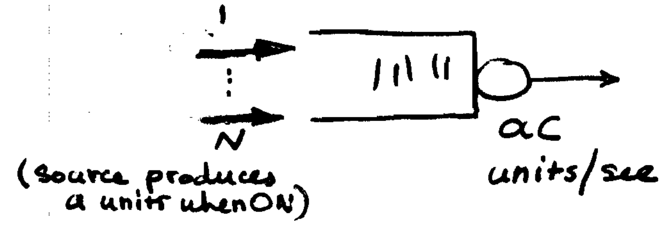
- If ℓ denotes the buffer size in cells the <u>normalized</u> <u>buffer</u> size is $\boxed{X = \frac{\ell}{V/a} \text{ (Normalized Buffer size)}}$

- Therefore $\mathbb{P}[\ell > i] = \mathbb{P}\left[X > \frac{iV}{a}\right]$

Fluid Flow Approximation : Equations

- $F_i(t,x) = \mathbb{P}[\text{Queuesize} \le x, \text{ at time } t, \; i \text{ sources ON}]$

Scaled Model



- $F_i(t+\Delta t, x) =$

$$= [N-(i-1)]\,\lambda\,\Delta t\, F_{i-1}(t,x) \quad (\text{One source activates})$$

$$+ \; (i+1)\,a\,\Delta t\, F_{i+1}(t,x) \quad (\text{One source deactivates})$$

$$+ \; \{1 - [(N-i)\lambda + ia]\,\Delta t\}\, F_i[t,\, x-(i-c)a\Delta t]$$

(No source change)

$$+ \; o(\Delta t)$$

- Note that relation above involves dependence on i $(0 \le i \le N)$, t, x.

Fluid Flow Approximations: Equations.

- Using limiting $\Delta t \to 0$ we can write:

$$\frac{\partial F_i(x,t)}{\partial t} = [N-(i-1)]\lambda F_{i-1}(t,x) + (i+1)a F_{i+1}(t,x) - [(N-i)\lambda + ia] F_i(t,x) - (i-c)a \frac{\partial F_i}{\partial x}(t,x)$$

- Assuming stationarity $\frac{\partial F_i}{\partial t} \xrightarrow[t\to\infty]{} 0$, $F_i(t,x) \xrightarrow[t\to\infty]{} F_i(x)$

$$(i-c)a \frac{dF_i(x)}{dx} = [N-(i-1)]\lambda F_{i-1}(x) - [(N-i)\lambda + ia] F_i(x) + (i+1)a F_{i+1}(x) \qquad 0 \le i \le N$$

- Boundary conditions

$$F_{-1}(x) = F_{N+1}(x) = 0$$

- Observe that we must have $c < N$ so that there are both emptying states ($i<c$) and filling states ($i>c$).

Fluid Flow Approximation: Solution

- Equation: $\dfrac{d\mathbb{F}(x)}{dx}\mathbb{D} = \mathbb{F}(x)\cdot\mathbb{M} \Leftrightarrow \dfrac{d\mathbb{F}(x)}{dx} = \mathbb{F}(x)\,\mathbb{M}'$

  where $\mathbb{M}' = \mathbb{M}\mathbb{D}^{-1}$

  - $\mathbb{F}(x) = [\,F_0(x), \ldots, F_N(x)\,]$ (1×N vector)

  - $D = \text{diag}[-ca, (1-c)a, \ldots, (N-c)a]$ [(N+1)×(N+1) matrix)

  - $$M = \begin{bmatrix} -N\lambda & N\lambda & & \cdots \\ a & -[a+(N-1)\lambda] & (N-1)\lambda & \cdots \\ 0 & 2a & -(N-2)\lambda & -2a \;\cdots \\ 0 & 0 & 3a & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

    ((N+1)×(N+1)) matrix

- Solution

$$\mathbb{F}(x) = \sum_{i=0}^{N} a_i \Phi_i e^{z_i x},$$

a_i: set of constant coefficients

Φ_i: eigenvectors of $M' = M D^{-1}$ i.e. $z_j \Phi_j = \Phi_j M'$.

z_i: Corresponding eigenvalues.

$0 \le i \le N$
(i.e we have N+1 eigenvalues)

Fluid Flow Approximation: Solution

- $$\boxed{F(x) = \pi + \sum_{i:[\mathrm{Re}\, z_i < 0]} a_i \Phi_i e^{z_i x} \quad \text{(Final Solution)}}$$
- π is s.th. : $\pi M = 0$ (steady state vector for arrival process)
- Calculation of the constants a_i

  – # of "negative" eigenvalues is $N - \lfloor c \rfloor$
  (See paper by ANIC [1982]) ($\lfloor \cdot \rfloor$: floor operator, e.g. $\lfloor 3.2 \rfloor = 3$).
  ⊛ This is possibly an interesting topic

  - Same as number of states i such that

  $$c < i \leq N \quad \text{(Overload states)}.$$

  – Boundary Condition for the determination of a_i.

  $$F_i(0) = 0 \qquad i > c$$ this generates a system of equations.

  i.e. $\mathbb{P}(\text{Queue is } 0,\ i \text{ is overload state}) = 0$

  $\downarrow$

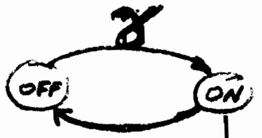
Caution:

This is NOT a stability condition $\longrightarrow$ $ia > ca$

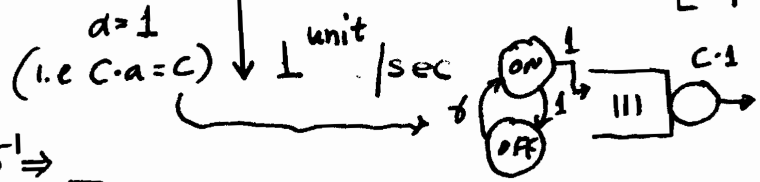
arrival rate (ia) Service rate (ca)

Fluid Flow Approximation: Example.

