

Lecture Notes 8

Random Processes

- Definition and Simple Examples
- Discrete Time Random Processes
 - IID
 - Random Walk Process
 - Markov Processes
 - Independent Increment Processes
 - Gauss-Markov Process
- Mean and Autocorrelation Function
- Gaussian Random Processes

Random Process

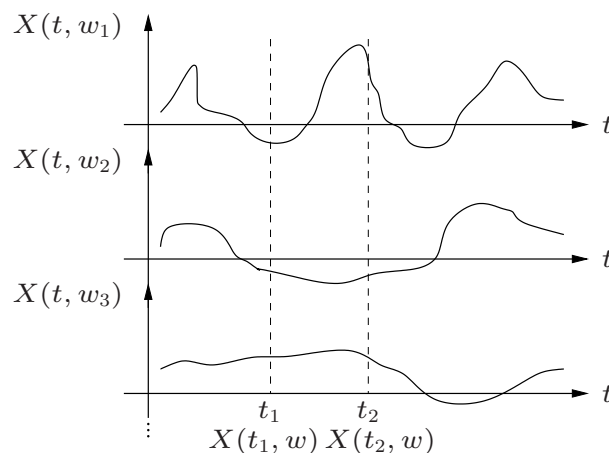
- A *random process* (or *stochastic process*) is an infinite indexed collection of random variables $\{X(t) : t \in \mathcal{T}\}$, defined over a common probability space
- The index parameter t is typically time, but can also be a spatial dimension
- Random processes are used to model random experiments that evolve in time:
 - Received sequence/waveform at the output of a communication channel
 - Packet arrival times at a node in a communication network
 - Thermal noise in a resistor
 - Scores of an NBA team in consecutive games
 - Daily price of a stock
 - Winnings or losses of a gambler

Questions Involving Random Processes

- Dependencies of the random variables of the process
 - How do future received values depend on past received values?
 - How do future prices of a stock depend on its past values?
- Long term averages
 - What is the proportion of time a queue is empty?
 - What is the average noise power at the output of a circuit?
- Extreme or boundary events
 - What is the probability that a link in a communication network is congested?
 - What is the probability that the maximum power in a power distribution line is exceeded?
 - What is the probability that a gambler will lose all his captial?
- Estimation/detection of a signal from a noisy waveform

Two Ways to View a Random Process

- A random process can be viewed as a function $X(t, \omega)$ of two variables, time $t \in \mathcal{T}$ and the outcome of the underlying random experiment $\omega \in \Omega$
 - For fixed t , $X(t, \omega)$ is a random variable over Ω
 - For fixed ω , $X(t, \omega)$ is a deterministic function of t , called a *sample function*

