Practical Problems in Networks and motivation for use of Probability Theory and Stochastic Processes

- Statistics of Queue Length ← Buffer sizing
  - ↑ Blocking → Information Loss.
- How long does it take for messages to be transmitted
  - ↑ Statistics of delay
    - ↑ QoS implications
- Traffic analysis ← How we can model traffic in ATM networks?
  - ↑ Are traffic models accurate descriptors of real Broadband-Multimedia traffic? (What "accurate" really mean)

Methods of Analysis:

- Probability Theory
- Statistics

} ← - Computational Techniques
- Approximation/Assumptions

↑

Simulation methodologies (very useful for complex systems).

Stochastic Processes - Brief review

X_t , $t \in \mathcal{I}$, or if $\mathcal{I} = \mathbb{R}^+$ $\{X_t\}_{t \geq 0}$

I: index set , finite or infinite (countable or uncountable)

X_t : random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ ← Prob. measure

↑ Sample Space ↑ Appropriate 6-field

i.e. $\omega \in \Omega \xrightarrow{X_t(\cdot)} X_t(\omega) \in S$

S: is the process State Space

Problems of interest: - If $Q \subset S$ then find $\mathbb{P}(X_t \in Q) \triangleq$
$\mathbb{P}(\omega: X_t(\omega) \in Q)$.

- If $Q_1, Q_2 \subset S$ then find $\mathbb{P}(X_{t_1} \in Q_1, X_{t_2} \in Q_2)$ = Joint probability

- Hitting probabilities: $\mathbb{P}(X_t \in Q$ for ~~any~~ some $t \in I)$.

- First passage probabilities: Statistics of $T_{B>A} - T_A$ where

  T_A: is the instant when $\{X_t\}_{t \geq 0}$ enters set $A \subset S$

  $T_{B>A}$: is the first instant $> T_A$ when process enters set $B \subset S$.

Notions of interest that students must be familiar with:

- Stationarity: $\{X_t\}_{t \geq 0}$ is stationary when for any $T, n, t_1, t_2 \ldots, t$

$$F_{X_{t_1+T}, X_{t_2+T}, \ldots X_{t_n+T}} = F_{X_{t_1}, X_{t_2}, \ldots X_{t_n}}.$$

- $\{X_t\}_{t\geq 0}$ has independent increments when for any $t_1 < t_2 < \dots \leq t_n$ and any $n \geq 1$ the differences

$$X_{t_2} - X_{t_1} \;,\; \dots \;,\; X_{t_n} - X_{t_{n-1}}$$

are independent random variables.

- $\{X_t\}_{t\geq 0}$ is a Markov Chain when it has the Markov Property i.e for any $t_0 < t_1 < \dots < t_m < \dots\; t_n$

$$F_{X_{t_n}, X_{t_{n-1}}, \dots X_{t_{m+1}} / X_{t_m} \dots X_{t_0}} \left(x_n, \dots, x_{m+1} / x_m , \dots x_0 \right) =$$

$$= F_{X_{t_n} \dots X_{t_{m+1}} / X_{t_m}} \left(x_n, \dots x_{m+1} / x_m \right).$$

- Any process with independent increments has the Markov property - opposite is NOT true.

- Special cases of processes with the Markov Property are the Random Walks (p. 84, 85).

- Markov Chain

~ It is a discrete time Stochastic Process i.e.

$$X_n \in S, \quad n = 1, 2, 3 \ldots$$

with S being a countable state space.

i.e. Markov Chains have both as index sets and state spaces countable sets.

- nth step transition probabilities

$$p_{ij}^{(n)}(m) \overset{\Delta}{=} p_{ij}^{(n)} = \mathbb{P}(X_{m+n} = j \,/\, X_m = i),$$

time homogenuity

Note: $p_{ij}^{(1)} = p_{ij}$

Properties: - $\mathbb{P}(X_0 = i_0, X_1 = i_1, \ldots X_n = i_n) =$

$$= \mathbb{P}(X_0 = i_0)\, p_{i_0, i_1} \cdot p_{i_1, i_2} \cdot \ldots p_{i_{n-1}, i_n}$$

if $\mathbb{P}^{(n)} = [\, p_{ij}^{(n)} \mid i, j \in S \,]$ is the nth step transition matrix

- $\mathbb{P}^{(n)}$ is a stochastic matrix (i.e. sum of the rows = 1).

- Chapman-Colmogorov Equation:

$$\mathbb{P}^{(m+n)} = \mathbb{P}^{m}\, \mathbb{P}^{n} \quad \text{or}$$

$$p_{ij}^{(m+n)} = \sum_{k \in S} p_{ik}^{(m)}\, p_{kj}^{(n)}$$

– Hence $P^{(n)} = P^n$.