- ON-OFF source (We consider 1 source).



$$M = \begin{bmatrix} -\gamma & \gamma \\ 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} c & \emptyset \\ \emptyset & 1-c \end{bmatrix}$$



$$D^{-1} = \begin{bmatrix} -\frac{1}{c} & \emptyset \\ \emptyset & \frac{1}{1-c} \end{bmatrix}$$

$M' = MD^{-1} \Rightarrow$

- $\Rightarrow M' = \begin{bmatrix} \frac{\gamma}{c} & \frac{\gamma}{1-c} \\ \frac{-1}{c} & \frac{-1}{1-c} \end{bmatrix}$,

eigenvalue calculation:

$$\left\{ zM' = z \right\} \Longleftrightarrow \left\{ \begin{array}{l} z = 0 \\ z = \frac{\gamma}{c} - \frac{1}{1-c} \end{array} \right.$$

- Server capacity is C units/sec.

- For $z < 0$ $\quad \underbrace{\frac{\gamma}{1+\gamma} < c}_{\text{Stability Condition}} \underbrace{< 1}_{\text{ON state is overload state.}}$

$\Downarrow$

- Loading

$$\rho = \frac{\frac{\gamma}{1+\gamma} \cdot 1}{C} < 1$$

$\frac{\gamma}{1+\gamma} \cdot 1$ ← Average # of units arriving/sec

< 1 ← stability.

C ← Units served per second.

Fluid Flow Approximation : Example

- z can be alternatevily written as:

$$z = -\frac{(1-\rho)(1+\gamma)}{(1-c)}$$

- Solution

$$\boxed{F(x) = \pi + a\,\Phi e^{zx}} \leftarrow \text{Need } \Phi, \pi, a.$$

- Let $\Phi = [\Phi_0, \Phi_1]$ then $z\Phi = \Phi M'$, for $z = \frac{\gamma}{c} - \frac{1}{1-c}$

Hence $\dfrac{\Phi_1}{\Phi_0} = \dfrac{c}{1-c}$. Let: $\boxed{\begin{array}{l}\Phi_1 = c \\ \Phi_0 = 1-c\end{array}}$

- $\boxed{\pi_0 = \dfrac{1}{1+\gamma}, \quad \pi_1 = \dfrac{\gamma}{1+\gamma}}$ (This is the eigenvector for $z=0$)

- Boundary condition for overload state:

$$F_1(0) = 0 \Rightarrow \pi_1 + a\Phi_1 = 0 \Rightarrow \boxed{a = \frac{-\gamma}{c(1+\gamma)}}$$

(Note: If you did not select Φ_1, Φ_0 as done previously, a would be different! However $a\Phi$ would be ok!)

Fluid Flow Approximation : Example

- Putting everything together:

$$\begin{bmatrix} F_0(x) \\ F_1(x) \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix} - \frac{\gamma}{c(1+\gamma)} \begin{bmatrix} 1-c \\ c \end{bmatrix} e^{-\frac{(1-\rho)(1+\gamma)}{1-c} x}$$

$$F(x) = \mathbb{P}(\text{Quesize} \leqq x)$$

$$= F_0(x) + F_1(x) \quad \text{(why? Hint: Total Probability Law)}$$

$$= 1 - \underbrace{\frac{\gamma}{c(1+\gamma)}}_{= \rho} e^{-\frac{(1-\rho)(1+\gamma)}{1-c} x}$$

$$G(x) = \mathbb{P}(\text{Quesize} > x) = \mathbb{P}(\text{Blocking})$$

(or Survivor Function)

