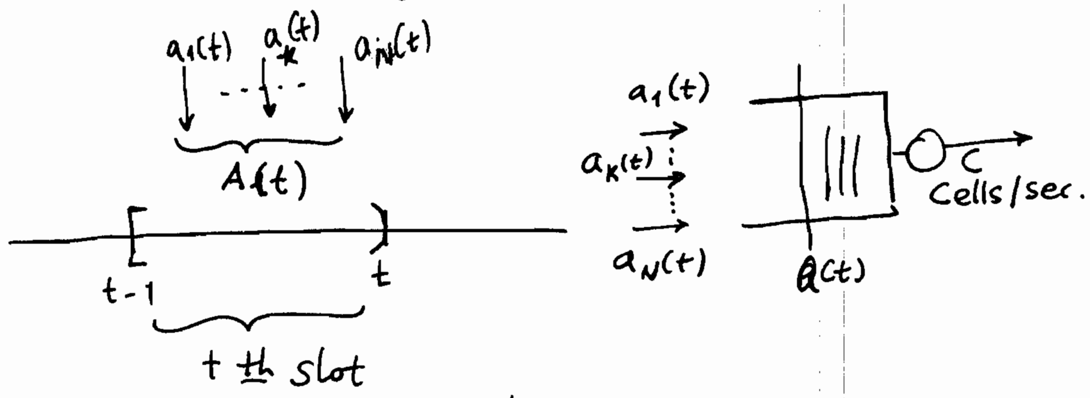
Notes on Effective Bandwidths

- We consider a buffer fed by multiple sources $a_k(t)$, $k=1,2,\ldots N$. where $a_k(t)$ in the interval $[t-1,t)$. We assume time is slotted and the duration is one time unit per slot. i.e. $a_k(t)$ denotes the # of packets/cells/bits etc during the t'th slot.



- Total arrivals in $[t-1,t)$ is $\sum_{i=1}^{K} a_i(t)$

- Total arrivals in $[t_1,t_2)$ from $k^{\underline{th}}$ stream is

$$A_k(t_1,t_2) = \sum_{t=t_1}^{t_2} a_k(t)$$

- Total arrivals in $[t_1,t_2)$ from all N streams is

$$A(t_1,t_2) = \sum_{k=1}^{N} A_k(t_1,t_2) = \sum_{k=1}^{N} \sum_{t=t_1}^{t_2} A_k(t_1,t_2)$$

- Goal: Characterize $\mathbb{P}(Q(t) > x)$ for large t, i.e $P(Q>x)$ and in particular find bounds s.th.

$$P(Q>x) \approx e^{-\theta x} \quad \text{for suitable } \underline{\theta > 0}.$$

- If we have a R.V. X then the moment generation function of X is $M_X(\theta) = E(e^{\theta X})$.

- Conside $A_K(t_1, t_2)$. It's moment generating function is then $E\left[e^{\theta A_K(t_1,t_2)}\right]$.

- We will make an assumption at this point:

Let $A_K(t_1, t_2)$ (which is a RV) be bounded somehow

One way to bound $A_K(t_1,t_2)$ is in the sense of bounding It's moment generation function by a function $\hat{A}_K(\theta, t_2-t_1)$ s.th.

$$E\left[e^{\theta A_K(t_1,t_2)}\right] \le e^{\theta \hat{A}_K(\theta, t_2-t_1)}$$

or

$$\ln E\left[e^{\theta A_K(t_1,t_2)}\right] \le \theta \underbrace{\hat{A}_K(\theta, t_2-t_1)}$$

This deterministic function is called an <u>envelope process</u>.

- An intuitive selection would be

$$\hat{A}_K(\theta, t_2-t_1) = \hat{a}_K(\theta)(t_2-t_1) + \hat{c}_K(\theta)$$

- This is the <u>linear</u> envelope process and $\hat{a}_K(\theta)$ will play a role in the definition of the so called "effective Bandwidth".