<u>Fact:</u> – $P^{(n)} = P^n \xrightarrow[n\to\infty]{} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \underline{\pi}$ where $\underline{\pi} \in R^{1\times n}$, i.e $\underline{\pi} = [\pi_1, \pi_2, \dots \pi_n]$

* If all the eigenvalues of P are $1, \lambda_1, \lambda_2, \dots \lambda_4$ and $|\lambda_i| < 1$

– $\underline{\pi} = \underline{\pi} P$. $\underline{\pi}$ = stationary probability vector

$$\pi_j = \lim_{n\to\infty} p_{ij}^{(n)} = P(X_n = j \text{ as } n \to \infty).$$

irrespective of initial condition i.

<u>Special cases:</u> If $|\lambda_j| = 1$, ie. $\lambda_i = e^{j\vartheta}$ then P^n will not converge and the Markov chain has no steady state!
If P is a block matrix it will be impossible for some states to be entered after some others. Simplest case is $P = I$ and there is no asymptotic solution.

<u>Note:</u> $1, \lambda_1, \dots, \lambda_j, \dots \lambda_s$ are the eigenvalues of matrix P

The λ's can be found by solving

$$f(\lambda) = |P - \lambda I| = 0 \Rightarrow c_0\lambda^n + c_1\lambda^{n-1} + \dots + c_{n-1}\lambda + c_n = 0$$

(Characteristic Equation)

With every eigenvalue λ_i there is an eigenvector X_i such that

$$P \cdot X_i = \lambda_i X_i$$

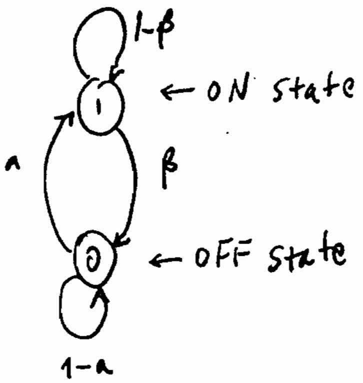
or $AX = X\Lambda \Rightarrow A = X\Lambda X^{-1}$ (diagonalization)

↑

matrix of eigenvectors

A note on the two state -ON.OFF- model

$\mathcal{J} = \{0,1\}$



$$\mathbb{P} = \begin{bmatrix} 1-a & a \\ \beta & 1-\beta \end{bmatrix}$$

We can write

$$\mathbb{P} = M \Lambda M^{-1}$$

$$M = \begin{bmatrix} 1 & -a \\ 1 & \beta \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} \beta & a \\ -1 & 1 \end{bmatrix} \frac{1}{a}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1-a-\beta \end{bmatrix}$$

Therefore $\mathbb{P}^n = M \Lambda^n M^{-1} = M \begin{bmatrix} 1 & 0 \\ 0 & (1-a-\beta)^n \end{bmatrix} M^{-1}$

$$= \frac{1}{a+\beta} \begin{bmatrix} \beta & a \\ \beta & a \end{bmatrix} + \frac{\overset{(1-a-\beta)^n}{\cancel{\omega}}}{a+\beta} \begin{bmatrix} a & -a \\ -\beta & \beta \end{bmatrix}$$

Hence as $n \to \infty$ $\mathbb{P}(X_n = 0 / X_0 = 0) = \mathbb{P}(X_n = 0 / X_1 = 0) = \mathbb{P}(X_n = 0) = \frac{\beta}{a+\beta}$

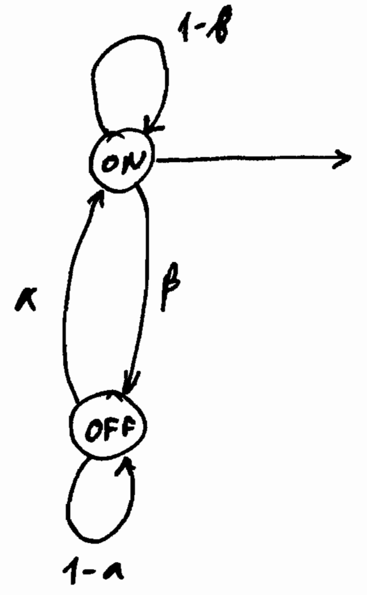
$$\mathbb{P}(X_n = 1 / X_0 = 0) = \mathbb{P}(X_n = 1 | X_0 = 0) = \mathbb{P}(X_n = 1) = \frac{a}{a+b}$$

If $a = \beta = 0$ dependency exist always in the initial state. No steady state solution