$$= \rho \, e^{-\frac{(1-\rho)(1+\gamma)}{1-c} x} .$$

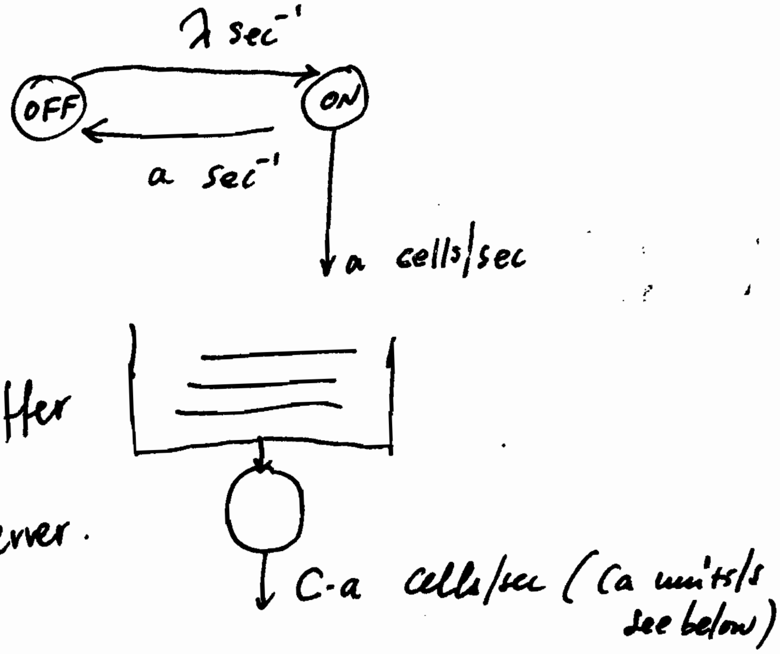
- ~~De-scaling:~~

$$\cancel{\mathbb{P}(L > i) = G[a i V] = \rho \, e^{-\frac{(1-\rho)(1+\gamma)}{1-c} a i V}}$$

~~(assume $a=1$)~~

Observations

System #1



Notice that:

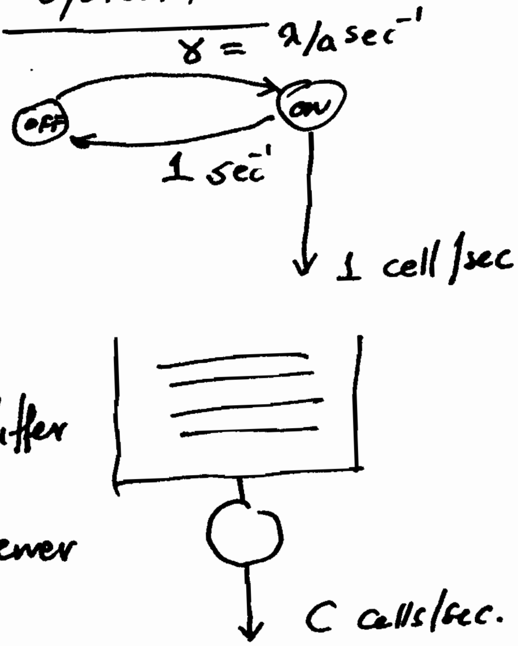
In $\frac{1}{a}$ secs (average ON time)

- $a\frac{1}{a} = 1$ cell will be produced i.e. 1 unit of information
- And the "normalized" service rate is $\frac{Ca}{1} = Ca$ units/sec (unchanged)
- Stability Condition:

$$\frac{\lambda}{\lambda+a}a < C\cdot a \Rightarrow$$

$$\Rightarrow \frac{\lambda}{\lambda+a} < C$$

System #2



Notice that:

In 1 sec (Average ON time)

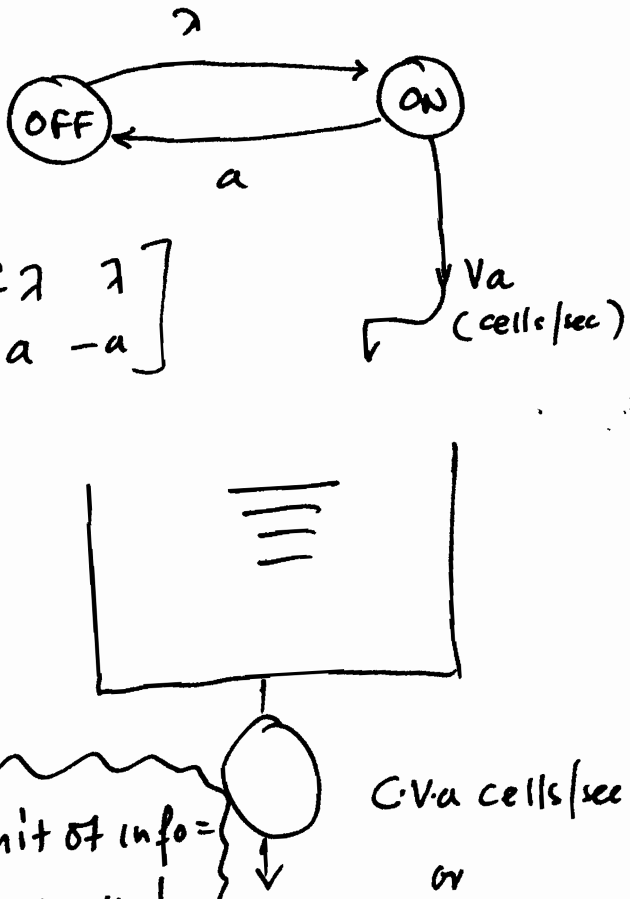
- 1 cell will be produced. i.e 1 unit of information
- The "normalized" service rate will be $\frac{C}{1} = C$ units/sec (unchanged)
- Stability Condition

$$\frac{\gamma\cdot 1}{\gamma+1} = \frac{\lambda/a\cdot 1}{\lambda/a+1} = \frac{\lambda\cdot 1}{\lambda+a} < C$$

systems are equivalent.