- We will now consider the buffer dynamics!

1) $Q(t) = \max(0, q(t-1) + a(t) - C) = (q(t-1) + a(t) - C)^+$

2) We apply the formula starting recursively from $Q(0)$ at $t=0$.

$$Q(1) = (Q(0) + a(1) - c)^+ = \max(0, a(1) - c)$$

$$Q(2) = (Q(1) + a(2) - c)^+ = \max(0, a(2) - c, a(1) + a(2) - 2c)$$

$$\vdots$$

$$Q(t) = \max(0, \overbrace{a(t)}^{A(t-1,t)} - c, \overbrace{a(t) + a(t-1)}^{A(t-2,t)} - 2c,$$

$$\dots \underbrace{a(t) + a(t-1) + \dots a(1)}_{A(0,t)} - tc)$$

i.e

$$\boxed{Q(t) = \max_{\{s:\, t \geq s \geq 0\}} (A(t-s,t) - sc)}$$

- Hence

$$E[e^{\vartheta Q(t)}] = E[e^{\vartheta \max_{(s:\, s \leq t)} \{A(t-s,t) - sc)\}}].$$

$$\geq \max_s E[e^{\vartheta (A(t-s,t) - sc)}].$$

Therefore $\boxed{E[e^{\vartheta Q(t)}] \leq \sum_{s=0}^{t} E[e^{\vartheta (A(t-s,t) - sc)}].}$

– Now let's go back to the traffic bounds we introduced!

$$A(t_1,t_2) = \sum_{k=1}^{N} A_k(t_1,t_2)$$

$$E\left[e^{\theta(A(t-s,t)-sc)}\right] = E\left[e^{\theta(\sum A_k(t-s,t)-sc)}\right] =$$

$$= e^{-\theta sc}\prod_{k=1}^{N} E\left[e^{\theta A_k(t-s,t)}\right] \leq e^{-\theta sc}\prod_{l=1}^{N} e^{\theta \hat{A}_k(\theta,s)}$$

$$= e^{-\theta sc}\, e^{\theta \sum \hat{A}_k(\theta,s)}$$

$$= e^{\theta(\hat{A}(\theta,s)-sc)}$$

Where the composite <u>envelope process</u> is $\hat{A}(\theta,s) = \sum_{k=1}^{N} \hat{A}_k(\theta,t)$

Considering the <u>linear</u> <u>envelope process</u> we get:

$$\hat{A}_k(\theta,t) = \hat{a}_k(\theta) + \hat{\sigma}_k(\theta)$$

and $$\hat{A}(\theta,t) = \hat{a}(\theta) + \hat{\sigma}(\theta)$$

with $$\boxed{\hat{a}(\theta) = \sum_{k=1}^{N} \hat{a}_k(\theta)} \quad (\$)$$

$$\hat{\sigma}(\theta) = \sum_{k=1}^{N} \hat{\sigma}(\theta).$$

– We can now write for $E\left[e^{Q(t)\vartheta}\right]$

$$E\left[e^{Q(t)\vartheta}\right] \leq \sum_{s=0}^{t} e^{\vartheta[\hat{a}(\theta, s) + \hat{\sigma}(\theta)]} e^{-\theta s c} = \sum_{s=0}^{t} e^{\vartheta(\hat{a}(\theta)s - sc)} e^{-\vartheta s c}$$

$$= e^{\vartheta \hat{\sigma}(\theta)} \sum_{s=0}^{t} e^{\vartheta s[\hat{a}(\theta) - c]} . \quad \circledast .$$

As $t \to \infty$ $E\left[e^{Q(t)\vartheta}\right]$ will be bounded if

$$\boxed{\hat{a}(\theta) < c} \quad \$\$$$

Equations \$ and \$\$ are very important!