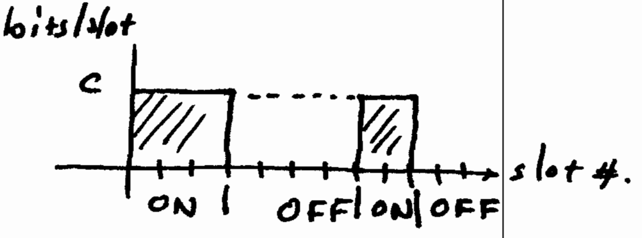
If $a = \beta = 1$ we have a periodic chain! However $\frac{(1-a-\beta)^n}{a+\beta} \begin{bmatrix} a & -a \\ -\beta & \beta \end{bmatrix}$ does not converge and P^n does not converge

Application

Simple Voice model:



constant bit rate of c bits/slot



Average traffic load:

Note: $E[\text{ON period}] = \frac{1}{b}$ slots

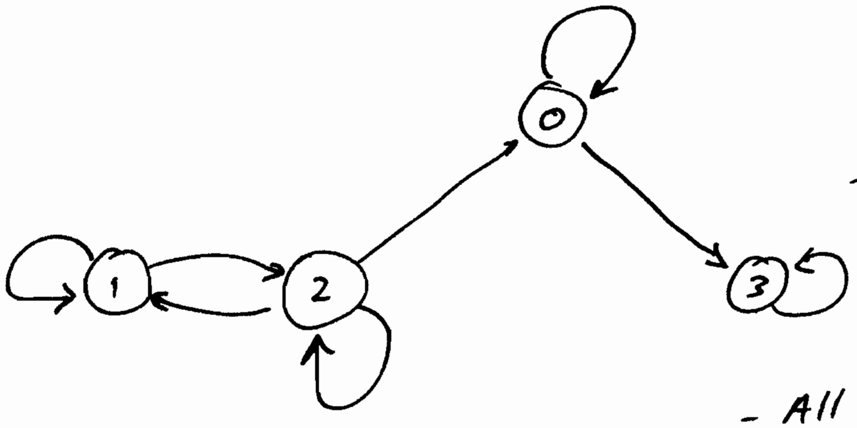
$E[\text{OFF period}] = \frac{1}{a}$ slots

$$\frac{\frac{1}{b}c}{\frac{1}{b}+\frac{1}{a}} = \frac{ac}{a+b} = \mathbb{P}(\text{ON})\cdot c$$

Note: If X represents the # of slots the Markov Chain will remain is state ON, <u>GIVEN that is starts at state ON</u>, until the process leaves the state ON is described by:

$$\mathbb{P}(X=k) = (1-p)^{k-1}p \quad , \quad k=1,2,\dots.$$

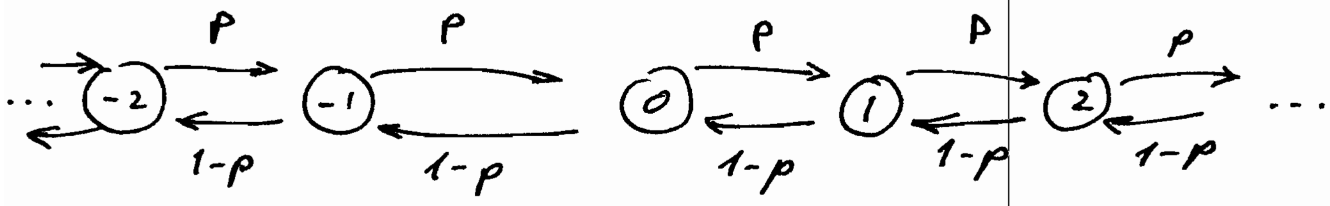
Examples:



- Three classes:

  {0}, {1,2}, {3}.

- All states have period 1.



Random Walk process

- If $p>0$ then the process has one class

  $\{0, \pm 1, \pm 2, \ldots\}$.

- The period of any state is d.

State Classification

- States i, j communicate i.e $i \leftrightarrow j$ when $\exists\, n, m$ s.th $p_{ij}^{(n)} > 0$, $p_{ji}^{(m)} > 0$.
- $\leftrightarrow$ is an equivalence relationship which over S corresponds to equivalence classes
- If a Markov chain has a state space S which consists of one equivalence class then it is called an irreducible MC
- Period d_i of a state i : A state i is periodic with period d_i if:

$p_{ii}^{(n)} = 0$ if $n \neq m d_i$ for any positive integer m

and d_i is the largest integer with this property, i.e d_i is the largest common factor of all n's of the form $n = m \cdot d_i$ for which $p_{ii}^{(n)} \neq 0$.

Recurrence Properties: Assume T_{ii} the first time the chain returns to state i when $X_0 = i$.