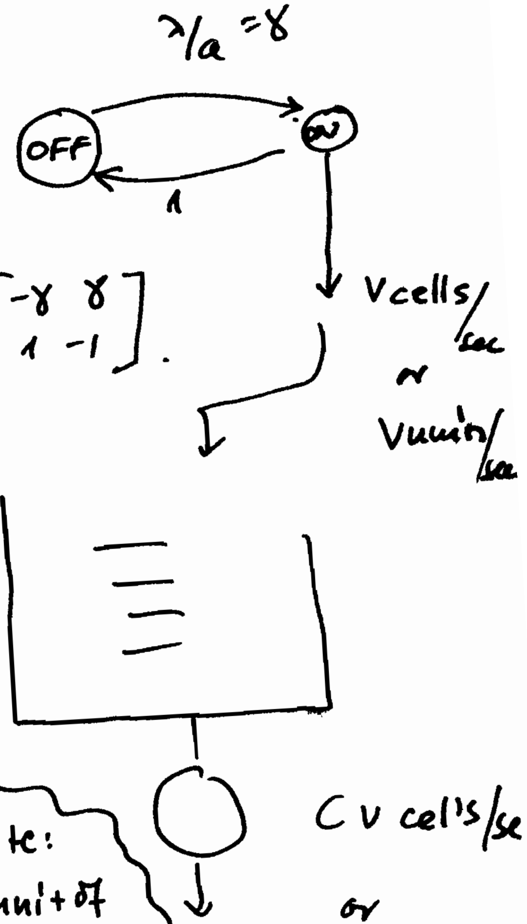
In general

Equivalent Systems.



$$M = \begin{bmatrix} -\lambda & \lambda \\ a & -a \end{bmatrix}$$

Note: 1 unit of info = $= \frac{Va}{a} = V$ cells!



$$M = \begin{bmatrix} -\gamma & \gamma \\ 1 & -1 \end{bmatrix}.$$

Note: 1 unit of info $= \frac{V}{1}$ cells $= V$ cells

Stability:

$$\frac{\lambda}{\lambda + a} Va < CVa$$

$$\frac{\lambda}{\lambda + a} < C$$

Stability

$$\frac{\gamma V}{\gamma + 1} = \frac{\frac{\lambda}{a} V}{\frac{\lambda}{a} + 1} < CV$$

$$\frac{\gamma}{\gamma + 1} < C$$

- Descaling:

Buffer size is "normalized" with respect to the information unit

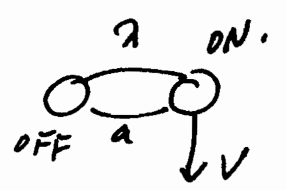
$$V \text{ cells/sec in } \frac{1}{a} \text{ sec's} = \frac{V}{a} \text{ cells} = \text{information unit.}$$

(or if we follow notation in p. 15a

$$Va \text{ cells/sec in } \frac{1}{a} \text{ secs} = V \text{ cells} = \text{information unit} \Big)$$

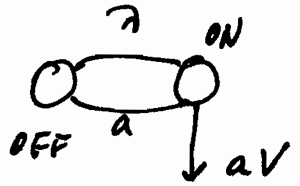
Therefore our buffer size was computed in "information units" following the fluid approximation. If L is in cells, Q in information units. (from fluid calculation)

$$\mathbb{P}(L > i) = \mathbb{P}\left(\frac{L}{V/a} > \frac{i}{V/a}\right) = \mathbb{P}\left(Q > \frac{ia}{V}\right) \text{ for}$$



or

$$\mathbb{P}\left(\frac{L}{V} > \frac{i}{V}\right) = \mathbb{P}\left(Q > \frac{i}{V}\right) \text{ for}$$



Therefore "de-normalizing" the example:

$$\mathbb{P}(L > i) = G(ai/v)$$

$$= \rho \, e^{\frac{(1-\rho)(1+\gamma)}{(1-c)} ai/v} .$$

Fluid Flow Approximation: Numerical Calulations

- Assume $\gamma = \frac{\lambda}{a} = \frac{2}{3}$. Then $\mathbb{P}(\text{source is ON}) = \frac{\lambda}{\lambda+a} = \frac{\gamma}{1+\gamma} = 0.$

Assume $\frac{1}{a} = 1.25$ (Average ON time talk spurt)

Assume $V = 170$ cells/sec (corresponds 64 kbps of digital voi... segmented into 53 octet ATM cel... (Justify!)

$\frac{V}{a} = 1.25 \cdot 170 = 215.5$ cells = 1 information unit!

Assume $C = 0.5$ i.e transmission capacity is $VC = 85$ cells/sec

Hence $\rho = \frac{\gamma}{1+\gamma} \cdot \frac{1}{c} = 0.8$ ← Heavy load! (Justify fluid approx.)

$\mathbb{P}(L>i) = 0.8\, e^{-0.003i}$ [Note $\frac{(1-0.8)(1+\frac{2}{3})}{1-0.5} \frac{1}{1.25} \cdot 170^{-1} \approx 0.003$]

For $\mathbb{P}(L>i) \approx 10^{-4} \Rightarrow i \cong 3000$ cells! (Large buffer)

If we increase capacity C to 0.8 then $\underline{\rho = 0.5}$ and

$$\mathbb{P}(L>i) \simeq 0.5\, e^{-0.02i}$$

For $\mathbb{P}(L>i) \approx 10^{-4} \Rightarrow i \cong 450$ cells! (Feasible buffer)

CAUTION: IS $\rho = 0.5$ heavy load? How accurate is fluid approximation? Further investigation may be necessary!

