Supplemental Notes on Linear System I/O Relations

Given a random process X_n (discrete time) or $X(t)$ (continuous time) (input), a *linear time invariant system* is characterized by an *impulse response* h_n (discrete time, Kronecker delta response) or $h(t)$ (continuous time, Dirac delta response).

For simplicity we assume the systems are *causal*, i.e., the impulse responses are 0 for negative arguments.

The output Y_n or $Y(t)$ is described by convolution sum (DT) or integral (CT):

$$Y_n = \sum_k X_{n-k}h_k = \begin{cases} \sum_{k=0}^{\infty} X_{n-k}h_k & \text{two-sided} \\ \sum_{k=0}^{n} X_{n-k}h_k & \text{one-sided} \end{cases}$$

$$Y(t) = \int X(t-s)h(s)ds = \begin{cases} \int_0^{\infty} X(t-s)h(s)ds & \text{two-sided} \\ \int_0^{t} X(t-s)h(s)ds & \text{one-sided} \end{cases}$$

The sum/integral is over all values of the dummy summation/integration variable for which X_{n-k} or $X(t-s)$ makes sense, e.g., the index ≥ 0 for the one sided case.

Limits cause complications, so often omitted until needed — then plug in.

In most engineering applications, consider two-sided, so limits simple.

Note: Sums and integrals of random processes are actually *limits of random variables*, e.g., an infinite convolution sum is defined as a limit of finite sums of random variables.

We omit all the technical details (no problem in DT FIR case, no need for limits).

The Questions

DT: Given an input random process X_n with mean function $\mu_X(n) = E(X_n)$ and autocorrelation function $R_X(t,s) = E(X_tX_s)$, what are the mean function $\mu_Y(n)$ and the autocorrelation function $R_Y(t,s)$?.

Simplify under assumption that X_n is WSS. Is Y_n then WSS?

CT: ditto

Also:If WSS, Find power spectral density $S_Y(f)$ in terms of $S_X(f)$ and transfer function $H(f)$: the Fourier transform of the autocorrelation:

$$S_X(f) = \begin{cases} \sum_k R_X(k) e^{-j2\pi f k} & \text{DT} \\ \int R_X(\tau) e^{-j2\pi f\tau} d\tau & \text{CT} \end{cases}$$

$$H(f) = \begin{cases} \sum_k h(k) e^{-j2\pi f k} & \text{DT} \\ \int h(t) e^{-j2\pi f t} dt & \text{CT} \end{cases}$$

Mean

$$E(Y_n) = \sum_k h_k E(X_{n-k}) = \sum_k h_k \mu_X(n-k)$$

If Y_n WSS, $\mu_X(n) = \mu_X$,

$$EY_n = \mu_X \sum_k h_k = \begin{cases} \mu_X \sum_{k=0}^{\infty} h_k = \mu_X H(0) & \text{two-sided} \\ \mu_X \sum_{k=0}^{n} h_k & \text{one-sided} \end{cases}$$

Note: Simpler in two-sided. If $\mu_X \neq 0$, then already it is clear that Y_n can not be WSS.

Autocorrelation

$$\begin{aligned} R_Y(k,j) &= E[Y_k Y_j] \\ &= E\left[\left(\sum_n h_n X_{k-n}\right)\left(\sum_m h_m X_{j-m}\right)\right] \\ &= \sum_n \sum_m h_n h_m E[X_{k-n} X_{j-m}] \\ &= \sum_n \sum_m h_n h_m R_X(k-n, j-m) \end{aligned}$$

If X_n is WSS, $R_X(t,s) = R_X(t-s)$ and

$$\begin{aligned} R_Y(k,j) &= \sum_n \sum_m h_n h_m R_X((k-j)-(n-m)) \\ &= \begin{cases} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_n h_m R_X((k-j)-(n-m)) & \text{two-sided} \\ \sum_{n=0}^{k} \sum_{m=0}^{j} h_n h_m R_X((k-j)-(n-m)) & \text{one-sided} \end{cases} \end{aligned}$$

Note for two-sided case, $R_Y(k,j) = R_Y(k-j) \Rightarrow Y_n$ is also WSS!!