$\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$ ↙

$f_i = \mathbb{P}(T_{ii} < \infty) < 1 \longrightarrow$ i is transient (i.e. $f_i = P($ever returning to state $i) < 1$

$f_i = \mathbb{P}(T_{ii} < \infty) = 1 \longrightarrow$ i is recurrent (i.e $f_i = \mathbb{P}($ever returning to state $i) = 1$

$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$ ↗

null recurrent if $E[T_{ii}] = \infty$

positive-recurrent $E[T_{ii}] < \infty$

Proof: Outline:

T_{ij} the first passage time from i to j. $f_{ij}^{(n)} = \mathbb{P}(T_{ij}=$

$$F_{ij}(z) = \sum_{n=1}^{\infty} f_{ij}^{(n)} z^n$$

$$G_{ij}(z) = \sum_{n=0}^{\infty} p_{ij}^{(n)} z^n$$

$$G_{ij}(z) - \delta_{ij} = F_{ij}(z)\, G_{jj}(z)$$ [Renewal Equation].

$$F_{ii}(1) = \sum_{n=1}^{\infty} \mathbb{P}(T_{ii}=n) = \mathbb{P}(T_{ii}<\infty)$$

i is ~~Transient~~ recurrent $\Longrightarrow F_{ii}(1) = 1 \Rightarrow 1 - \frac{1}{G_{ii}(1)} = 1 \Rightarrow$

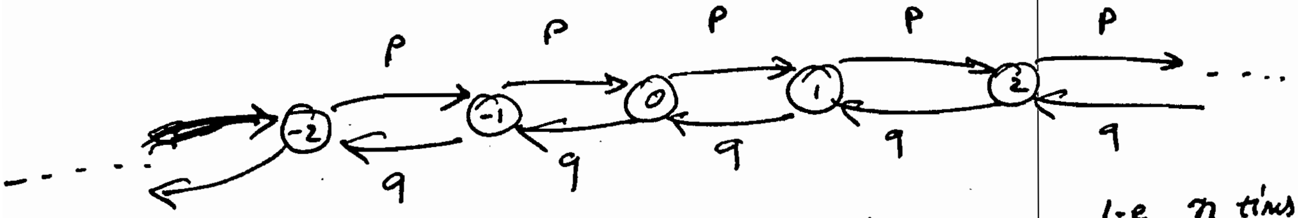
$$\Rightarrow \frac{1}{G_{ii}(1)} = 0 \Rightarrow G_{ii}(1) = \infty \Rightarrow$$

$$\Rightarrow \sum_{n=0}^{\infty} p_{ii}^{n} = \infty$$

If i is transient similar proof.

Example (Check book by Leon Garcia p.481)

Random walk in one dimension with no barriers:



$q = 1-p$

i.e n times $+1$
n times -1

$$p_{00}^{(2n)} = \underbrace{pq}_{2step} + \underbrace{2p^2q^2}_{4steps} + \dots = \binom{2n}{n} p^n \underbrace{(1-p)^n}_{q}$$

Note:
Stirling's formula
$n! \approx \sqrt{2\pi}\, n^{n+1/2} e^{-n}$

$$\binom{2n}{n} p^n (1-p)^n \sim \frac{[4p(1-p)]^n}{\sqrt{\pi n}}$$

$$\sum_{n=1}^{\infty} p_{00}^{(2n)} \sim \sum_{n=1}^{\infty} \frac{[4p(1-p)]^n}{\sqrt{\pi n}}$$

$p=\frac{1}{2}$ series diverges $\Rightarrow$ State 0 is recurrent
(in this case $p(1-p)=0.25$)

$p \neq 1/2$ series converges $\Rightarrow$ state 0 is transient.
(in this case $p(1-p)<0.25$)

In fact for $p=\frac{1}{2}$ the chain is <u>null</u> <u>recurrent</u>

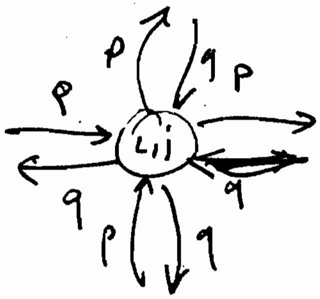
Since $F_{ii}(z) = 1-(1-z^2)^{1/2}$ (see over)

$$F'_{00}(1) = \lim_{z \to 1} \frac{z}{(1-z^2)^{1/2}} = \infty = E[T_{00}]$$

Note:
$F_{ii}(z) = \sum f_{ii}^{(n)} z^{n-1}$
$F'_{ii}(1) = E[T_{ii}]$.