- If # of multiplexed sources is increased and capacity C is scaled proportionaly (CONSTANT LOAD) it is expected a decrease in the buffer size for constant blocking.

- Since $F(x)$ is a sum of negative exponentials the exponential with the SMALLEST negative eigenvalue will dominate. Let r be this eigenvalue, it can be shown:

$$r = \frac{(1-\rho)(1+\gamma)}{1-\frac{C}{N}} \qquad \text{for} \quad \rho = \frac{\gamma}{1+\gamma}\frac{N}{C} < 1.$$

- Then it can be shown that for large N

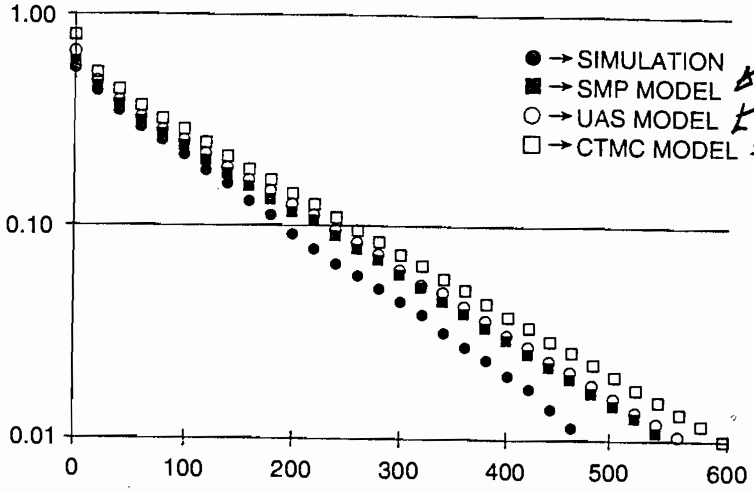
$$G(x) = \mathbb{P}(Q > x) \approx A_N \rho^N e^{-rx}$$

where A_N is a constant. (ANICK).

- Hence $$\mathbb{P}(\mathcal{L} > i) \sim A_N \rho^N e^{-rai/v}.$$

- Observe that ρ and r depend on $\frac{N}{C}$ which is a constant (N, C Increase proportionaly)
- Observe exponential decrease of $\mathbb{P}(\mathcal{L}>i)$ according to ρ^N!

$\mathbb{P}(L>x)$



Semi Markov model

fluid flow

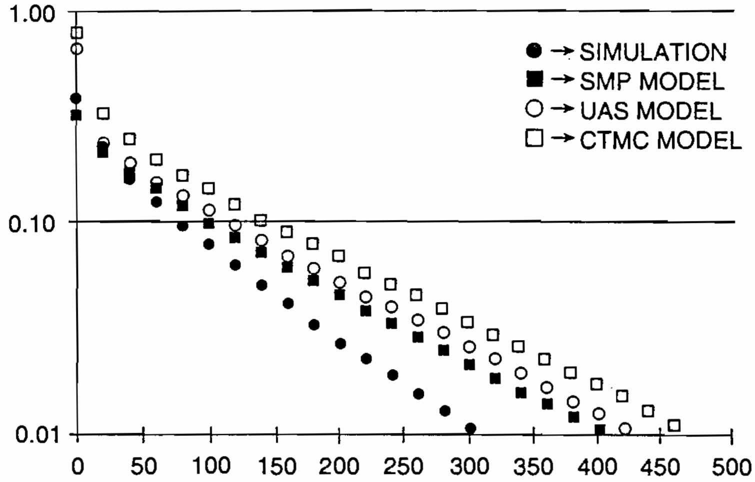
M.M.P.P.