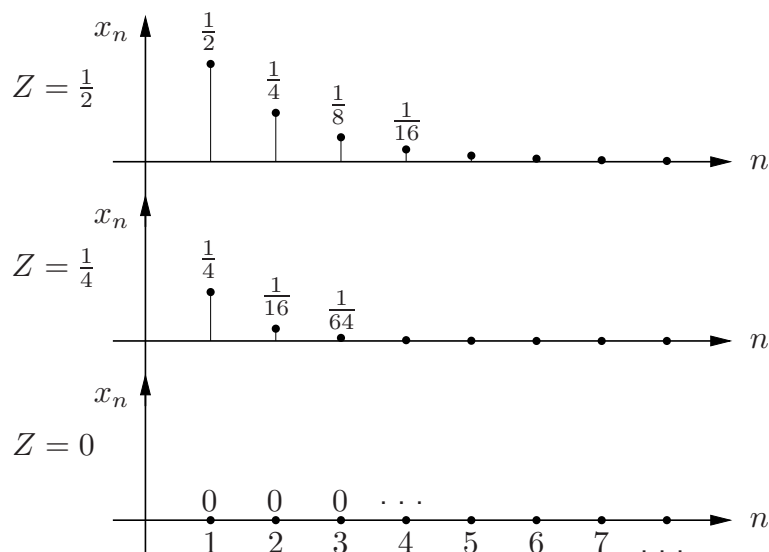


Discrete Time Random Process

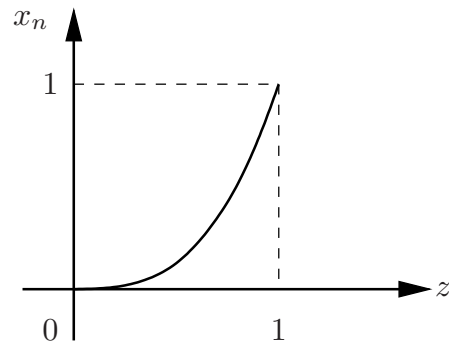
- A random process is said to be *discrete time* if \mathcal{T} is a countably infinite set, e.g.,
 - $N = \{0, 1, 2, \dots\}$
 - $Z = \{\dots, -2, -1, 0, +1, +2, \dots\}$
- In this case the process is denoted by X_n , for $n \in \mathcal{N}$, a countably infinite set, and is simply an infinite sequence of random variables
- A sample function for a discrete time process is called a *sample sequence* or *sample path*
- A discrete-time process can comprise discrete, continuous, or mixed r.v.s

Example

- Let $Z \sim U[0, 1]$, and define the discrete time process $X_n = Z^n$ for $n \geq 1$
- Sample paths:



- First-order pdf of the process: For each n , $X_n = Z^n$ is a r.v.; the sequence of pdfs of X_n is called the *first-order pdf* of the process



Since X_n is a differentiable function of the continuous r.v. Z , we can find its pdf as

$$f_{X_n}(x) = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n} x^{\frac{1}{n}-1}, \quad 0 \leq x \leq 1$$

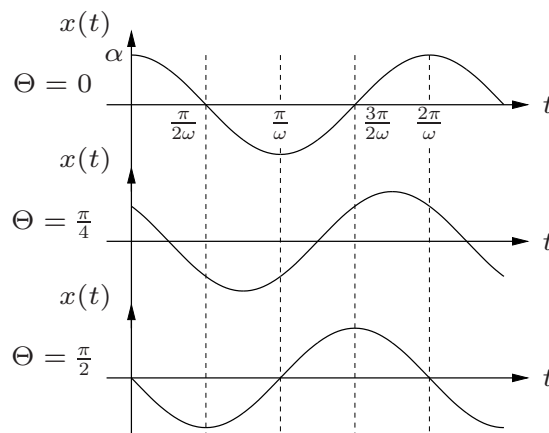
Continuous Time Random Process

- A random process is *continuous time* if \mathcal{T} is a continuous set
- Example: *Sinusoidal Signal with Random Phase*

$$X(t) = \alpha \cos(\omega t + \Theta), \quad t \geq 0$$

where $\Theta \sim U[0, 2\pi]$ and α and ω are constants

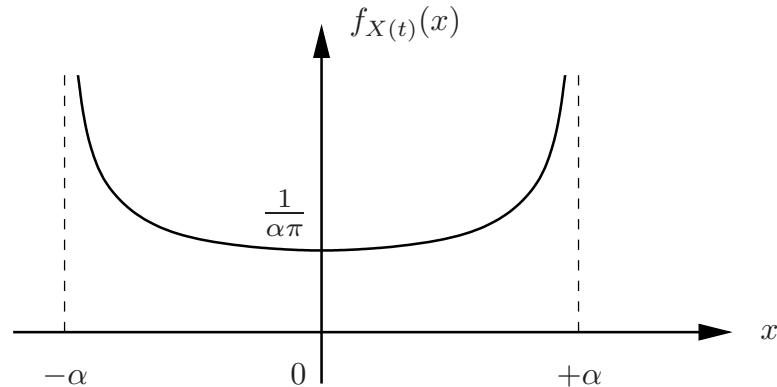
- Sample functions:



- The first-order pdf of the process is the pdf of $X(t) = \alpha \cos(\omega t + \Theta)$. In an earlier homework exercise, we found it to be

$$f_{X(t)}(x) = \frac{1}{\alpha\pi\sqrt{1 - (x/\alpha)^2}}, \quad -\alpha < x < +\alpha$$

The graph of the pdf is shown below



Note that the pdf is independent of t . (The process is *stationary*)

Specifying a Random Process

- In the above examples we directly defined the random process by describing the set of sample functions (sequences, paths) and explicitly providing a probability measure over the set of events (subsets of sample functions)
- This way of describing a random process has very limited applicability, and is suited only for very simple processes
- A random process is typically specified by specifying all its n -th order cdfs (pdfs, pmfs), i.e., the joint cdf (pdf, pmf) of the samples

$$X(t_1), X(t_2), \dots, X(t_n)$$

for every order n and for every set of n points $t_1, t_2, \dots, t_n \in \mathcal{T}$
(Kolmogorov specification)

- The following examples of important random processes will be specified (directly or indirectly) in this manner

Important Discrete Time Processes

- IID process: $\{X_n : n \in \mathcal{N}\}$ is an IID process if the r.v.s X_n are i.i.d.
Examples:
 - Bernoulli process: $X_1, X_2, \dots, X_n, \dots$ i.i.d. $\sim \text{Bern}(p)$
 - Discrete-time white Gaussian noise (WGN): X_1, \dots, X_n, \dots i.i.d. $\sim \mathcal{N}(0, N)$
- Here we specified the n -th order pmfs (pdfs) of the processes by specifying the first-order pmf (pdf) and stating that the r.v.s are independent
- It would be quite difficult to define the process by directly describing the probability measure over the subsets of the sample space

The Random Walk Process

- Let $Z_1, Z_2, \dots, Z_n, \dots$ be i.i.d., where

$$Z_n = \begin{cases} +1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

- The *random walk process* is defined by

$$X_0 = 0$$

$$X_n = \sum_{i=1}^n Z_i, \quad n \geq 1$$

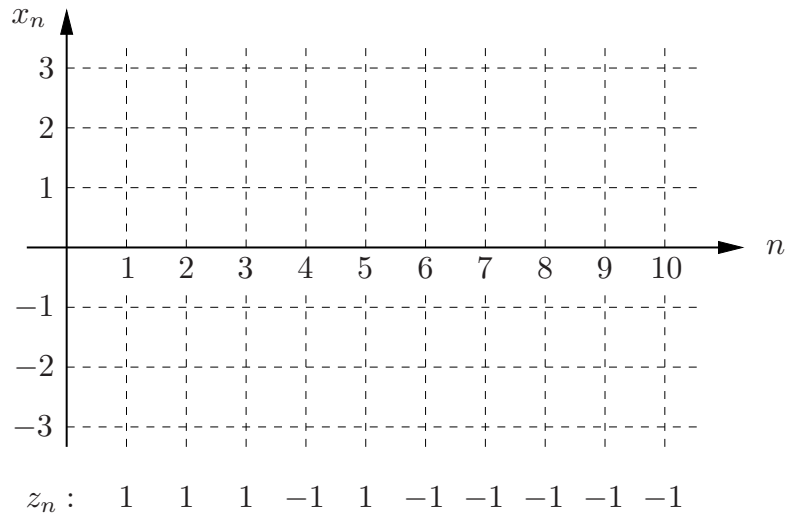
- Again this process is specified by (indirectly) specifying all n -th order pmfs
- Sample path: The sample path for a random walk is a sequence of integers, e.g.,

$$0, +1, 0, -1, -2, -3, -4, \dots$$

or

$$0, +1, +2, +3, +4, +3, +4, +3, +4, \dots$$

Example:



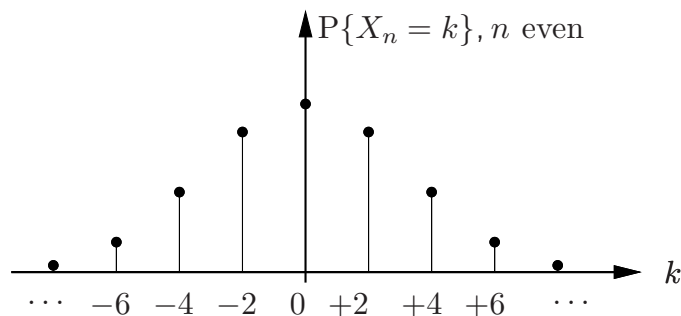
- First-order pmf: The first-order pmf is $P\{X_n = k\}$ as a function of n . Note that
 - $k \in \{-n, -(n-2), \dots, -2, 0, +2, \dots, +(n-2), +n\}$ for n even
 - $k \in \{-n, -(n-2), \dots, -1, +1, +3, \dots, +(n-2), +n\}$ for n odd

Now if we let a be the number of $+1$'s in n steps, then the number of -1 's is $n - a$, and we find that

$$k = a - (n - a) = 2a - n \Rightarrow a = \frac{n + k}{2}$$

Thus

$$\begin{aligned} P\{X_n = k\} &= P\{\frac{1}{2}(n + k) \text{ heads in } n \text{ independent coin tosses}\} \\ &= \binom{n}{\frac{n+k}{2}} \cdot 2^{-n} \quad \text{for appropriate parity of } k \end{aligned}$$



Markov Processes

- A random process X_n is said to be *Markov* if the “future and the past are conditionally independent given the present”
- Mathematically this can be rephrased in several ways. For example, if the r.v.s $\{X_n : n \geq 1\}$ are discrete, then the process is Markov iff

$$p_{X_{n+1}|\mathbf{X}^n}(x_{n+1}|\mathbf{x}^n) = p_{X_{n+1}|X_n}(x_{n+1}|x_n)$$

for every n and every $\mathbf{x}^{n+1} = (x_1, x_2, \dots, x_n, x_{n+1})$

- IID processes are Markov
- The random walk process is Markov. To see this consider

$$\begin{aligned} P\{X_{n+1} = x_{n+1} | \mathbf{X}^n = \mathbf{x}^n\} &= P\{X_n + Z_{n+1} = x_{n+1} | \mathbf{X}^n = \mathbf{x}^n\} \\ &= P\{X_n + Z_{n+1} = x_{n+1} | X_n = x_n\} \\ &= P\{Z_{n+1} = x_{n+1} - x_n\} \\ &= P\{X_{n+1} = x_{n+1} | X_n = x_n\} \end{aligned}$$

Independent Increment Processes

- A random process $\{X_n : n \geq 0\}$ is said to be *independent increment* if the *increment* random variables

$$X_{n_1}, X_{n_2} - X_{n_1}, \dots, X_{n_k} - X_{n_{k-1}}$$

are independent for all sequences of indices such that $n_1 < n_2 < \dots < n_k$

- Example: Random walk is an independent increment process because

$$X_{n_1} = \sum_{i=1}^{n_1} Z_i, \quad X_{n_2} - X_{n_1} = \sum_{i=n_1+1}^{n_2} Z_i, \quad \dots, \quad X_{n_k} - X_{n_{k-1}} = \sum_{i=n_{k-1}+1}^{n_k} Z_i$$

are independent because they are functions of independent random vectors

- The independent increment property makes it easy to find the n -th order pmfs of a random walk process from knowledge only of the first-order pmf

- Example: Find $P\{X_5 = 3, X_{10} = 6, X_{20} = 10\}$ for random walk process $\{X_n\}$

Solution: We use the independent increment property as follows

$$\begin{aligned} P\{X_5 = 3, X_{10} = 6, X_{20} = 10\} &= P\{X_5 = 3, X_{10} - X_5 = 3, X_{20} - X_{10} = 4\} \\ &= P\{X_5 = 3\}P\{X_5 = 3\}P\{X_{10} = 4\} \\ &= \binom{5}{4}2^{-5}\binom{5}{4}2^{-5}\binom{10}{7}2^{-10} = 3000 \cdot 2^{-20} \end{aligned}$$