Generalization:

Two-dimensional random walk



$$P^{2n}_{(0,0),(0,0)} = \left[\binom{2n}{n}\frac{1}{2^{2n}}\right]^2$$

$$\sim \left(\frac{1}{(\pi n)^{1/2}}\right)^2 = \frac{1}{\pi n}$$

$$\sum_{n=1}^{\infty} \frac{1}{\pi n} = \infty$$ (Null-recurrent)

In three dimensions however

$$P^{2n}_{(0,0,0)(0,0,0)} \sim \left(\frac{1}{\pi n}\right)^{3/2}$$

$$\sum_{m=1}^{\infty} p^{2n}_{(0,0,0)(0,0,0)} < \infty$$ and the chain becomes

transient!

Steady State Probabilities: Ergodic Theorems

let $E[T_{ii}] = \mu_i = \begin{cases} \infty & \text{for } i \text{ being transient} \\ \infty & \text{" " " null-recurrent} \\ \sum_{n=1}^{\infty} n f_{ii}^{(n)} & \text{for } i \text{ positive-recurrent.} \end{cases}$

Kolmogorov's Thm:

A MC which is irreducible, aperiodic with a state space S:

a) $p_{ij}^{(n)} \rightarrow \mu_j^{-1} \geq 0$ as $n \rightarrow \infty$

b) $\pi = \pi P \iff$ chain is positive-recurrent and $\pi_j = \mu_j^{-1}$

Markov Processes

X_t , $t \in T$ continuous index set normally $[0,\infty)$.

$X_t \in \mathcal{S}$, $\mathcal{S}$ is a discrete set.

$$\mathbb{P}(X_{s+t} = j \,/\, X_u = x_u ,\ u \le s) = \mathbb{P}(X_{s+t} = j \,|\, X_s = x_s).$$

Example: Poisson (arrival) process:

Let $N_{t\tau}$ be the # of arrivals in the interval $(t,\tau]$.

Then if $\{N_{0,t} , t \geq 0\}$.

$$\left\{ \begin{array}{l} \bullet \text{ is time homogeneous i.e } \\ \qquad \mathbb{P}(N_{\tau,\tau+t} = k) = \mathbb{P}(N_{0,t}) \\ \bullet \text{ has independent increments} \\ \quad \text{i.e. } \{N_{0,\tau}\} \text{ indep of } \{N_{\tau,\tau+t}\} \\ \bullet \quad \frac{\mathbb{P}(N_{t,t+\tau} \geq 2)}{\tau} \longrightarrow 0 \text{ as } \tau \to 0 \end{array} \right\} \iff \left\{ \begin{array}{l} \mathbb{P}(N_{t,t+h} = 1) = \lambda h + o(h) \\ \mathbb{P}(N_{t,t+h} = 0) = 1 - \lambda h + o(h) \\ \mathbb{P}(N_{t,t+h} = 2) = o(h) \\ N_{0,t},\ N_{t,t+h} \text{ are independent} \end{array} \right\}$$

for some $\lambda \geq 0$

then $N_{0,t}$ is called a Poisson process

{ Sometimes $N_{0,t}$ is denoted as N_t, $t \geq 0$. }.

Facts for Poisson Process

pendent Observer property:

X_t = state of queue at t.

Arrival in $(\tau-h, \tau]$. i.e. $N_{\tau-h,\tau} \geqq 1$. (Event $\{N_{t-h,\tau} \geq 1\}$ is independent of history up to time $t-h$ because of ~~memoryless~~ Markovian property).

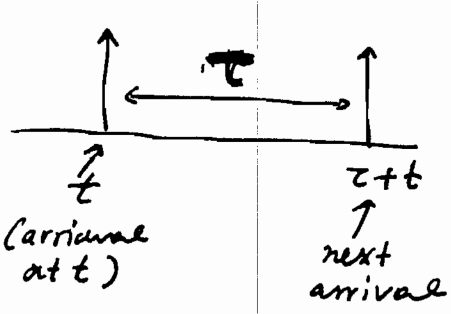
$$\mathbb{P}(X_{\tau-h} = x \mid N_{\tau-h,\tau} \geq 1) = \mathbb{P}(X_{\tau-h} = x)$$

Letting: $h \to 0$ $\mathbb{P}(X_\tau = x \mid \text{arrival at time } \tau) = \mathbb{P}(X_\tau = x)$

i.e the distribution of X_t is independent of the event that there is an arrival at time t _

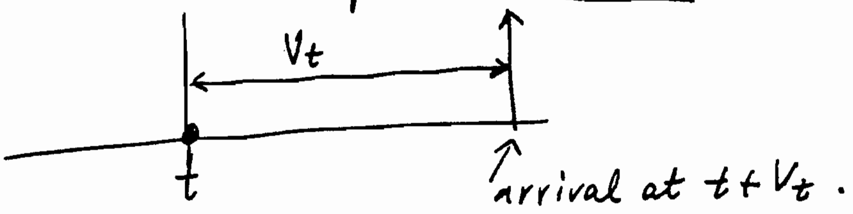
Also from the memoryless property:

$$\mathbb{P}(T > t+\tau \mid T > t) = \mathbb{P}(T > \tau).$$



i.e. given that an arrival happens at t the interarrival time τ is independently distributed of the event at t!