buffer size x

$\rho = 0.85$

$N = 8$

$\mathbb{P}(L>x)$

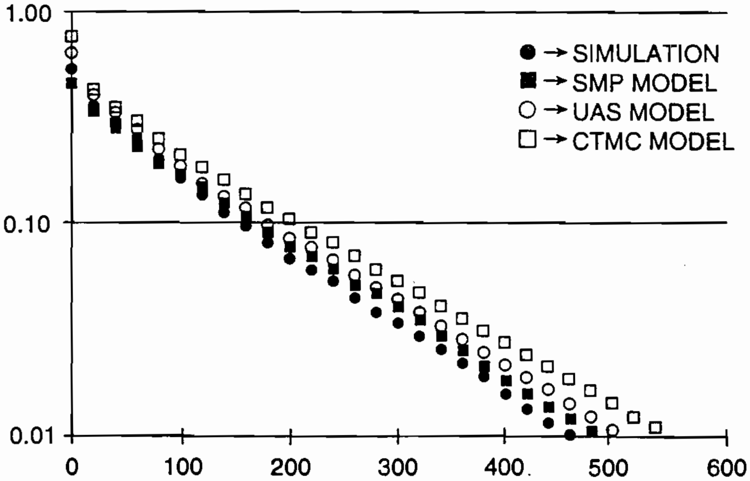


buffer size x

$\rho = 0.85$

$N = \cancel{15}$ 30

$\mathbb{P}(L>x$



buffer size x

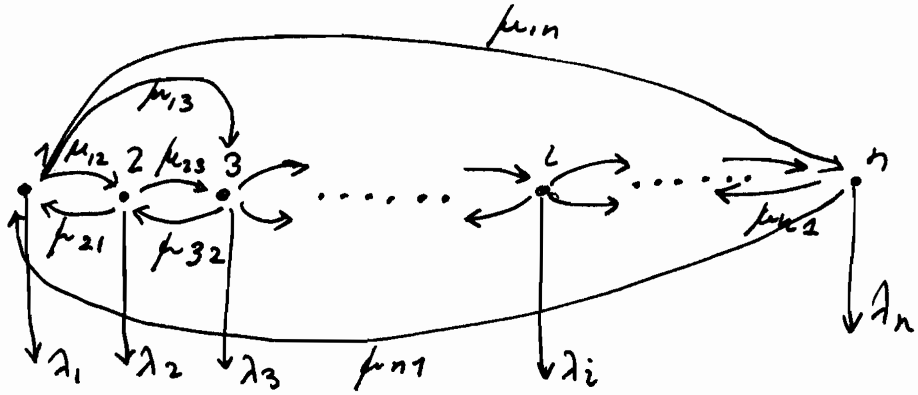
$\rho = 0.85$

$N = \cancel{30}$ 15

Voice packet Modelling and Multiplexing: Matrix Geometric Methodology

MARKOV MODULATED POISSON PROCESS (MMPP)

- The MMPP is defined as a model with n states which are visited according to a given rate (or probability transition) matrix. Being at state $i \in [1,n]$ the traffic generated is Poisson with rate λ_i.



- Steady state probability vector:

$$\pi = [\pi_1, \pi_2, \pi_3, \ldots, \pi_i \ldots \pi_n]$$

$\pi_i = \mathbb{P}[\text{State} = i]$ when in steady state.

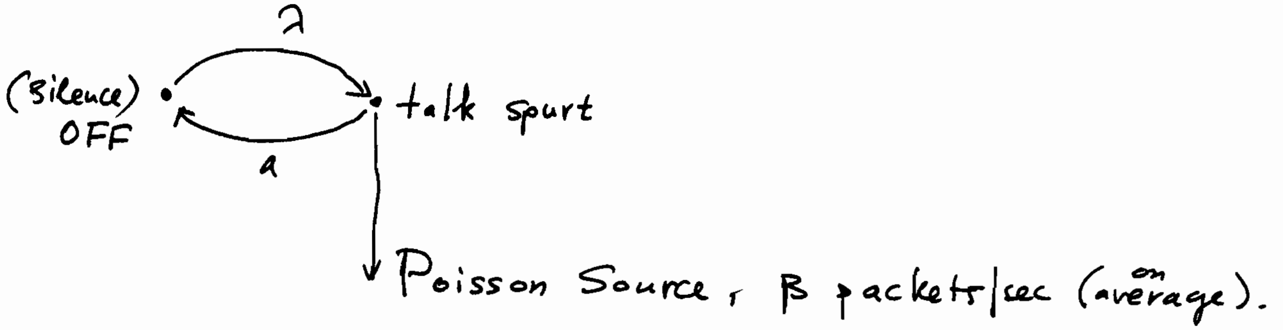
- Rate Matrix:

$$M = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \ddots & & \\ & & \ddots & \mu_{nn} \end{bmatrix} \quad \mu_{ii} = -\sum_{j \neq i} \mu_{ij}$$

- $\pi M = 0$, $\sum_{i=0}^{n} \pi_i = 1$ (Determination of steady state prob. vector).

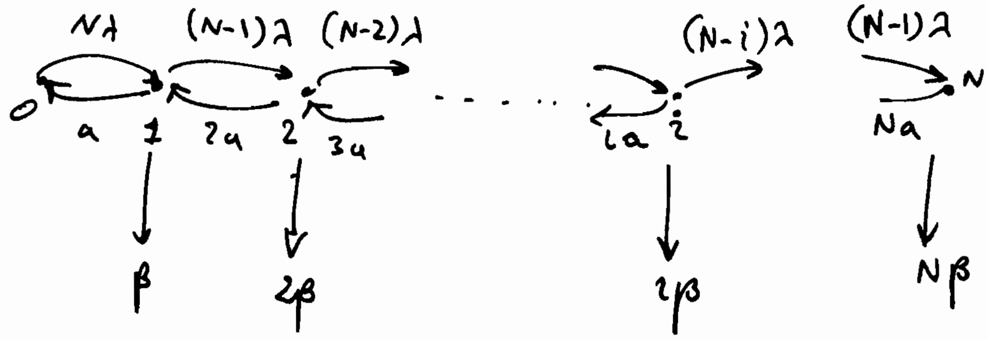
Markov Modulated Poisson Process

- We will proceed with the voice minisource model we considered before.