- In general if a process is independent increment, then it is also Markov. To see this let X_n be an independent increment process and define

$$\Delta \mathbf{X}^n = [X_1, X_2 - X_1, \dots, X_n - X_{n-1}]^T$$

Then

$$\begin{aligned} p_{X_{n+1}|\mathbf{X}^n}(x_{n+1} | \mathbf{x}^n) &= P\{X_{n+1} = x_{n+1} | \mathbf{X}^n = \mathbf{x}^n\} \\ &= P\{X_{n+1} - X_n + X_n = x_{n+1} | \Delta \mathbf{X}^n = \Delta \mathbf{x}^n, X_n = x_n\} \\ &= P\{X_{n+1} = x_{n+1} | X_n = x_n\} \end{aligned}$$

- The converse is not necessarily true, e.g., IID processes are Markov but not independent increment

Gauss-Markov Process

- Let Z_1, Z_2, \dots be i.i.d. $\sim \mathcal{N}(0, N)$, i.e., a WGN process.

The Gauss-Markov process is a *first-order autoregressive process* defined by

$$\begin{aligned} X_1 &= Z_1 \\ X_n &= \alpha X_{n-1} + Z_n, \quad n > 1, \end{aligned}$$

where the parameter α is usually assumed to satisfy $|\alpha| < 1$ (in order to ensure overall stationary behavior, e.g., if $\alpha = 1$ this is a random walk and the variance of X_n will grow with time). If $\alpha = 1$, the process is an independent increment process.

- Note: This process is obtained by passing WGN through a discrete time linear time invariant system (more on this later)
- Clearly the Gauss-Markov process is Markov. It is not, however, an independent increment process unless $\alpha = 1$.

Mean and Autocorrelation Functions

- For a random vector \mathbf{X} the first and second order moments are
 - mean $\boldsymbol{\mu} = \mathbf{E}(\mathbf{X})$
 - correlation matrix $R_{\mathbf{X}} = \mathbf{E}(\mathbf{X}\mathbf{X}^T)$ or covariance matrix $\Sigma_{\mathbf{X}}$
- For a random process $X(t)$ the first and second order moments are
 - *mean* function: $\mu_X(t) = \mathbf{E}(X(t))$ for $t \in \mathcal{T}$
 - *autocorrelation* function: $R_X(t_1, t_2) = \mathbf{E}(X(t_1)X(t_2))$ for $t_1, t_2 \in \mathcal{T}$
- The *autocovariance function* of a random process is defined as

$$C_X(t_1, t_2) = \mathbf{E}[(X(t_1) - \mathbf{E}(X(t_1)))(X(t_2) - \mathbf{E}(X(t_2)))]$$

The autocovariance function can be expressed using the mean and autocorrelation functions as

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

Examples

- IID process:

$$\begin{aligned} \mu_X(n) &= \mathbf{E}(X_1) \\ R_X(n_1, n_2) &= \mathbf{E}(X_{n_1}X_{n_2}) = \begin{cases} \mathbf{E}(X_1^2) & n_1 = n_2 \\ (\mathbf{E}(X_1))^2 & n_1 \neq n_2 \end{cases} \end{aligned}$$

- Random phase signal process:

$$\begin{aligned} \mu_X(t) &= \mathbf{E}(\alpha \cos(\omega t + \Theta)) = \int_0^{2\pi} \frac{\alpha}{2\pi} \cos(\omega t + \theta) d\theta = 0 \\ R_X(t_1, t_2) &= \mathbf{E}(X(t_1)X(t_2)) \\ &= \int_0^{2\pi} \frac{\alpha^2}{2\pi} \cos(\omega t_1 + \theta) \cos(\omega t_2 + \theta) d\theta \\ &= \int_0^{2\pi} \frac{\alpha^2}{4\pi} [\cos(\omega(t_1 + t_2) + 2\theta) + \cos(\omega(t_1 - t_2))] d\theta \\ &= \frac{\alpha^2}{2} \cos(\omega(t_1 - t_2)) \end{aligned}$$