If the observation point is <u>random</u> a similar result holds:



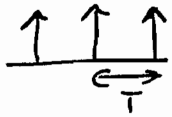
$$\mathbb{P}(V_t \leq u \mid N_s = n_s,\ s \leq t) =$$

$$= \mathbb{P}(N_{t,t+u} > 0 \mid N_s = n_s,\ s \leq t)$$

$$= \mathbb{P}(N_{t,t+u} > 0)$$ indep. increments and time homogeneity

$$= 1 - e^{-\lambda u}\ !$$

- The Poisson Process is a Markov Process (Prove it!)
- $N_{o,t}$ is a Poisson R.V. i.e. $\mathbb{P}(N_{ot}=k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$



- Interarrival times are *exponential* random variables i.e. $\mathbb{P}(T \le t) = 1 - e^{-\lambda t}$, and they are *memoryless*

  i.e $P(T > t+\tau / T > \tau) = e^{-\lambda t} = \mathbb{P}(T > t)$.

- Sums or (probabilistic splits) of Poisson processes are Poisson processes.

Some facts on Continous Time MC

- $p_{ij}(t) = \mathbb{P}(X_{s+t} = j / X_s = i)$ $s,t \in [0,\infty)$ $(i,j) \in S$.

  $\uparrow$

  Homogeneity: indep. of s.

- $\mathbb{P}(t) = [\, p_{ij}(t) \,]$ is a stoch. matrix

- $P(t)$ is right continous at $t=0$ i.e.

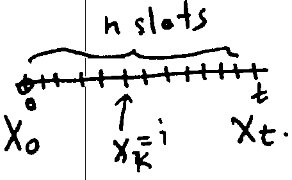
  $P(t) \to I$ as $t \to 0$.

- Chapman-Kolmogorov:

  $\mathbb{P}(s+t) = \mathbb{P}(s)\mathbb{P}(t)$

- For all $t \geq 0$, $p_{ii}(t) > 0$ $(i \in S)$

<u>Pf:</u> $p_{ii}(t) \geq \mathbb{P}\left[X_t = i, \overset{=i}{X_{t(1-\frac{1}{n})}}, \overset{=i}{X_{t(1-\frac{2}{n})}} \,/\, X_0 = i \right]$



$$= \mathbb{P}\left[\bigcap_{k=1}^{n} X_{\frac{kt}{n}} = i \,\middle|\, X_0 = i \right]$$

$$= \left[p_{ii}\left(\frac{t}{n}\right)\right]^n.$$ As $n \to \infty$ $\forall t > 0$

$$p_{ii}\left(\frac{t}{n}\right) \to \underline{1 > 0}.$$

- $p_{ij}(t)$ is unif. continuous on $[0, \infty)$ and uniform in j.

i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.th: $\left| p_{ij}(t + \Delta t) - p_{ij}(t) \right| < \varepsilon$ for $|\Delta t| < \delta(\varepsilon)$

δ is <u>indep.</u> of $\{t\}$ (unif. continuity)

If a state i is entered at time t and the next state transition takes place at time $t+T$ the T is called the <u>holding time</u> of i.

By the Markov Property the time of the next time of change is independent of the time of the previous change of state. In other words state holding times are memoryless therefore are exponentially distributed.

$$\mathbb{P}(X_{t+h} = j \,/\, X_t = i) = q_{ij} h + o(h)$$

— q_{ij} is the <u>instantaneous transition rate</u>

or the generator of the <u>Markov Process</u>

— q_{ij} can be arranged in matrix form $Q = \begin{array}{cc} & j \\ i & \begin{bmatrix} & | & \\ - & q_{ij} & - \\ & | & \end{bmatrix} \end{array}$

— Facts: $\left[\text{let } \mathbb{P}(t) = \begin{array}{cc} & j \\ i & \begin{bmatrix} & | & \\ - & p_{ij}(t) & - \\ & | & \end{bmatrix} \end{array} \right].$

$$* \ \mathbb{P}(t) = e^{tQ} \triangleq \sum_{n=0}^{\infty} \frac{(tQ)^n}{n!}$$

$$\left[\begin{array}{l} \text{Since } \mathbb{P}(s+t) = \mathbb{P}(s)\mathbb{P}(t) \Rightarrow \\ \qquad \Rightarrow \mathbb{P}(h) = \mathbb{I} + Qh + o(h) \ \text{ for some} \\ \qquad (p_{ij}(h) = \delta_{ij} + q_{ij}h + o(h)) \quad \text{matrix } Q \end{array} \right.$$