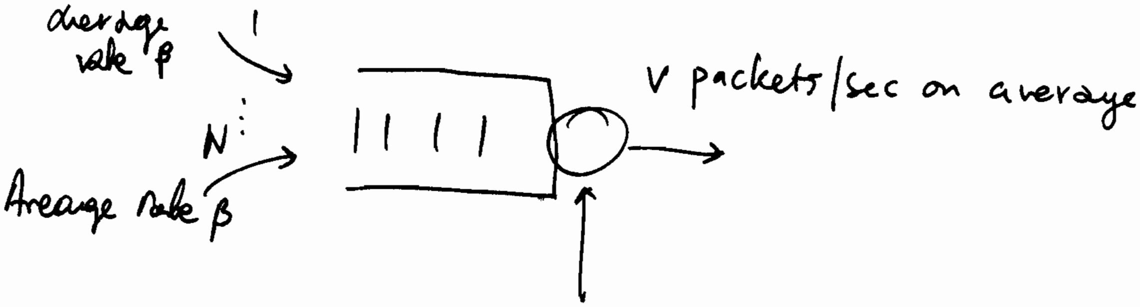
- 

- Assumption of Poisson traffic generation when model is ON. This is an <u>unrealistic</u> assumption (since voice model assumes CBR cell generation) however this assumption leads to a MMPP. (NOTE: The matrix geometric methodology that will be shortly outlined <u>may</u> <u>be</u> <u>extended</u> to Semi-Markov or discrete time models with CBR generation).

- Based on assumptions above the MMPP model will be:



- Multiplexer assumption:



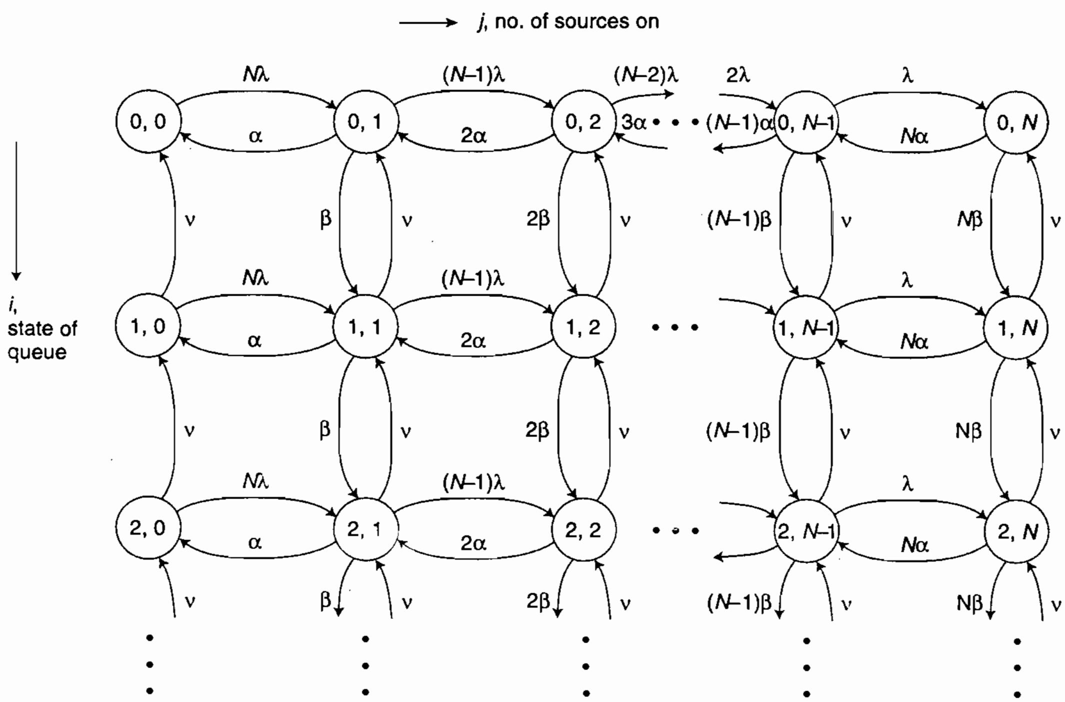
— Service is exponential with mean $\frac{1}{v}$ secs/packet
Again <u>non</u> realistic assumption.

- <u>Non</u> <u>realistic</u> assumptions give birth to the following Markov Birth-Death process.

$p_{00} = (1-N\lambda)p_{00} + \alpha p_{01} + \nu p_{10}$

or

$N\lambda p_{00} = \alpha p_{01} + \nu p_{10}$



- We want to compute:

$$p_{ij} = \mathbb{P}(\text{Queue Length} = i,\ j \text{ sources } \underline{\underline{\text{on}}}).$$

- Then we can find

$$\mathbb{P}(\text{Quelength} = i) = p_i = \sum_{j=0}^{N} p_{ij}$$

- Writing the balance equations in a systematic way we can show that the vector

$$\mathbf{p}_i = [p_{i0}, p_{i1}, \dots, p_{iN}] \quad (i^{\text{th}} \ \underline{\underline{\text{level}}} \ \text{probability vector})$$

satisfies the following equations:

For $i=0$.

$$\mathbf{p}_0 = \mathbf{p}_0 B_0 + \mathbf{p}_1 B_1$$

where

$$B_0 = \begin{bmatrix} 1-N\lambda & N\lambda & 0 & \cdots & 0 \\ a & [1-a-(N-1)\lambda-\beta] & (N-1)\lambda & \cdots & 0 \\ 0 & 2a & [1-2a-(N-2)\lambda-2\beta] & \cdots & 0 \\ 0 & 0 & 3a & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & \lambda \\ & & & & (1-Na-N\beta) \end{bmatrix}$$

$$B_1 = \operatorname{diag}[v, v, v, \ldots].$$

- Observe that B_0 involves parameters of the source model and describes the evolution of the MMPP.

Observe that B_1 describes the service process.