Given \$\$ then from $\circledast$ we can write:

$$\lim_{t \to \infty} E\left[e^{Q(t)\vartheta}\right] \leq \beta(\vartheta)$$

where $\beta(\vartheta) = e^{\vartheta \hat{\sigma}(\theta)} \sum_{s=0}^{\infty} e^{+\vartheta s[\hat{a}(\theta) - c]} = \dfrac{e^{\vartheta \hat{\sigma}(\theta)}}{1 - e^{-\vartheta(c - \hat{a}(\theta))}}$

- We should not forget that we are looking for a bound of $\mathbb{P}(Q>x)$

- In probability theory a bound called <u>Chernoff</u> bound has been devlloped ~~for~~ for any R.V Q with a ~~known~~ <u>MGF</u>.

Namely:

$$\mathbb{P}(Q>x) \le e^{-\theta x} E\left[e^{Q\theta}\right] \quad \text{for any } \underline{\theta>0}$$

- But we already have a bound for $E\left[e^{Q\theta}\right] \le \beta(\theta)$

i.e

$$\mathbb{P}(Q>x) \le e^{-\theta x} \underbrace{\frac{e^{\theta \hat{\sigma}(\theta)}}{1-e^{-\theta(c-\hat{a}(\theta))}}}_{\beta(\theta)}$$

$$= e^{-\theta x + \ln \beta(\theta)}.$$

Now given ~~...~~ $\hat{a}(\theta) < C$ and hence $\beta(\theta)$ finite for large x sth $\theta x >> 1$ the contribution of $\beta(\theta)$ will be small and

$$\mathbb{P}(Q \ge x) \approx e^{-\theta x} \quad \theta x >> 1$$

However this assumption may <u>NOT</u> <u>always</u> be correct.

- ~~with~~ Using Engineering arguments we can take this as correct and in this case

$$P_L = P(Q \geq x) \cong e^{-\vartheta x} \Rightarrow \vartheta = \frac{1}{x} \ln(1/P_L).$$

- How can we determine $\hat{a}(\vartheta, t)$ now?

We know $E\left(e^{\vartheta A_k(t)}\right) \leq e^{\vartheta \hat{A}_k(\vartheta,t)} \Rightarrow$

$$\Rightarrow \quad \hat{A}_k(\vartheta,t) \geq \frac{1}{\vartheta} \ln E\left[e^{\vartheta A_k(t)}\right].$$

take $\hat{A}_k(\vartheta,t) = \hat{a}_k(\vartheta)t + \hat{\sigma}_k(\vartheta)$

Then

$$\hat{a}_k(\vartheta)t + \hat{\sigma}_k(\vartheta) \geq \frac{1}{\vartheta} \ln E\left[e^{\vartheta A_k(t)}\right].$$

or

$$\hat{a}_k(\vartheta) + \frac{\hat{\sigma}_k(\vartheta)}{t} \geq \frac{1}{\vartheta t} \ln E\left[e^{\vartheta A_k(t)}\right].$$

Take limits for $t \to \infty$

and

$$\hat{a}_k(\vartheta) \geq \lim_{t \to \infty} \frac{1}{\vartheta t} \ln E\left[e^{\vartheta A_k(t)}\right].$$

Furthermore we know that:

$$\sum \hat{a}_k(\vartheta) = \hat{a}(\vartheta)$$

Therefore

$$\hat{a}(\vartheta) \geq \sum a_k(\vartheta) = \lim_{t\to\infty} \frac{1}{t}\frac{1}{\vartheta} \ln E\left[e^{\vartheta A(t)}\right]$$

$$\left(\lim_{t\to\infty} \sum \frac{1}{t} \hat{A}_k(\vartheta, t)\right)$$

Therefore if

$$\underbrace{\lim_{t\to\infty} \frac{1}{t}\frac{1}{\vartheta} \ln\left[E\, e^{\vartheta A(t)}\right]}_{\text{effective BW or equivalent capacity.}} \leq \hat{a}(\vartheta) < C$$

Then

$$\mathbb{P}(Q \geq x) \sim e^{-\vartheta x}.$$