<u>Therefore:</u>
$$Q = \lim_{h \to 0} \frac{\mathbb{P}(h) - I}{h} = \lim_{h \to 0} \frac{\mathbb{P}(h) - \mathbb{P}(0)}{h}$$

$$\left[(q_{ii} - p_{ii}'(0)) \sim \right] = \mathbb{P}'(0).$$

Fact: $\sum_{j \in S} q_{ij} = \lim_{t \to 0} \frac{d}{dt} \left\{ \sum_{j \in S} p_{ij}(t) \right\} = \lim_{t \to 0} \frac{d}{dt} 1 = 0$

Let $q_i = -q_{ii} = \sum_{(\neq)} q_{ij}$

Also from elementary considerations:

$$\left. \begin{array}{l} p_{ij}(h) = \overset{1+}{} q_{ij}h + o(h) \\ p_{ii}(h) = 1 - \sum q_{ij}h + o(h) \end{array} \right\} \Rightarrow q_{ii} = -\sum_{j \neq i} q_{ij} \Rightarrow \boxed{\sum_j q_{ij} = 0}$$

$\cancel{\sum_{j \neq i} q_{ij} h + o(h) = 0 \ \forall h}$

$\cancel{\sum_j q_{ij} = 0}$

- Facts:

i.e. Residence time in state i follows an exponential distribution

$$\left\{\begin{array}{l}
\mathbb{P}(X_{t+h}=j \mid X_t=i) = q_{ij}h + o(h) \to \underline{\underline{\mathbb{P}(T_{ij}<t) = 1-e^{-q_{ij}t}}} \\
\mathbb{P}(X_{t+h}=i / X_t=i) = 1-\sum_{i\neq j} q_{ij}h + o(h) \\
\quad = 1+q_i h + o(h) \Rightarrow \\
\quad \Rightarrow \mathbb{P}(X_{t+h}\neq i / X_t=i) = q_i h + o(h) \Rightarrow \\
\qquad \Rightarrow \mathbb{P}(T_i<t) = \underline{\underline{1-e^{-q_i t}}}.
\end{array}\right.$$

$$\left.\begin{array}{l}
- \quad \mathbb{P}'(t) = \mathbb{P}(t)\mathbb{Q}, \text{ initial condition } \mathbb{P}(0)=I! \\
\qquad\qquad (\text{bounded } q_i \text{ i.e. } q_i<M<\infty). \\
- \quad \mathbb{P}'(t) = \mathbb{Q}\mathbb{P}(t), \quad '' \quad '' \quad '' \quad '' \\
\qquad\qquad (\text{finite } q_i<\infty)
\end{array}\right\} \Leftarrow \left\{\begin{array}{l}
\text{Forward Colmog. Eq} \\
\text{Backward } '' \quad ''
\end{array}\right\}$$

The Embedded Markov Chain

Continous time Markov ~~Chain~~ Process

X_t, $t > 0$.

Let $0, \tau_1, \tau_2, \tau_3 \dots, \tau_n, \dots$ be transition instants.

Define $Z_n = X_{\tau_n^+}$, $n = 1, 2, 3, \dots$

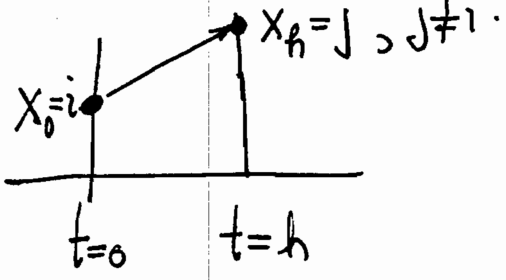
$\{Z_n, n = 1, 2, 3, \dots\}$ is called the embedded Markov Chain.

$Z_0 = X_0$

Transition probabilities: (p. 125 of text)

$i \neq j$ $\quad r_{ij} = \mathbb{P}\left[Z_n = j / Z_{n-1} = i\right] = \mathbb{P}\left[Z_1 = j / Z_0 = i\right]$

$$= \lim_{h \to 0} \mathbb{P}\left[X_h = j / X_h \neq i, X_0 = i\right]$$



$$= \lim_{h \to 0} \frac{\mathbb{P}(X_h = j, X_h \neq i \mid X_0 = i)}{\mathbb{P}(X_h \neq i / X_0 = i)} = \lim_{h \to 0} \frac{p_{ij}(h)}{1 - p_{ii}(h)}$$

$$= \frac{q_{ij}}{q_i}$$ (By Hospital's rule).