Double-convolution, a complicated mess! As in linear systems theory, take Fourier transform to simplify:

$$
\begin{aligned}
&S_Y(f) \\
&= \sum_k \left(\sum_n \sum_m h_n h_m R_X(k-(n-m)) \right) e^{-j2\pi fk} \\
&= \sum_n \sum_m h_n h_m \left(\sum_k R_X(k-(n-m)) e^{-j2\pi f(k-(n-m))} \right) \\
&\quad \times e^{-j2\pi f(n-m)} \\
&= \left(\sum_n h_n e^{-j2\pi fn} \right) \left(\sum_m h_m e^{+j2\pi fm} \right) S_X(f) \\
&= |H(f)|^2 S_X(f),
\end{aligned}
$$

An *extremely* important result for linear time-invariant systems with random signals.

Continuous time

Same manipulations with integrals instead of sums:

$$
\begin{aligned}
E(Y(t)) &= \int E[X(t-s)]h(s)\,ds = \int \mu_X(t-s)h(s)\,ds \\
R_Y(t,s) &= \int d\alpha \int d\beta R_X((t-s)-(\alpha-\beta))h(\alpha)h(\beta)
\end{aligned}
$$

If $X(t)$ is WSS this becomes

$$
\begin{aligned}
E(Y(t)) &= \mu_X \int h(t)\,dt = \mu_X H(0) \\
R_Y(t,s) &= \int d\alpha \int d\beta R_X((t-s)-(\alpha-\beta))h(\alpha)h(\beta) \\
S_Y(y) &= |H(f)|^2 S_X(f)
\end{aligned}
$$

Power Spectral Density

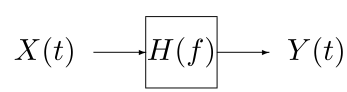
For a WSS process X_n or $X(t)$:

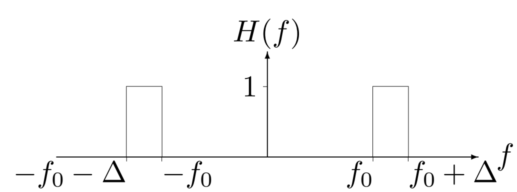
$$
S_X(f) = \begin{cases} \sum_k R_X(k) e^{-j2\pi fk} & \text{discrete time} \\ \int R_X(\tau) e^{-j2\pi f\tau}\,d\tau & \text{continuous time} \end{cases},
$$

$$
R_X(\tau) = \begin{cases} \int_{-1/2}^{1/2} S_X(f) e^{j2\pi f\tau}\,df & \text{discrete time} \\ \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau}\,d\tau & \text{continuous time} \end{cases}.
$$

Why the name *power spectral density*? Consider CT case:

Suppose $X(t)$ a voltage and you want to know the average power across a unit resistor in a particular frequency band $F = \{f : f_0 \leq |f| < f_0 + \Delta f\}$. Can find by putting $X(t)$ through a band pass filter with transfer function shown below and then compute total power in output process, say $Y(t)$





Then

$$\begin{aligned} E[Y(t)^2] &= R_Y(0) = \int S_Y(f)\, df \\ &= \int |H(f)|^2 S_X(f)\, df \\ &= \int_F S_X(f)\, df = \int_{f:f_0 \le |f| < f_0+\Delta} S_X(f)\, df. \end{aligned}$$

i.e., we integrate the psd of $X(t)$ over a frequency band to find the average power over that band, $S_X(f)$ is a *density* of average power in the frequency domain.

Note: This means that $S_X(f) \ge 0$ for all f since integrating it over any arbitrarily small region must give a nonnegative quantity.