- Generalizing we can easily show the following.

vector elements (each containing $N+1$ elements)

Let $\mathbf{p} = [\mathbf{p}_0, \mathbf{p}_1, \ldots\ldots \mathbf{p}_i \ldots]$

level 0 prob. vector, level 1 prob. vector, etc.

$\boxed{\text{level represents quesesize!}}$

- Manipulation of balance equations result in the following recursion:

$$\mathbf{p}_i = \mathbf{p}_{i-1} A_0 + \mathbf{p}_i A_1 + \mathbf{p}_{i+1} A_2 \qquad i \geq 1$$

- Equation above can be summarized in

$$\mathbf{p} = \mathbf{p}\,\mathbf{P}$$

where $\mathbf{P}$ is of the following form:

$$\mathbf{P} = \underset{\text{Quesize level}}{} \left\{ \begin{bmatrix} B_0 & A_0 & \cdots & \cdots & \cdots & \cdots \\ B_1 & A_1 & A_0 & \cdots & \cdots & \cdots \\ 0 & A_2 & A_1 & A_0 & \cdots & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \right.$$

- Non diagonal matrix block elements represent queue size variations. (B_1, A_0, A_2)

- Diagonal matrix block elements represent transitions (evolution) of MMPP traffic model. (A_1

- In a more general framework the matrix $\mathbb{P}$ will have the following form:

$$\mathbb{P} = \begin{bmatrix} B_0 & A_0 & 0 & 0 & \cdots & 0 & \cdots \\ B_1 & A_1 & A_0 & 0 & \cdots & 0 & \cdots \\ B_2 & A_2 & A_1 & A_0 & \cdots & 0 & \cdots \\ B_3 & A_3 & A_2 & A_1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}$$

A_i, B_i $m \times m$ matrices.

- $\mathbb{P}$ is a stochastic matrix (rows sum to 1)

<u>Goal:</u> Solve $x = x\mathbb{P}$

$$\sum x_i = 1 \quad \left(\text{or} \quad x e^T = 1\right).$$

unit column vector.

- Goal of the matrix-geometric ~~equation~~ method is to find an algorithm for solving equations above.

- $\boldsymbol{x}$ is partitioned as follows:

$$\boldsymbol{x} = [\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \overset{i^{\text{th}}\text{ level}}{\boldsymbol{x}_i}, \dots]$$

vectors with appropriate dimension.

- We get from $\boldsymbol{x} = \boldsymbol{x}\mathbb{P}$

$$\boldsymbol{x}_0 = \sum_{v=0}^{\infty} \boldsymbol{x}_v B_v$$

$$\boldsymbol{x}_k = \sum_{v=0}^{\infty} \boldsymbol{x}_{k+v-1} A_v$$

$$\sum_{k=0}^{\infty} \boldsymbol{x}_k e^T = 1 \text{ (normalizing condition)}$$

- <u>SOLUTION</u>: Neuts has shown that

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i R \quad \text{(Matrix-Geometric Solution!)}$$

where R is a matrix ($m \times m$) satisfying:

$$R = \sum_{k=0}^{\infty} R^k A_k$$

(In the voice multiplexer: $R = A_0 + RA_1 + R^2A^2$).

Matrix Geometric Techniques:

(Derivation of x_0)

- **Goal:** To find matrix R which will provide x_i by

$$x_{i+1} = x_i \cdot R.$$

- Also note that: $x_0 = \sum_{v=0}^{\infty} x_v B_v.$

- Assuming that R is known: then for x_0 it is true that

$$x_0 = x_0 \sum_{v=0}^{\infty} R^v B_v = x_0 B(R)$$

- It can be shown that $B(R) = \sum_{v=0}^{\infty} R^v B_v$ is a stochastic matrix and x_0 is the left eigenvector of $B(R)$.

- HOWEVER x_0 can not be determined UNIQUELY and another "normalizing" equation for x_0 may be needed:

$$x e^T = 1 \Rightarrow \sum_{i=0}^{\infty} x_i e_m^T = 1 \Rightarrow x_0 \sum_{i=0}^{\infty} R^i e_m^T = 1$$

(e^T = unity vector = $[1,1,1,\ldots,1,\ldots]$)

$$\Rightarrow \boxed{x_0 [I-R]^{-1} e_m^T = 1}$$

(Normalizing equation for x_0).

(Fall' 2006) we need $|R|<1$ for convergence

Matrix Geometric Techniques :

(Derivation of R).

- We know that:

$$R = \sum_{k=0}^{\infty} R^k A_k$$

Neuts has shown that the recursion $X_{l+1} = \sum_{k=0}^{\infty} X_l^k A_k$ converges monotonicaly to R i.e $X_l \xrightarrow[l \to \infty]{} R$

However, convergence may be <u>slow</u> (if system load is high) and <u>divergence</u> may occur when eigenvalues of R lie outside the unit disk (finite buffer systems).

- Another method is to rewrite equation

$$R = \sum_{k=0}^{\infty} R^k A_k$$

as

$$R(I - A_1) = \sum_{\substack{k=0 \\ k \neq 1}}^{\infty} R^k A_k$$

$$R = \left[\sum_{\substack{k=0 \\ k \neq 1}}^{\infty} R^k A_k \right] [I - A_1]^{-1}$$

and solve the equation above iteratively (not a real gain in doing so).

- Check other methods in paper (ICC 92).