Also: $r_{ii} = 0$

See over for facts! →

$$- \pi_j = \lim_{t \to \infty} \overset{P}{\cancel{\Pi}}_{ij}(t)$$

A M.P is irreducible, transient, null recurrent or positive recurrent based on respective property of its EMBEDDED MC !

Therefore:

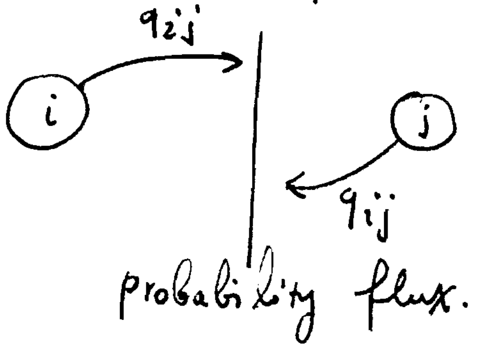
If the ~~[illegible]~~ M.P is transient or null recurrent → $\pi_j = 0$

i.e. there is no steady state

The process is positive recurrent if $0 = \pi Q$ (real proof on p. 126)

Balance Equations: $$\sum_{j \neq i} \pi_i q_{ij} = \sum_{j \neq i} \pi_j q_{ji}$$

$\forall i$



[Local balance Equations]

Important

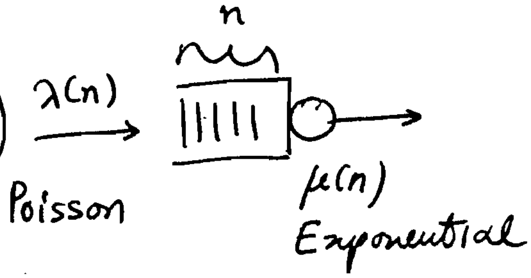
Note: If $\pi_i q_{ij} = \pi_j q_{ji}$ the local balance equations are always satisfied. In this case the process is called reversible

[Global Balance Equations]: $$\overbrace{\sum_{i \in A} \sum_{j \notin A} \pi_i q_{ij}}^{\text{out-flux}} = \overbrace{\sum_{i \in A} \sum_{j \notin A} \pi_j q_{ij}}^{\text{in-flux}}$$

ACS

Example:

(Markovian Queue)



- Birth and death process: $S = \{0, 1, 2, 3, \dots\}$

- If $X_t = j$ is the population of the queue at time t:

$$X_{t+h} = \begin{cases} j+1 & \lambda(j)h + o(h) \\ j-1 & \mu(j)h + o(h) \\ j & 1 - \lambda(j) - \mu(j) + o(h) \\ \text{other} & o(h) \end{cases}$$

if $X_t = 0$ then $X_{t+h} = \begin{cases} 1 \text{ with } - \lambda(0)h + o(h). \\ 0 \text{ with prob } 1 - \lambda(0)h. \\ 0 \text{ otherwise.} \end{cases}$

Therefore:

$$Q = \begin{bmatrix} -\lambda(0) & \lambda(0) & 0 & 0 \\ \mu(1) & -[\lambda(1)+\mu(1)] & \lambda(1) & 0 \\ 0 & \mu(2) & -[\lambda(2)+\mu(2)] & \lambda(2) \\ 0 & 0 & \mu(3) & \cdots \end{bmatrix}$$

- $\pi Q = 0 \Rightarrow \quad \lambda(j-1)\pi(j-1) = \mu(j)\pi(j) \quad ; \quad \sum_{j=0}^{\infty} \pi(j) = 1.$

$$\Downarrow$$

$$\boxed{\lambda_{j-1}\pi_{j-1} - (\mu_j + \lambda_j)\pi_j + \mu_{j+1}\pi_{j+1} = 0.}$$

↖ Simpler equations.

- Solution

$$\pi_i = \frac{p_j}{\sum_{k=0}^{\infty} p_k} \, ; \qquad \left\{ \sum_{K=0}^{\infty} p_K < \infty \text{ for the solution to exist!} \right\}.$$

$$p_0 = 1 \, , \quad p_j = \prod_{k=1}^{j} \frac{\lambda_{k-1}}{\mu_k} \quad j > 0.$$

- Simple M|M|1 queue.

$$\lambda(j) = \lambda \, , \quad \mu(j) = \mu .$$

$$\pi(n) = (1-\rho)\rho^n \, , \quad \rho = \frac{\lambda}{\mu}$$

Utilization of the server:

$$U = \underset{\substack{\| \\ \text{Prob. server is busy}}}{1 - \pi(0)} = \rho . = \frac{\lambda}{\mu} .$$