- Random walk:

$$\mu_X(n) = \mathbb{E} \left(\sum_{i=1}^n Z_i \right) = \sum_{i=1}^n 0 = 0$$

$$\begin{aligned} R_X(n_1, n_2) &= \mathbb{E}(X_{n_1} X_{n_2}) \\ &= \mathbb{E} [X_{n_1} (X_{n_2} - X_{n_1} + X_{n_1})] \\ &= \mathbb{E}(X_{n_1}^2) = n_1 \quad \text{assuming } n_2 \geq n_1 \\ &= \min\{n_1, n_2\} \quad \text{in general} \end{aligned}$$

- Gauss-Markov process (with $|\alpha| < 1$):

$$\begin{aligned} \mu_X(n) &= \mathbb{E}(X_n) = \mathbb{E}(\alpha X_{n-1} + Z_n) \\ &= \alpha \mathbb{E}(X_{n-1}) + \mathbb{E}(Z_n) = \alpha \mathbb{E}(X_{n-1}) = \alpha^{n-1} \mathbb{E}(Z_1) = 0 \end{aligned}$$

To find the autocorrelation function, for $n_2 > n_1$ we write

$$X_{n_2} = \alpha^{n_2-n_1} X_{n_1} + \sum_{i=0}^{n_2-n_1-1} \alpha^i Z_{n_2-i}$$

Thus

$$R_X(n_1, n_2) = \mathbb{E}(X_{n_1} X_{n_2}) = \alpha^{n_2-n_1} \mathbb{E}(X_{n_1}^2) + 0,$$

since X_{n_1} and Z_{n_2-i} are independent, zero mean for $0 \leq i \leq n_2 - n_1 - 1$

Next, to find $\mathbb{E}(X_{n_1}^2)$, consider

$$\begin{aligned} \mathbb{E}(X_1^2) &= N \\ \mathbb{E}(X_{n_1}^2) &= \mathbb{E}[(\alpha X_{n_1-1} + Z_{n_1})^2] = \alpha^2 \mathbb{E}(X_{n_1-1}^2) + N \end{aligned}$$

Thus

$$\mathbb{E}(X_{n_1}^2) = \frac{1 - \alpha^{2n_1}}{1 - \alpha^2} N,$$

and finally the autocorrelation function:

$$R_X(n_1, n_2) = \alpha^{|n_2-n_1|} \frac{1 - \alpha^{2 \min\{n_1, n_2\}}}{1 - \alpha^2} N$$

Gaussian Random Processes

- A Gaussian random process (GRP) is a random process $X(t)$ such that

$$[X(t_1), X(t_2), \dots, X(t_n)]^T$$

is a GRV for all $t_1, t_2, \dots, t_n \in \mathcal{T}$

- Since the joint pdf for a GRV is specified by its mean and covariance matrix, a GRP is specified by its mean $\mu_X(t)$ and autocorrelation $R_X(t_1, t_2)$ functions
- Example: The discrete time WGN process is a GRP
- Example: The Gauss-Markov process with $|\alpha| \leq 1$ is a GRP. To see why, note that since $X_1 = Z_1$ and $X_k = \alpha X_{k-1} + Z_k$ where Z_1, Z_2, \dots are i.i.d. $\mathcal{N}(0, N)$,

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \alpha & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha^{n-2} & \alpha^{n-3} & \cdots & 1 & 0 \\ \alpha^{n-1} & \alpha^{n-2} & \cdots & \alpha & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{bmatrix}$$

is a linear transformation of a GRV and is therefore a GRV. If $\alpha = 1$ this process is called *the discrete time Wiener process*.